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Analysis of one-third harmonic generation in waveguides

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This paper investigates one-third harmonic generation (OTHG) by analytical methods with a nondepletion approximation and with an exact solution in continuous wave conditions. The nondepletion method shows that OTHG with a small initial power is confined in a very small range except that the overlapping integrals of the pump and signal follow a certain relation. The efficiency depends only on the initial conditions. Increasing pump power only shortens the interaction length to reach the maximum conversion. Furthermore, we exactly explore OTHG by the elliptic functions whose expressions depend on the types of roots derived from the integral and the initial power. We find that the output power level is limited by the initial conditions and the structure of the waveguide, while the pump power only determines the period. There is a constant value Γ that is determined by U0, β0, and the overlap integrals, where U0 and β0 are the initial power of the signal and the initial phase difference between pump and signal, respectively. We found that a highly efficient conversion only occurs when Γ is larger than a specific value Γc, called a critical value. A Γc provides a relation between U0 and β0, so a set of critical conditions of U0 and β0 is obtained. A highly efficient conversion may be supported if U0 is larger than the power in this set. We investigated some typical structure parameters and found the minimum initial power supporting high conversion efficiency. In OTHG, the variation curve has a sharp peak pattern, which means that a variation of the initial phase difference leads to a great change of the conversion. We established a way to get the smallest initial power with a large phase tolerance. Finally, we find a relation among the overlapping integrals and phase mismatching that can support a high conversion efficiency with a small initial power. This study gives valuable suggestions on the experimental design. © 2014 Optical Society of America

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1. INTRODUCTION

One-third harmonic generation (OTHG), also called third-order parametric fluorescence, is a challenging area that is related to the generation of entangled photon triplets. OTHG is the inversion process of third-harmonic generation (THG) that has been studied in detail and optimized [1]. However, an OTHG experiment is considered significantly more complicated because of the small value of third-order susceptibilities and the phase detuning by Kerr effects [2]. Spontaneous OTHG, in which the initial power of the signal is zero, is a quantum process that generates three entangled photons from vacuum. It has been studied in detail theoretically but has not been demonstrated in experiments [3–6]. An experiment with nonzero initial signals, which is a stimulated process, has been demonstrated in KTP crystal. It was claimed that classical theory can be used to describe the process if the input power of the initial signal is high enough [4,7]. Afshar et al. have studied the efficiencies of THG and OTHG and observed that high conversion efficiency only occurred when the ratio of initial power and pump power was larger than a specific value, i.e., 2% [8]. However, they did not provide the reason for their observation, nor did they provide a way to obtain the ratio. Therefore, more work is necessary to clarify the process of OTHG, such as the effects of the initial phase and the requirement on initial power to support highly efficient conversion and the optimum experiment conditions.

Waveguides may be good devices to demonstrate OTHG in that the overlapping integrals of the signal and pump alter according to the waveguide structure. The continuous wave (CW) condition is a usual experimental condition for OTHG and THG [8,9]. Here, we focus on the CW OTHG process based on waveguides with a small initial signal power using classical theory in CW conditions. By classical coupled equations, the property of signal light evolution is discussed to give valuable suggestions for the experiment design. First, the nondepletion method is used to give a clear concept of OTHG with a very low initial power. Further, based on the former works of the exact solution for four-wave mixing [10], complete analytical solutions are also obtained to study possible high conversion with a very low initial condition. Finally, the influence of the waveguide structure and the initial conditions on OTHG will
be discussed. A set of optimum experimental conditions will be suggested.

2. NONDEPLETION MODEL

CW OTHG is governed by the coupled equations that are derived from classical Helmholtz equations along the z direction by the slowly varying envelope approximation:

$$\frac{dA_1}{dz} = i\gamma_0(J_1|A_1|^2 + 2J_2|A_3|^2)A_1 + J_3(A_1^*J_2A_3^*e^{i\delta\beta}) \tag{1}$$

$$\frac{dA_3}{dz} = i\gamma_0((6J_2|A_1|^2 + 3J_5|A_3|^2)A_3 + J_3A_1^*e^{i\delta\beta}) \tag{2}$$

where the subscripts, 1 and 3, indicate the signal wave with a frequency of $\omega$ and the pump wave with frequency of $3\omega$, respectively. $A$ is the complex amplitude and $\gamma_0 = 2\pi n_0^3/\lambda$. $\delta\beta = \beta(3\omega) - 3\beta(\omega)$. $J_1$, $J_2$, $J_3$, and $J_5$ are field overlap integrals determined by the waveguide structure \cite{8,11}, which correspond to the terms for signal self-phase modulation (SPM), cross-phase modulation, pump–signal overlap, and pump SPM \cite{12}.

When the nondepletion assumption is implemented, $|A_1|^2 \ll |A_3|^2$, and Eqs. (1) and (2) can be simplified as

$$\frac{dA_1}{dz} = i\gamma_0[2J_2|A_3|^2A_1 + J_3(A_1^*J_2A_3^*)e^{i\delta\beta}] \tag{3}$$

$$\frac{dA_3}{dz} = i3\gamma_0|A_3|^2A_3. \tag{4}$$

Here, the complex amplitudes are decomposed into their absolute amplitudes and phase terms as $A_1 = \rho_1e^{i\phi_1}$ and $A_3 = \sqrt{P_3}e^{i\phi_3}$, where $\rho_1$ is the amplitude of the signal wave, $P_3$ is the pump power with $\rho_1 \ll \sqrt{P_3}$, and $\phi_1$ and $\phi_3$ are the phase terms of $\omega$ and $3\omega$, respectively. Simple manipulations show that $\phi_3 = 3\gamma_0J_3P_3z$, and the nondepletion coupled equations transform into a set of equations governing $\rho_1$ and $\theta = \delta\beta z - 3\phi_1 + \phi_3$:

$$\frac{d\rho_1}{dz} \approx -\gamma_0J_3\rho_1^2\sqrt{P_3}\sin \theta, \tag{5}$$

$$\frac{d\theta}{dz} = P_3(\frac{\delta\beta}{P_3} + 3\gamma_0J_3 + 6\gamma_0J_2 - 3\gamma_0J_3\frac{\rho_1}{\sqrt{P_3}}\cos \theta) \approx P_3(\frac{\delta\beta}{P_3} + 3\gamma_0J_3 - 2J_3) = P_3\delta\beta \theta. \tag{6}$$

where $\delta\beta = \delta\beta/P_3 + 3\gamma_0(J_3 - 2J_3)$. The solution is obtained by integration as

$$\rho_1(z) = \frac{P_0\delta\beta\rho_1(0)}{P_3\delta\beta - \gamma_0J_3\rho_1(0)/\sqrt{P_3}\cos[\rho_3\delta\beta z + \theta_0] - \cos \theta_0} = \frac{\rho_1(0)}{1 - \frac{\rho_0G_0}{\sqrt{P_3}}[\cos(\rho_3\delta\beta z + \theta_0) + \cos \theta_0]. \tag{7}$$

where $\theta_0 = \phi_3(0) - 3\phi_1(0)$. The amplitude is a periodic function with period $2\pi/\delta\beta$. If $\delta\beta < 0$, the maximum value of $\rho_1$ is

$$\rho_1(\frac{\pi}{\delta\beta}) = \frac{\rho_1(0)}{1 - \frac{2\rho_0G_0}{\sqrt{P_3}}}. \tag{8}$$

when $\theta_0 = 0$.

Meanwhile, $\rho_1(0)$ is the same maximum when $\theta_0 = \pi$. If $(2\gamma_0J_3\rho_1(0))/\sqrt{P_3}\delta\beta \ll 1$,

$$\rho_1(\frac{\pi}{\delta\beta}) \approx \rho_1(0) + \frac{2\gamma_0J_3}{\sqrt{P_3}\delta\beta}\rho_1(0)^2. \tag{9}$$

The maximum conversion is inversely proportional to $\delta\beta$ and $\sqrt{P_3}$. From Eq. (8), the maximum value will decrease by improving the pump power, which is quite different from other nonlinear phenomena. Accordingly, the maximum conversion efficiency is achieved for longer interaction lengths.

Therefore, we can get some conclusions from the nondepletion result that the highest conversion is determined by $\theta_0$, and is inversely proportional to $\sqrt{P_3}$. A large $J_3$ or a small $J_3 - 2J_3$ will result in a high conversion. However, the nondepletion assumption is not valid when $2\gamma_0J_3\rho_1(0)/\sqrt{P_3}\delta\beta = 1$, which is a singular point in Eq. (8).

If perfect phase matching is achieved, we can get a relation of overlap integrals with a specific conversion rate, which is $\mu = (\rho_1(\pi/P_3\delta\beta))/\rho_1(0)$. Therefore, the structure parameter can be written as

$$\frac{J_3 - 2J_2}{J_3} \approx \frac{4}{3} - \frac{1}{\rho_1(0)} \frac{\rho_1(0)}{\sqrt{P_3}}. \tag{10}$$

This equation can give a guideline for the experimental design. If a high conversion is expected, we have $1/\mu \ll 1$ that leads to

$$\frac{J_3 - 2J_2}{J_3} \approx \frac{4}{3} \frac{\rho_1(0)}{\sqrt{P_3}}. \tag{11}$$

We also note that the period is proportional to the inverse of $J_3 - 2J_2/J_3$, which means a long interaction is required to get higher conversion efficiencies. If the perfect phase matching condition cannot be attained, high conversion can also be achieved by

$$\frac{\delta\beta}{P_3} + 3\gamma_0J_3 = \frac{1}{\mu} \frac{\rho_1(0)}{\sqrt{P_3}}. \tag{12}$$

So if the structure is determined, Eq. (12) can be satisfied to get a high conversion by changing $\delta\beta$. The above conclusions give us a suggestion of the experimental conditions.

3. EXACT SOLUTION OF OTHG

To get an exact solution of the OTHG equations, the following substitutions are adopted: $A_1 = \sqrt{U}Pe^{i\phi}$ and $A_3 = \sqrt{(1 - U)}Pe^{i\phi}$, where $P$ is the total power of $\omega_1$ and $\omega_2$, and $U$ is the ratio of signal power to $P$, which is called the power for convenience. By definitions $\theta = \delta\beta z - 3\phi_1 + \phi_3$ and $\zeta = \gamma_0e^{i\phi}z$, the coupled equations transform into

$$\frac{dU}{dz} = -2\sqrt{U}\beta(U - 1) \sin \theta. \tag{13}$$
\[
\frac{d\theta}{dz} = (\Delta S + b) + \frac{\cos \theta \sqrt{\frac{\gamma_1}{\eta}}}{\sin \theta \sqrt{U^3(1 - U)}} + (a - b)U, \tag{14}
\]

where \(\Delta S, a, \) and \(b\) are defined as
\[
\Delta S = \frac{\delta \beta}{\gamma_0 J_2^P}, \tag{15}
\]
\[
a = \frac{3(J_2 - J_1)}{J_3}, \tag{16}
\]
\[
b = \frac{3(J_5 - 2J_2)}{J_3}. \tag{17}
\]

Equation (14) is solved by straight integration leading to
\[
\Gamma = \sqrt{U^3(1 - U)} \cos \theta - \frac{a - b}{4} U^2 - b + \frac{\Delta S}{2} U, \tag{18}
\]

where \(\Gamma\) is a constant determined by the initial conditions. Equation (18) is substituted into Eq. (13) to get
\[
\zeta = \int_{U_0}^{U(\zeta)} \frac{1}{\sqrt{f(U)}} dU. \tag{19}
\]

\[
f(U) = 4(1 - U)U^3 - 4 \left(\frac{1}{4} (a - b) U^2 + \Gamma + \frac{1}{2} U(b + \Delta S)\right)^2. \tag{20}
\]

Equation (19) is an elliptic integral, which is dependent on the four roots of \(f(U) = 0\). The sign of the integral is determined by the opposite sign of \(\sin \theta_0\). The roots may be all real or contain two complex conjugates and two real numbers, which depend on the value of \(\Gamma\) and \(a, b\). As defined, \(U\) is in the interval \([0,1]\).

From the integration of Eq. (19), we can understand why spontaneous OTHG is not supported in classical theory. If the initial power, \(U_0\), is zero, the solution is not allowed. So spontaneous OTHG is not supported by classical theory and must be described by quantum mechanics.

From the property of Eq. (19) and its physical limitations, \(U\) varies in an interval determined by the two real roots, where \(U_0\) locates. Accordingly, there are two conditions for the roots of \(f(u)\): (1) two of them are complex conjugate, whereas the other two are real and (2) all four roots are real. In the former condition, \(\eta_1 = \eta_3^*, \eta_3 < U_0 < \eta_4\), the solution is given as
\[
U = \frac{\eta_4 M + \eta_3 N - (\eta_4 M - \eta_3 N)cn\left(\frac{\zeta_0}{\eta_3}, k\right)}{M + N + (M - N)cn\left(\frac{\zeta_0}{\eta_3}, k\right)}, \tag{21}
\]
where \(cn(x, k)\) is the Jacobi elliptic function and other items are defined as
\[
m = \text{Re}\left(\frac{\sqrt{-\eta_1^2 - \eta_2^2}}{2}\right). \tag{22}
\]

When all four roots are real, the solution is expressed by Jacobi elliptic function \(cn\). The roots are \(\eta_1 < \eta_2 < \eta_3 < \eta_4\). If \(\eta_1 < U_0 \leq \eta_2\), the solution is given as
\[
U(\zeta) = \frac{\eta_1 \eta_4 - \eta_2 \eta_3 + (\eta_2 \eta_4 - \eta_3 \eta_1)sn^2\left(\frac{\zeta_0}{\eta_4}, k\right)}{\eta_4 - \eta_2 + (\eta_2 - \eta_1)sn^2\left(\frac{\zeta_0}{\eta_4}, k\right)}, \tag{31}
\]
where \(sn\) is the Jacobi elliptic function; \(\zeta_0\) and \(\zeta_0\) are the same as Eqs. (28) and (29); and \(\zeta_0, g, \) and \(k\) are redefined as
\[
\zeta_0 = \zeta_c F\left(\sin^{-1}\left(\frac{(\eta_1 - \eta_2)(U_0 - \eta_1)}{(\eta_2 - \eta_1)(U_1 - U_0)}\right), k\right), \tag{32}
\]
\[
g = \frac{2}{\sqrt{(\eta_4 - \eta_2)(\eta_3 - \eta_1)}}, \tag{33}
\]
\[
k = \frac{(\eta_4 - \eta_2)(\eta_2 - \eta_1)}{(\eta_4 - \eta_2)(\eta_3 - \eta_1)}, \tag{34}
\]
\(g\) is the same for all the solutions with four real roots. If \(\eta_2 \leq U_0 \leq \eta_3\), the solution is given as
\[
U = \frac{\eta_4 \eta_3 - \eta_3 \eta_2 + (\eta_2 \eta_4 - \eta_3 \eta_1)sn^2\left(\frac{\zeta_0}{\eta_3}, k\right)}{\eta_4 - \eta_3 + (\eta_2 - \eta_3)sn^2\left(\frac{\zeta_0}{\eta_3}, k\right)}, \tag{35}
\]
where \(\zeta_0, k, \) and \(\zeta_0\) are defined as
\[
k = \frac{(\eta_3 - \eta_2)(\eta_4 - \eta_1)}{(\eta_4 - \eta_2)(\eta_3 - \eta_1)} \tag{36}
\]
\[ \zeta_0 = \xi F \left( \sin^{-1} \frac{(\eta_4 - \eta_1)(\eta_4 - U_0)}{(\eta_4 - \eta_1)(\eta_4 - U_0)} k \right) \]  

(37)

If \( \eta_3 \leq U_0 < \eta_4 \), the solution is given as

\[ U = \frac{\eta_4 \eta_1 - \eta_4 \eta_3 + (\eta_4 \eta_1 - \eta_4 \eta_3) \sin^2 \left( \frac{\pi \xi_0 + \theta_0}{\xi} \right)}{\eta_1 - \eta_3 + (\eta_3 - \eta_1) \sin^2 \left( \frac{\pi \xi_0 + \theta_0}{\xi} \right)} \]  

(38)

where \( \zeta_0 \) and \( k \) are defined as

\[ k = \frac{(\eta_4 - \eta_3)(\eta_2 - \eta_1)}{(\eta_4 - \eta_2)(\eta_3 - \eta_1)} \]  

(39)

\[ \zeta_0 = \xi F \left( \sin^{-1} \frac{(\eta_4 - \eta_1)(\eta_4 - U_0)}{(\eta_4 - \eta_1)(\eta_4 - U_0)} k \right) \]  

(40)

4. DISCUSSION

A. OTHG in Certain Structure

OTHG can reach a high conversion only when the initial power is larger than a certain value, which is also reported by Ref. [8]. Here, the initial conditions supporting high conversion under certain waveguide structures will be discussed by the results in Section 3. As shown previously, the conversion process is related to initial conditions.

We assume the field overlapping integrals as \( J_5 = 6.0879 \times 10^{10} \text{m}^{-2}, J_4 = 3.3468 \times 10^{10} \text{m}^{-2}, J_2 = 3.2930 \times 10^{10} \text{m}^{-2} \), and \( J_3 = 2.9317 \times 10^{10} \text{m}^{-2} \), which leads to \( a = 3.18 \) and \( b = -0.38 \). The four roots depend on the initial value, \( \Gamma \), that is determined by \( U_0 \) and cos \( \theta_0 \). As \( U_0 \) is between 0 and 1, the roots are the functions of \( \Gamma \) by the general root expression of \( f(u) \). There are two complex conjugate roots unless \( \Gamma \) is between 0 and 0.1. There are two real roots and the absolute of imaginary of the complex roots are plotted in Fig. 1. We found that there are four real roots only when \( \Gamma \) is in the interval \( [0.0008, 0.0018] \). By studying Eq. (18), we found that \( U_0 \) is between the two lower roots or between the two upper roots.

The point at which \( \eta_3 \) and \( \eta_4 \) converge is a critical point that determines the critical value, \( \Gamma_c \). If \( \Gamma \) is larger than \( \Gamma_c \), \( \eta_3 < \eta_4 \). As \( \Gamma \) varies just between \( \eta_3 \) and \( \eta_4 \), a set of roots with small \( \eta_3 \) and large \( \eta_4 \) results in a high conversion efficiency with a small initial power. For clarity, the range of \( \Gamma \) and the roots are shown in Table 1.

Table 1 gives the range of \( \Gamma \) and \( U_0 \) for simplicity.

If \( U_0 \) is between the lower roots, a high conversion efficiency is not supported because the highest conversion limit is \( \eta_2 \), which is always small, shown as the gray area in Fig. 1. By Eq. (18), \( \Gamma_c \) can determine a set of \( U_0 \) with \( \theta_0 \) shown in Fig. 2(a). \( U_0 \) is in the interval \( [0.0035, 0.013] \). Accordingly, the minimum initial power, \( U_0 \), supported a high conversion efficiency is derived as 0.003 with \( \theta_0 = 0 \), where the roots are \( 0.01270 - i 0.0018 \) and \( 0.01270 + i 0.0018 \), 0.0035, 0.7178. So \( U \) varies between 0.0035 and 0.7178. A high conversion may be expected. As shown in Ref. [8], a high conversion is supported only when \( U_0 \) is larger than 0.02. Here, we can extend the high conversion range of \( U_0 \) to 0.0035. However, the initial phase, \( \theta_0 \), plays a key role in the conversion type when \( U_0 \) is between 0.0035 and 0.013.

The first three roots with the cosine of different \( \theta_0 \) are plotted in Figs. 2(b)–2(d). The points in Figs. 2(b)–2(d) are the critical points. There are four real roots if \( U_0 \) is smaller than the corresponding \( U_0 \) of the points, whereas two real roots are there when \( U_0 \) is larger. At these points, \( \Gamma \) is calculated as 0.0008.

If the \( \theta_0 \) slightly changes from 0, the critical initial power becomes large, which means that \( \Gamma \), larger than \( \Gamma_c \), does not support high conversion anyway. So the phase difference between pump and signal must be controlled strictly, which is surely difficult. Therefore, a relatively large value of \( U_0 \) between 0.0035 and 0.013 is chosen for a reasonable phase tolerance.

By calculating with the results in Section 3, Fig. 3(a) shows the evolution of \( U \) on the conditions as \( U_0 = 0.012 \), and \( \theta_0 \) equals \( \pi, 0, \pi/2, 2\pi/3, \pi/3 \), respectively. The positive \( \theta_0 \) shares the same evolution function as the negative one after shifting of \( \zeta \) becomes \( -\zeta \). We can see that the initial phase difference influences the position of highest conversion. Usually, the phase difference between two lasers is stochastic. So the position of highest conversion is random at a large distance. This suggests that the phase difference be controlled strictly to get a high conversion with a specific interaction length. The corresponding solutions, simulating
the exact coupled wave equations, as Eqs. (1) and (2) by the ordinary differential equations (ODE) numerical method, are shown in Fig. 3(b), which is consistent with Fig. 3(a).

If the initial power is very small, such as 0.0001, the solution of \( f(u) \) has four real roots. As shown in Fig. 1, \( U_0 \) locates between the two smallest roots. Accordingly, the variation of signal is denoted by Eq. (31). Figure 4(a) shows that the variation pattern changes with \( \theta_0 \). When \( \theta_0 = 0 \), the power of the signal will transfer into the pump, which is a THG process rather than OTHG. This is similar to the result derived from the nondepletion method. Figure 4(b) gives the corresponding result by ODE numerical simulation of the exact equations (1) and (2).

In summary, for a specific structure that leads to \( \Delta S \approx 0 \), the critical value is obtained by analyzing the roots of \( f(u) = 0 \). By Eq. (18), a set of \( U_0 \) and \( \theta_0 \) is determined. For a good tolerance of phase shifting, a medium value should be selected as the initial power.

**B. Optimization of Experiment**

Here, we begin to discuss the way to decrease the critical value in order to support highly efficient conversion.

As shown in Section 2 by the nondepletion method, we find that the value of \( J_3 - 2J_3 P_3 \), proportional to \( b \) as shown in Eq. (17), plays a key role in OTHG with low initial power. If it gets a small initial value, a high conversion may be obtained with a large interaction length. In particular, the conversion is infinity when it is near zero or near \(-Δ/S_3\), which is beyond the approximation of the nondepletion method. However, we can also get information that the conversion efficiency may be higher if \( b \) is near zero when a perfect phase match is achieved as \( ΔS = 0 \).

If \( b = 0 \), \( f(u) \) transforms to

\[
f(U) = 4(1 - U)U^3 - \left( \frac{a}{4} U^2 + \Gamma \right)^2.
\]

Furthermore, we consider the situation in which \( a \) is also zero. \( \Gamma \) is zero by the initial phase difference as \( \pi/2 \); the solution is obtained as

\[
U = \frac{1}{2} \left( \frac{2}{a} \right)^{1/3}.
\]
where $\zeta_0 = 2\sqrt{1/U_0} - 1$. $\zeta_0$ is the position of maximum conversion, which is inversely related to $U_0$.

If $\Gamma \neq 0$, it follows the procedure in Section 3. We use the small initial power in Section 3 as $U_0 = 0.0001$. By Eq. (41), we can get the power evolution with a maximum conversion up to 100% with a long distance and a sharp peak. If $a \neq 0$, it also follows the procedure in Section 3. If $a$ varies between $-8$ and $8$, and $\Gamma$ varies between $-2.06024$ and $2.06024$, the two roots are functions of $a$ and $\Gamma$, as shown in Fig. 5.

We can find that there is a large difference between $\eta_3$ and $\eta_4$ when $a < 1$, which implies that an efficient conversion may be obtained in this region. Using the same initial condition as above, we can find the maximum of the third root is much smaller than the initial value, $U_0$, which means $U$ can reach a large value. A smaller $a$ leads to a small period. Figure 6(b) gives two examples with $a$ as 8 and 0.1, respectively, which are evaluated by the exact coupled wave equations [Eqs. (1) and (2)] with the ODE numerical method. Figure 6(a) is $\eta_4$ depending on $a$. As discussed above, $\eta_4$ is the upper limit of $U$, so a large value of $a$ will lead to a small conversion.

According to the result in Eq. (11), a high conversion may be expected when $b$ equals $-\sqrt{U_0}$. Accordingly, we get $b = -0.01$ as the initial power 0.0001. $a$ is assumed as 1. As shown in Fig. 7, the variation of signal is plotted by the ODE numerical method, in which we confirm that a highly efficient conversion occurs.

However, to design a waveguide satisfying the requirement is difficult. From Eq. (18), if $\Delta S$ is not zero, dependence of $\Gamma$
is larger than $b$ is supposed, there are 2 to 4 causes the critical value $a=0.1$ and $\theta_0=0.5$. When $b=\Delta S$ is satisfied, a high conversion can be obtained.

Numerical simulation of Eqs. (1) and (2) confirms that a highly efficient conversion is supported by both values of $b$ as 0.1 and 4, shown in Fig. 8. However, a larger value of $b$ leads to a longer length to get the maximum conversion. A small value of $b$ is desired in a variation with short length. To get a highly efficient OTHG experiment, we should let the sum of $\Delta S$ and $b$ be as small as possible. In Fig. 8(b), the smaller real root is plotted with dependence of $\Delta S$. Highly efficient conversion may be supported at the interval of $\Delta S$ as $[0.075, 0.14]$ in that $\eta_3$ is smaller than $U_0$. The variation patterns of $U$ with different $\Delta S$ is shown in Fig. 8(c). It is obvious that highly efficient conversion is supported by all these values of $\Delta S$.

For an experimental demonstration, an initial signal power and a pump power are determined. The waveguide should be designed to let the value of $b$ be as small as possible. And the mismatching parameter, $\delta b$, should be tuned to equal $b$ to support a highly efficient conversion during the experiment. So we suggest a guideline to construct the structure parameter supporting a high conversion OTHG from a low initial power. $a$ and $b$ determined by $J_1$ to $J_5$ should be designed as small as possible. The mismatching factor is considered to be around the negative value of $b$.

5. CONCLUSION

Using the nondepletion and exact solution, we analyze the OTHG process to find the dependence on the initial condition and structure parameter. The critical value $\Gamma_c$ that derives a set of critical $U_0$ and $\theta_0$ is found. High conversion can be achieved only from the initial power larger than the specific ratio of the whole power for a specific structure, whereas the conversion is very low. $\Delta S + b \approx 0$ causes the critical value $\Gamma_c$ to be low, which supports a high conversion efficiency with a small initial phase. This work can give a useful direction to the experimental demonstration of OTHG.

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