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<th>Spectral properties for a new composition of a matrix and a complex representation</th>
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A way to compose a matrix $P$ and a finite dimensional representation $\rho$ of $C$ via a map $h$ into a new matrix $P \ast_h \rho$ is defined. Several results about the spectrum, eigenvectors, kernel and rank of $P \ast_h \rho$ are proved.


1. Introduction

In two recent papers [4, 5] it has been pointed out the interest of the symmetric matrix

$$P(\omega) := \begin{bmatrix} P_1(\omega) & P_2(\omega) \\ P_2^T(\omega) & P_1(\omega) \end{bmatrix}$$

for the design of some signal filters, where $P_1(\omega)$ and $P_2(\omega)$ are the square matrices of order $N$ whose entries are

$$(P_1(\omega))_{i,j}^N := (i + j - 2) \cos((i - j)\omega), \quad (P_2(\omega))_{i,j}^N := (i + j - 2) \sin((i - j)\omega).$$

In particular, it has been conjectured that the spectrum of $P(\omega)$, i.e. its eigenvalues and their multiplicities, is actually independent of $\omega$. In this paper we prove this fact as a consequence of a more general result (Theorem 2 and Proposition 2 here below). Indeed, we introduce a procedure which generalizes the construction of $P(\omega)$ and we prove the conjecture for each matrix we obtain in this way. Several other results exploring the connection of the new operation with other ways to combine matrices into a new matrix are given.

2. Results

Let $\rho$ be an $r$ dimensional linear representation of $C$, i.e. a map $C \to GL(r, C)$ satisfying the condition

$$(P \ast_h \rho)_{i,j} := (P \ast_h \rho)_{(1,1)} \cdots (P \ast_h \rho)_{(r,1)} \cdots (P \ast_h \rho)_{(1,r)} \cdots (P \ast_h \rho)_{(r,r)}$$

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where each block \((P \ast_h \rho)_{(I,J,:)}\) is itself an \(n \times m\) matrix and is defined as
\[
(P \ast_h \rho)_{(I,J,:)} := P_{i,j} \rho_{I,J}(h(i,j)) \quad i = 1, \ldots, n, \quad j = 1, \ldots, m.
\]

**Remark 1.** There is a canonic way to build \(\rho\): let \(T\) be an arbitrary square matrix in \(\mathcal{M}(r, \mathbb{C})\), and take
\[
\rho(x) := \exp(xT) := \sum_{k=0}^{\infty} \frac{x^k}{k!} T^k, \quad \forall x \in \mathbb{C}.
\]
Every regular (analytic) representation is of this form (see [1, Ch. 6 Appendix A] and [3, Ch. 8]).

**Remark 2.** It is immediate to realize that (2) is satisfied if and only if \(h(i, j) = g(i) - g(j)\) for some map \(g : \mathbb{N} \to \mathbb{C}\).

**Remark 3.** The matrix \(P(\omega)\) is of the form \(P \ast_h \rho\) with
\[
P \in \mathcal{M}(N, \mathbb{C}) : \quad (P)_{i,j} := i + j - 2,
\]
\[
\rho : \mathbb{C} \to GL(2, \mathbb{C}) : \quad \rho(x) := \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix},
\]
\[
h(i, j) := (i - j)\omega.
\]

The following theorem gives a first set of properties for \(P \ast_h \rho\) in terms of analogous properties for \(P\).

**Theorem 1.**

1. \((\mu P + \nu Q) \ast_h \rho = \mu P \ast_h \rho + \nu Q \ast_h \rho\) for every \(P, Q \in \mathcal{M}(n \times m, \mathbb{C})\) and for every \(\mu, \nu \in \mathbb{C}\). Moreover, \(P \ast_h \rho = 0\) if and only if \(P = 0\).

2. Let \(P\) be diagonal, then \(P \ast_h \rho\) is the direct sum of \(r\) copies of \(P\) and therefore it is diagonal too. In particular, \(\mathbb{I}_n \ast_h \rho = \mathbb{I}_n\).

3. \((P \ast_h \rho)^* = P^* \ast_{-h} \rho^*\). In particular, if \(P\) is a square matrix and the restriction of \(\rho\) on the range of \(h\) is unitary (i.e., if \(\rho^*(x) = \rho(-x)\) for every \(x\) in the range of \(h\)), then \(P \ast_h \rho\) is self-adjoint if and only if \(P\) is self-adjoint.

4. Let \(P \in \mathcal{M}(n \times l, \mathbb{C})\) and \(Q \in \mathcal{M}(l \times m, \mathbb{C})\), then
\[
(P \ast_h \rho) \cdot (Q \ast_h \rho) = (P \cdot Q) \ast_h \rho.
\]

5. Let \(P\) be a square matrix. The minimal polynomials of \(P\) and \(P \ast_h \rho\) are equal.

6. \(P \ast_h \rho\) is diagonalizable if and only if \(P\) is diagonalizable, and a complex number \(\lambda\) is an eigenvalue for \(P \ast_h \rho\) if and only if it is an eigenvalue for \(P\).

7. \(P \in GL(n, \mathbb{C})\) if and only if \(P \ast_h \rho \in GL(nr, \mathbb{C})\), with \((P \ast_h \rho)^{-1} = P^{-1} \ast_h \rho\).

**Proof.**

1. The linearity of \(P \ast_h \rho\) as a function of \(P\) is evident; it implies that \(0 \ast_h \rho = 0\).

Suppose that \(P_{i,j} \rho_I,J(h(i,j)) = 0\) for every \(I, J = 1, \ldots, r\) and every \(i = 1, \ldots, n, j = 1, \ldots, m\), and that by absurd \(P_{i,j} \neq 0\) for a couple of indexes \(i, j\). Then \(\rho_I,J(h(i,j)) = 0\) for every \(I\) and \(J\), which is impossible because \(\rho(h(i,j)) \in GL(r, \mathbb{C})\).
2. Let $P$ be diagonal, so that $P_{i,j} = a_i \delta_{i,j}$, then:
\[
(P \ast h)_{i,j} = a_i \delta_{i,j} \rho_{I,J}(h(i,j)) = a_i \delta_{i,j} \rho_{I,J}(h(i,i)) = a_i \delta_{i,j} \rho_{I,J}(0)
\]
because $h$ is an odd map, and this is $a_i \delta_{i,j} \delta_{I,J}$, because $\rho(0) = I_r$.

3. The equality $(P \ast h \rho)^* = P^* \ast -h \rho^*$ is an immediate consequence of the definition of the $\ast$-product. Now suppose that $\rho^*(-h(i,j)) = \rho(h(i,j))$, then $(P \ast h \rho)^* = P^* \ast h \rho$ so that this is equal to $P \ast h \rho$ if and only if $P^* - P \ast h \rho = 0$, i.e. if and only if $P^* = P$, by Item 1.

4. The proof is a direct consequence of Relations (1–2). In fact, for every couple of indexes $I, J = 1, \ldots, r$ and $i = 1, \ldots, n$, $j = 1, \ldots, m$ we have
\[
((P \ast h \rho) \cdot (Q \ast h \rho))_{i,j} = \sum_{k,K} P_{i,k} \rho_{I,K}(h(i,k)) Q_{k,j} \rho_{K,J}(h(k,j))
\]
By (1) the inner sum is $\rho_{I,J}(h(i,k) + h(k,j))$, i.e. $\rho_{I,J}(h(i,j))$, by (2). Thus
\[
((P \ast h \rho) \cdot (Q \ast h \rho))_{i,j} = \sum_k P_{i,k} Q_{k,j} \rho_{I,J}(h(i,j))
\]
which is the claim.

5. The formula in Item 4 implies that $f(P \ast h \rho) = f(P) \ast h \rho$ for every polynomial $f \in \mathbb{C}[x]$, so that $f(P \ast h \rho)$ is null if and only if $f(P)$ is null as well, by Item 1. The claim follows by the definition of the minimal polynomial of a matrix $A$ as the monic generator of the ideal of complex polynomials $f$ for which $f(A) = 0$.

6. A matrix is diagonalizable if and only if its minimal polynomial $f \in \mathbb{C}[x]$ factorizes in $\mathbb{C}[x]$ as product of distinct linear polynomials. Therefore the first claim follows by Item 5. Moreover, the eigenvalues coincide with the roots of the minimal polynomial, therefore Item 5 implies also the second part of this claim.

7. A matrix is invertible if and only if 0 is not an eigenvalue. Therefore the first part of the claim follows by Item 6. The formula for $(P \ast h \rho)^{-1}$ is an immediate consequence of Items 2 and 4.

\[\square\]

Item 6 of the previous theorem already shows that the spectrum of $P \ast h \rho$ and that one of $P$ contain the same points, but their spectral structures are even more strictly related. In fact, the next two theorems prove that also the eigenvectors of $P \ast h \rho$ can be easily deduced by those ones of $P$. We start with a general result which is of some independent interest.

**Theorem 2.** Let $P \in \mathcal{M}(n \times m, \mathbb{C})$. Then $\dim \ker(P \ast h \rho) = r \dim \ker(P)$ and $\rank(P \ast h \rho) = r \rank(P)$. 

Proof. Let $s$ denote the rank of $P$. The definition of rank implies the existence of a permutation $P \in \text{GL}(n, \mathbb{C})$ and a permutation $P' \in \text{GL}(m, \mathbb{C})$ such that

$$P' := P P' = \begin{bmatrix} P' & * \\ P & * \\ & * \\ & & * \end{bmatrix}$$

with $P''$ in $\text{GL}(s, \mathbb{C})$. By Theorem 1 Item 7 the matrices $P \rho_h$ and $P' \rho_h$ are invertible, so that the rank of $P' \rho_h$ is equal to that one of $P \rho_h$. Moreover, the matrix $P'' \rho_h$ is a submatrix in $P' \rho_h$, therefore $\text{rank}(P' \rho_h) \geq \text{rank}(P'' \rho_h)$ and the rank of $P'' \rho_h$ is $sr$ because it is in $\text{GL}(sr, \mathbb{C})$. As a consequence we have proved that

$$(3) \quad \text{rank}(P \rho_h) \geq r \text{rank}(P).$$

Let $v_1, \ldots, v_k$ be a basis for the kernel of $P$. Let $V := [v_1 \mid \ldots \mid v_k]$ be the matrix having the vectors $v_j$ for $j = 1, \ldots, k$ as columns. By (3) applied to $V$ we get that $\text{rank}(V \rho_h) \geq kr$. The columns in $V \rho_h$ belong to the kernel of $P \rho_h$, by Item 4 in Theorem 1; this proves that

$$(4) \quad \dim \ker(P \rho_h) \geq r \dim \ker(P).$$

Adding (3) and (4) and recalling the rank-nullity theorem we conclude that

$$mr = \text{rank}(P \rho_h) + \dim \ker(P \rho_h) \geq r \text{rank}(P) + r \dim \ker(P) = mr$$

which proves that the equality holds in (3) and (4). \hfill \Box

Let $P$ be a square matrix. For each $\lambda \in \mathbb{C}$ let $E_\lambda$ denote the kernel of $P - \lambda I_n$ (i.e. the $\lambda$-eigenspace of $P$ when $\lambda$ belongs to the spectrum of $P$), and analogously let $E_{\lambda, \rho}$ denote the kernel of $P \rho_h - \lambda I_{mr}$.

**Proposition 1.** For every $\lambda \in \mathbb{C}$, $\dim E_{\lambda, \rho} = r \dim E_\lambda$. In particular, $P$ and $P \rho_h$ have the same eigenvalues, and the multiplicity of every $\lambda$ as eigenvalue for $P \rho_h$ is $r$ times its multiplicity as eigenvalue for $P$. Moreover, if the columns in $V \in \mathcal{M}(m \times \dim E_\lambda, \mathbb{C})$ are a basis for $E_\lambda$, then the columns of $V \rho_h$ are a basis for $E_{\lambda, \rho}$.

**Proof.** In fact, $E_{\lambda, \rho} = \ker(P \rho_h - \lambda I_{mr}) = \ker((P - \lambda I_n) \rho_h)$ so that the claims follow by the previous theorem. \hfill \Box

**Remark 4.** We can rephrase the claims of Proposition 1 by saying that the spectrum (i.e., the eigenvalues and the dimension of each eigenspace) of $P \rho_h$ is independent of $h$ and depends on $\rho$ only via its dimension; this claim already suffices to completely determine the spectrum of $P \rho_h$ since one sees immediately that $P \rho_h$ collapses to the direct sum of $r$ copies of $P$ when $h$ is taken equal to 0 identically. The conjectured independence of the spectrum of $P(\omega)$ of $\omega$ in [5], therefore, is evidently only a special case of the independence of the spectrum of $P \rho_h$ on $h$ claimed in Proposition 1 when it is restated in this way.

Let $V$ and $W$ be two matrices, respectively in $\mathcal{M}(n \times v, \mathbb{C})$ and $\mathcal{M}(n \times w, \mathbb{C})$. Then we can form the new matrix $[V \mid W]$ in $\mathcal{M}(n \times (v + w), \mathbb{C})$ by joining the columns of $W$ to those ones of $V$. In general, the matrices $[V \rho_h \mid W \rho_h]$ and $[V \mid W] \rho_h$ are distinct, but they are quite strictly related. We begin with a simple computation, which is useful in applications. Suppose that the columns of $V$ and $W$ be eigenvectors for a matrix $P$, so that $PV = VD_V$ and $PW = WD_W$ with $D_V$
and $D_W$ diagonal matrices. Then, using two times the multiplicativity property for the $*_h$-product (Item 4 of Theorem 1), we get
\[
(P *_h \rho) \cdot [V | W] *_h \rho = (P \cdot [V | W]) *_h \rho = ([PV | PW]) *_h \rho = ([VD_V | WD_W]) *_h \rho = ([V | W] \cdot (D_V \oplus D_W)) *_h \rho = ([V | W] *_h \rho) \cdot ((D_V \oplus D_W) *_h \rho).
\]
The matrix $(D_V \oplus D_W) *_h \rho$ is diagonal and in particular is the direct sum of $r$ copies of $D_V \oplus D_W$ (by Item 2 of Theorem 1, because $D_V \oplus D_W$ is diagonal by the assumption), hence we have proved the following claim.

**Proposition 2.** With the previous notations, the columns of $[V | W] *_h \rho$ are eigenvectors for $P *_h \rho$, and if $\lambda_V$ and $\lambda_W$ denote the eigenvalues for the columns of $V$ and $W$ (i.e., the main diagonals of $D_V$ and $D_W$), then the eigenvalues corresponding to the columns of $[V | W] *_h \rho$ are the sequence $\lambda_V, \lambda_W, \lambda_V, \lambda_W, \ldots, \lambda_V, \lambda_W$ ($r$ couples).

Probably, the typical use of this computation will be in ‘tandem’ with Proposition 1, to produce a set of eigenvectors for $P *_h \rho$. We illustrate this through a simple example as follows. Consider $P$ with $N = 2$, $\rho$, $h$ and $P(\omega)$ as given in Remark 3, thus
\[
P = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, \quad P(\omega) = \begin{bmatrix} 0 & \cos \omega & 0 & -\sin \omega \\ \cos \omega & 2 & \sin \omega & 0 \\ 0 & \sin \omega & 0 & \cos \omega \\ -\sin \omega & 0 & \cos \omega & 2 \end{bmatrix}.
\]
The eigenvalues of $P$ (and hence also $P(\omega)$, with multiplicity 2) are $\lambda_\pm := 1 \pm \sqrt{2}$ with $v_\pm := \begin{bmatrix} 0 \\ 1 \pm \sqrt{2} \end{bmatrix}$ as corresponding eigenvectors. By extending each $v_\pm$ through $v_\pm *_h \rho$ we form the matrix
\[
Q(\omega) := [v_+ *_h \rho \mid v_- *_h \rho] = \begin{bmatrix} 1 & (1 + \sqrt{2}) \cos \omega & (1 + \sqrt{2}) \sin \omega & (1 - \sqrt{2}) \cos \omega & (1 - \sqrt{2}) \sin \omega \\ 0 & 1 & 1 & 0 & 1 \\ -(1 + \sqrt{2}) \sin \omega & (1 + \sqrt{2}) \cos \omega & -(1 - \sqrt{2}) \sin \omega & (1 - \sqrt{2}) \cos \omega \end{bmatrix}
\]
for which $P(\omega)Q(\omega) = Q(\omega) \text{diag}\{\lambda_+, \lambda_+, \lambda_, \lambda_-, \lambda_, \lambda_+\}$. Analogously, we can form the other matrix
\[
Q'(\omega) := [v_+ \mid v_-] *_h \rho = \begin{bmatrix} 1 & \cos \omega & 0 & -\sin \omega \\ (1 + \sqrt{2}) \cos \omega & (1 - \sqrt{2}) \cos \omega & (1 \pm \sqrt{2}) \sin \omega & 0 \\ 0 & \sin \omega & \cos \omega & 1 \\ -(1 - \sqrt{2}) \sin \omega & -(1 \pm \sqrt{2}) \cos \omega & 0 & 1 \end{bmatrix},
\]
for which $P(\omega)Q'(\omega) = Q'(\omega) \text{diag}\{\lambda_+, \lambda_, \lambda_, \lambda_-, \lambda_-, \lambda_+\}$. Both the procedures give a basis of eigenvectors for $P(\omega)$ from a basis of eigenvectors for $P$: for $Q(\omega)$ this is a consequence of Proposition 1, for $Q'(\omega)$ it is a consequence of Theorem 2 (or even by Item 7 in Theorem 1). These approaches to the construction of eigenvectors for $P(\omega)$ are in general convenient for applications: to obtain the eigenvectors directly from $\ker(P *_h \rho - \lambda I_4)$ one would need to solve a system of equations involving polynomials in $\sin \omega$ and $\cos \omega$ and this could be a computationally quite difficult task. In fact, although there are computational algebraic methods for solving a system of equations in a polynomial ring with several independent variables $z_1, \ldots, z_k$, e.g., Gröbner
bases [2], there is not a constructive method for solving a system of equations in the polynomial ring in $\sin \omega$ and $\cos \omega$ because now the variables $z_1 := \sin \omega$ and $z_2 := \cos \omega$ are algebraically dependent.

In the previous example it is easy to check that $\det Q(\omega) = -8$, $\det Q'(\omega) = 8$ and $\det([v_+|v_-]) = \sqrt{8}$; the simple relation between these determinants is not casual, but it is a consequence of a general relation which we explore now.

Let $V$, $W$ be generic $n \times v$ and $n \times w$ matrices. Then,

$$([V | W] *_h \rho)_{(I,J,i,j)} = \begin{cases} V_{i,j} \rho_{I,J}(h(i,j)) & \text{if } j \leq v, \\ W_{i,j-v} \rho_{I,J}(h(i,j)) & \text{if } j > v. \end{cases}$$

This proves that there exists a permutation $P$ such that

$$[V | W] *_h \rho \cdot P^{-1} = [V *_h \rho | S],$$

where

$$S \in \mathcal{M}(nr \times wr, \mathbb{C}), \quad \text{with} \quad S_{(I,J,i,j)} := W_{i,j} \rho_{I,J}(h(i,j+v)).$$

Noting that $h(i,j+v)$ can be written as $h(i,j) + h(j,j+v)$ and using the multiplicativity of the representation $\rho$, we get

$$S_{(I,J,i,j)} = W_{i,j} \rho_{I,J}(h(i,j) + h(j,j+v))$$

$$= \sum_K W_{i,j} \rho_{I,K}(h(i,j)) \rho_{K,J}(h(j,j+v))$$

$$= \sum_K (W *_h \rho)_{(I,K,i,j)} \rho_{K,J}(h(j,j+v)).$$

In matricial form this equality can be written as

$$S = (W *_h \rho) \cdot B$$

where

$$B = \begin{bmatrix} B_{1,1} & \cdots & B_{1,r} \\
\vdots & \cdots & \vdots \\
B_{r,1} & \cdots & B_{r,r} \end{bmatrix}, \quad B_{I,J} := \text{diag}\{\rho_{I,J}(h(1,v+1)), \ldots, \rho_{I,J}(h(w,v+w))\}.$$

This structure proves the existence of two permutations $Q$, $Q'$ in $\text{GL}(rw, \mathbb{C})$ such that

$$B = Q \bigoplus_{j=1}^w \rho(h(j,j+v))Q'.$$

Their definition makes evident that $Q$ and $Q'$ depend on $\rho$ only via its order; when $\rho$ is the trivial representation both $B$ and $\bigoplus_{j=1}^w \rho(h(j,j+v))$ collapse to the identity, thus proving that $Q' = Q^{-1}$. As a consequence we have proved that

$$S = W *_h \rho \cdot Q \bigoplus_{j=1}^w \rho(h(j,j+v))Q^{-1}.$$

With (5), this equality proves the following formula.

**Theorem 3.** With the previous notations, we have

$$[V | W] *_h \rho = [V *_h \rho | W *_h \rho] \cdot (\mathbb{I}_{rv} \oplus \bigoplus_{j=1}^w \rho(h(j,j+v))Q^{-1}) \cdot P.$$

In particular, the ranks of $[V \mid W] *_{h} \rho$ and of $[V *_{h} \rho \mid W *_{h} \rho]$ are equal and when $v + w = n$, i.e., when $[V \mid W]$ is a square matrix, we have
\[
\det([V *_{h} \rho \mid W *_{h} \rho]) = (-1)^{vw(w)} \det([V \mid W] *_{h} \rho) \det(\rho(\sum_{j=1}^{w} h(j + v, j))).
\]

Proof. The formula for $\det([V *_{h} \rho \mid W *_{h} \rho])$ is a direct consequence of (6), apart the computation of the determinant of $\mathcal{P}$, for which we need the following explicit description coming directly from its definition in (5). Split the integers $\{1, \ldots, vr\}$ in $r$ consecutive blocks denoted as $n_1, \ldots, n_r$ having $v$ integers each one, and analogously split the integers $\{vr + 1, \ldots, vr + wr\}$ in $r$ consecutive blocks denoted as $m_1, \ldots, m_r$, having $w$ integers each one. Then $\mathcal{P}$ is the ‘shuffle’ permutation which moves the blocks according to the following rule:
\[
\mathcal{P} : (n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r) \mapsto (n_1, m_1, n_2, m_2, \ldots, n_r, m_r).
\]
It is now easy to verify that $\det(\mathcal{P}) = (-1)^{vw(w)}$. \hfill \Box

Remark 5. The permutation $Q$ in Theorem 3 can be concretely described as follows. Each integer $n$ in $\{0, \ldots, vr - 1\}$ can be uniquely written both as $a + bw$ and as $a' + b'r$, with $0 \leq a, b < w$ and $0 \leq a', b < r$. The map $a + bw \mapsto a + br$ is therefore a well defined bijection of $\{0, \ldots, vr - 1\}$ in itself: $Q$ is the matrix representing this permutation.

Theorem 3 here above explains the equality $\det Q(\rho) = - \det Q'(\rho)$ in our previous example. As we will see now, the other equality $\det Q'(\rho) = (\det([v_+ \mid v_-]))^2$ is a consequence of a general formula relating the characteristic polynomial of $P *_{h} \rho$ to that one of $P$ (see next Theorem 4). We will deduce this formula via the Jordan decomposition of $P$ and using the following proposition describing the behavior of the $*_{h}$-product with respect to a direct sum in its first argument.

Proposition 3. Let $P \in \mathcal{M}(p, \mathbb{C})$ and $Q \in \mathcal{M}(q, \mathbb{C})$. Then, there is a permutation $\mathcal{P} \in \text{GL}((p + q)r, \mathbb{C})$ such that
\[
\mathcal{P} \cdot ((P \oplus Q) *_{h} \rho) \cdot \mathcal{P}^{-1} = (P *_{h} \rho) \oplus (Q *_{h_{P}} \rho),
\]
where $h_{P}(i, j) := h(i + p, j + p)$.

Proof. We have
\[
(P \oplus Q)_{i,j}p_{I,J}(h(i, j)) = \begin{cases} P_{i,j}p_{I,J}(h(i, j)) & \text{if } i, j \leq p, \\ Q_{i-p,j-p}p_{I,J}(h_{P}(i - p, j - p)) & \text{if } i, j > p, \\ 0 & \text{otherwise,} \end{cases}
\]
for every $I$ and $J$. Thus according to the definition of the $*_{h}$-product, we see that $(P \oplus Q) *_{h} \rho$ can be obtained by permuting columns and rows of $(P *_{h} \rho) \oplus (Q *_{h_{P}} \rho)$, i.e.
\[
(P \oplus Q) *_{h} \rho \mathcal{P}' = (P *_{h} \rho) \oplus (Q *_{h_{P}} \rho)
\]
for two suitable permutations $\mathcal{P}$ and $\mathcal{P}'$. The formula also shows that these permutations depend on $P$ and $Q$ only via their orders, thus substituting $P$ and $Q$ with the identities of the same order and using the conclusion in Item 2 of Theorem 1, we get that
\[
\mathcal{P} \mathcal{P}' = \mathcal{P}((\mathbb{I}_{p+q} *_{h} \rho) \mathcal{P}') = \mathcal{P}((\mathbb{I}_{p} \oplus \mathbb{I}_{q} *_{h} \rho) \mathcal{P}')
\]
\[ (I_p \ast_h \rho) \oplus (I_q \ast_{hp} \rho) = I_{pr} \oplus I_{qr} = I_{(p+q)r}, \]

thus proving that \( P' = P^{-1} \) in (7).

**Remark 6.** The argument in the proof of Proposition 3 also shows that \( P' \) coincides with the permutation having the same name already described in the proof of Theorem 3:

\[ P : (n_1, n_2, \ldots, n_r, m_1, m_2, \ldots, m_r) \rightarrow (n_1, m_1, n_2, m_2, \ldots, n_r, m_r), \]

where \( n_1, \ldots, n_r \) are a partition of \( \{1, \ldots, pr\} \) in blocks of consecutive integers having \( p \) integers each one, and \( m_1, \ldots, m_r \) a partition of \( \{pr + 1, \ldots, pr + qr\} \) in blocks of consecutive integers having \( q \) integers each one.

**Theorem 4.** The characteristic polynomial of \( P \ast_h \rho \) is the \( r \)th power of that one of \( P \).

**Proof.** Let \( \oplus_l \oplus_m (\lambda_l I_{n_l,m} + J_{n_l,m}) \) be the Jordan canonical decomposition of \( P \), where \( \lambda_l \) are the distinct eigenvalues of \( P \) and \( \{J_{n_l,m}\}_m \) are the Jordan blocks corresponding to the eigenvalue \( \lambda_l \). Then,

\[ P \text{ is similar to } \oplus_l \oplus_m (\lambda_l I_{n_l,m} + J_{n_l,m}) \]

and by Items 4 and 7 of Theorem 1

\[ P \ast_h \rho \text{ is similar to } (\oplus_l \oplus_m (\lambda_l I_{n_l,m} + J_{n_l,m})) \ast_h \rho. \]

By Proposition 3

\[ P \ast_h \rho \text{ is similar to } \oplus_l \oplus_m (\lambda_l I_{n_l,m} + J_{n_l,m} \ast_{h_l,m} \rho), \]

where each \( h_{l,m} \) is a suitable map satisfying (2). Since

\[ I_{rp} = \oplus_l \oplus_m (I_{n_l,m} \ast_{h_{l,m}} \rho), \]

(by Theorem 1, Item 2) we get that

\[ x^r \ast_{hp} \rho \text{ is similar to } \oplus_l \oplus_m (x - \lambda_l) I_{n_l,m} - J_{n_l,m} \ast_{h_{l,m}} \rho). \]

Consider a matrix of the form \( (\lambda n + J_n) \ast_h \rho \). By Item 5 of Theorem 1 its minimal polynomial is \( (x - \lambda)^n \). Thus its characteristic polynomial must be a power of \( (x - \lambda)^n \), and hence is \( (x - \lambda)^{nr} \), i.e. the \( r \)th power of the characteristic polynomial of \( \lambda n + J_n \). The claim now follows by (8), by multiplicativity. \( \square \)

**References**


