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Generalized Balanced Tournament Designs with Block Size Four

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Abstract

In this paper, we remove the outstanding values \( m \) for which the existence of a GBTD(4, \( m \)) has not been decided previously. This leads to a complete solution to the existence problem regarding GBTD(4, \( m \))s.

Keywords: generalized balanced tournament design; holey generalized balanced tournament design; starter-adder

1 Introduction

A set system is a pair \( \mathcal{S} = (X, \mathcal{B}) \), where \( X \) is a finite set of points and \( \mathcal{B} \) is a collection of subsets of \( X \). Elements of \( \mathcal{B} \) are called blocks. The order of \( \mathcal{S} \) is \( |X| \), the number of points. Let \( K \) be a set of positive integers. A set system \( (X, \mathcal{B}) \) is said to be \( K \)-uniform if \( |B| \in K \) for all \( B \in \mathcal{B} \). Let \( (X, \mathcal{B}) \) be a set system and \( S \subseteq X \). A partial \( \alpha \)-parallel class over \( X \setminus S \) of \( (X, \mathcal{B}) \) is a set of blocks \( \mathcal{A} \subseteq \mathcal{B} \) such that each point of \( X \setminus S \) occurs in exactly \( \alpha \) blocks of \( \mathcal{A} \), and each point of \( S \) occurs in no block of \( \mathcal{A} \). A partial \( \alpha \)-parallel class over \( X \) is simply called an \( \alpha \)-parallel class. We adopt the convention that if \( \alpha \) is not specified, then it is taken to be one, so that a parallel class refers to a 1-parallel class. A set system \( (X, \mathcal{B}) \) is said to be resolvable if \( \mathcal{B} \) can be partitioned into parallel classes.

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A balanced incomplete block design of order \( v \), block size \( k \), and index \( \lambda \), denoted by \((v, k, \lambda)\)-BIBD, is a \( \{k\} \)-uniform set system \((X, B)\) of order \( v \) such that every 2-subset of \( X \) is contained in precisely \( \lambda \) blocks of \( B \). A resolvable \((km, k, k-1)\)-BIBD \((X, B)\) is called a generalized balanced tournament design (GBTD), or simply a GBTD \((k, m)\), if the \( m(km-1) \) blocks of \( B \) are arranged in an \( m \times (km-1) \) array such that

(i) the set of blocks in each column is a parallel class, and

(ii) each point of \( X \) is contained in at most \( k \) cells of each row.

GBTDs were introduced by Lamken [3] and are useful in the construction of many combinatorial designs, including resolvable, near-resolvable, doubly resolvable, and doubly near-resolvable balanced incomplete block designs (see [2, 3]). More recently, GBTDs have also found applications in near constant-composition codes [12], and codes for power line communications [1].

Schellenberg et al. [8] showed that a GBTD \((2, m)\) exists for all positive integers \( m \neq 2 \). Lamken [4] showed that a GBTD \((3, m)\) exists for all positive integers \( m \neq 2 \). For \( k = 4 \), Yin et al. [12] obtained the following.

**Theorem 1** (Yin et al. [12]). A GBTD \((4, m)\) exists for all positive integers \( m \geq 5 \), except possibly when \( m \in \{28, 32, 33, 34, 37, 38, 39, 44\} \).

The purpose of this paper is to remove all the remaining eight possible exceptions in Theorem 1 and to show that a GBTD \((4, m)\) exists for \( m = 4 \) but not for \( m \in \{2, 3\} \).

**Theorem 2.** For each \( m \in \{4, 28, 32, 33, 34, 37, 38, 39, 44\} \), a GBTD \((4, m)\) exists. For \( m = 2 \) and \( 3 \), a GBTD \((4, m)\) does not exist.

A GBTD \((4, 1)\) exists trivially. Combining Theorem 1 and Theorem 2, the existence of GBTD \((4, m)\) is now completely determined.

**Theorem 3.** A GBTD \((4, m)\) exists if and only if \( m \geq 1 \) and \( m \neq 2, 3 \).

We first present a non-existence result.

**Proposition 1.1.** A GBTD \((k, 2)\) does not exist for all \( k \geq 2 \).

**Proof:** Suppose \((X, B)\) is a \((2k, k, k-1)\)-BIBD whose blocks are organized into a \( 2 \times (2k-1) \) array to form a GBTD \((k, 2)\). Since \((X, B)\) is a \((2k, k, k-1)\)-BIBD, each point in \( X \) appears in exactly \( 2k-1 \) blocks, and hence each point must appear either \( k \) times in the first row and \( k-1 \) times in the second row, or vice versa.

Consider a point \( x \in X \) that appears \( k \) times in the first row and \( k-1 \) times in the second row. Let \( y \in X \) be distinct from \( x \). The cells in the first row can be classified as follows:

(i) Type-\(xy\): a cell that contains both \( x \) and \( y \).
(ii) Type-\(x\bar{y}\): a cell that contains \(x\) but not \(y\).

(iii) Type-\(\bar{x}y\): a cell that contains \(y\) but not \(x\).

(iv) Type-\(\bar{x}\bar{y}\): a cell that contains neither \(x\) nor \(y\).

Let \(\alpha\) and \(\beta\) be the number of type-\(xy\) cells and type-\(\bar{x}y\) cells in the first row, respectively. Then we have the table

\[
\begin{array}{|c|c|c|c|}
\hline
& Type-xy & Type-\bar{x}y & Type-\bar{x}y \\
\hline
\# cells in first row & \alpha & k - \alpha & \beta \\
\hline
\# cells in second row & k - 1 - \beta & \beta & k - \alpha \\
\hline
\end{array}
\]

where the second row is obtained from the first by the following observation: if a cell is of type-\(xy\) (respectively, type-\(\bar{x}y\), type-\(\bar{x}y\), type-\(\bar{x}y\)) in the first row, then the cell in the second row of the corresponding column is of type-\(\bar{x}y\) (respectively, type-\(xy\), type-\(\bar{x}y\), type-\(\bar{x}y\)). On the other hand, the total number of type-\(xy\) cells is \(k - 1\), since \((X, B)\) is a BIBD of index \(k - 1\). Hence, we have \(\alpha + (k - 1 - \beta) = k - 1\), implying \(\alpha = \beta\).

Considering the number of occurrences of \(y\) in the first row and the number of occurrences of \(y\) in the second row of the GBTD give the inequalities

\[
\begin{align*}
\alpha + \beta & \leq k, \\
2k - 1 - \alpha - \beta & \leq k,
\end{align*}
\]

from which, and \(\alpha = \beta\) shown earlier, follow that

\[
\alpha = \lfloor k/2 \rfloor.
\]

Table T1 can therefore be revised to

\[
\begin{array}{|c|c|c|c|}
\hline
& Type-xy & Type-\bar{x}y & Type-\bar{x}y \\
\hline
\# cells in first row & \lfloor k/2 \rfloor & \lceil k/2 \rceil & \lfloor k/2 \rfloor \\
\hline
\# cells in second row & \lfloor k/2 \rfloor & \lceil k/2 \rceil & \lfloor k/2 \rfloor \\
\hline
\end{array}
\]

Counting in two ways the number of elements in the set

\[
\{(y, C) : y \in X, y \neq x, \text{ and } C \text{ is a cell of type-}xy \text{ in the second row}\}
\]

gives

\[
(2k - 1)(\lfloor k/2 \rfloor - 1) = (k - 1)^2,
\]

which is a contradiction. \(\square\)
2 Existence of GBTD\((4, m)\)s with \(m = 3\) and 4

For a positive integer \(n\), the set \(\{1, 2, \ldots, n\}\) is denoted by \([n]\). Let \(\Sigma\) be a set of \(q\) symbols. A \(q\)-ary code of length \(n\) over \(\Sigma\) is a subset \(C \subseteq \Sigma^n\). Elements of \(C\) are called codewords. The size of \(C\) is the number of codewords in \(C\). For \(i \in [n]\), the \(i\)th coordinate of a codeword \(u \in C\) is denoted \(u_i\), so that \(u = (u_1, u_2, \ldots, u_n)\).

The symbol weight of \(u \in \Sigma^n\), denoted \(\text{swt}(u)\), is the maximum frequency of appearance of a symbol in \(u\), that is,

\[
\text{swt}(u) = \max_{\sigma \in \Sigma} |\{i : u_i = \sigma \in [n]\}|.
\]

A code has constant symbol weight \(w\) if all of its codewords have symbol weight exactly \(w\). The (Hamming) distance between \(u, v \in \Sigma^n\) is \(d_H(u, v) = |\{i \in [n] : u_i = v_i\}|\), the number of coordinates at which \(u\) and \(v\) differ. A code \(C\) is said to have distance \(d\) if \(d_H(u, v) \geq d\) for all distinct \(u, v \in C\). A \(q\)-ary code of length \(n\), constant symbol weight \(w\), and distance \(d\) is referred to as an \((n, d, w)\)-symbol weight code. An \((n, d, w)\)-symbol weight code with maximum size is said to be optimal.

Chee et al. [1] established the following relation between a GBTD and a symbol weight code.

**Theorem 4** (Chee et al. [1]). A GBTD\((k, m)\) exists if and only if an optimal \((km - 1, k(m - 1), k)\)-symbol weight code exists.

In view of Theorem 4, to prove the nonexistence of a GBTD\((4, 3)\), it suffices to show that there does not exist a ternary code of length 11, constant symbol weight four, and size 12, that is of equidistance eight. Consider the Gilbert graph \(G = (V, E)\), where \(V = \{u \in [3]^{11} : \text{swt}(u) = 4\}\) and two vertices \(u, v \in V\) are adjacent in \(G\) if and only if \(d_H(u, v) = 8\). Then there exists a ternary code of length 11, constant symbol weight four, and size 12, that is of equidistance eight if and only if there exists a clique of size 12 in \(G\). It is not hard to see that \(G\) is vertex-transitive so that we can just search for a clique of size 11 in \(G'\), the subgraph of \(G\) induced by the set of vertices \(\{v \in V : d_H(u, v) = 8\}\) for some fixed \(u \in V\). This induced subgraph \(G'\) has 8001 vertices and 7233060 edges. We solve this clique-finding problem using Cliquer, an implementation of Östergård’s clique-finding algorithm by Niskanen and Östergård [6]. The result is that the largest clique in \(G'\) has size 10. Consequently, we have the following.

**Proposition 2.1.** There does not exist a GBTD\((4, 3)\).

There exists, however, a GBTD\((4, 4)\). Unfortunately, a GBTD\((4, 4)\) is too large to be found by the method of clique-finding above. Instead, to curb the search space, we assume the existence of some automorphisms acting on the GBTD\((4, 4)\) to be found. Let \((X, B)\) be a GBTD\((4, 4)\), where \(X = \mathbb{Z}_4 \times \mathbb{Z}_4\). If \(B \subseteq X\) and \(x \in X\), \(B + x\) denotes the set \(\{b + x : b \in B\}\). If \(A\) is an array over \(X\) and \(x \in X\), \(A + x\) denotes the array obtained by adding \(x\) to every entry of \(A\). For succinctness, a point \((x, y) \in \mathbb{Z}_4 \times \mathbb{Z}_4\) is sometimes written \(xy\).
The GBTD$(4,4)$ we construct contains the $4 \times 3$ subarray

\[
A_0 = \begin{bmatrix}
{00, 02, 20, 22} & {11, 13, 31, 33} & {10, 12, 30, 32} \\
{01, 03, 21, 23} & {00, 02, 20, 22} & {11, 13, 31, 33} \\
{10, 12, 30, 32} & {01, 03, 21, 23} & {00, 02, 20, 22} \\
{11, 13, 31, 33} & {10, 12, 30, 32} & {01, 03, 21, 23}
\end{bmatrix}
\]

The blocks in $A_0$ contain exactly the 2-subsets $\{ab, cd\} \subseteq X$, where $a + c \equiv b + d \equiv 0 \mod 2$, each thrice. The remaining $4 \times 12$ subarray of the GBTD$(4,4)$ is built from a set of 12 base blocks $S = \{B_{i,j} : i \in [3] \text{ and } 0 \leq j \leq 3\}$ as follows. Let

\[
A_1 = \begin{bmatrix}
B_{1,0} & B_{2,0} & B_{3,0} \\
B_{1,1} & B_{2,1} & B_{3,1} \\
B_{1,2} & B_{2,2} & B_{3,2} \\
B_{1,3} & B_{2,3} & B_{3,3}
\end{bmatrix}
\]

Then the $4 \times 12$ subarray is given by

\[
A_1 \begin{bmatrix}
A_0 \\
A_1 + (0, 1) \\
A_1 + (0, 2) \\
A_1 + (0, 3)
\end{bmatrix}
\]

For

\[
A_0 \\
A_1 \\
A_1 + (0, 1) \\
A_1 + (0, 2) \\
A_1 + (0, 3)
\]

to be a GBTD$(4,4)$, the following conditions are imposed:

(i) $\bigcup_{i=0}^{3} B_{i,j} = Z_4 \times Z_4$, for $i \in [3]$, so that every column is a parallel class.

(ii) For each $j$, $0 \leq j \leq 3$, each element of $Z_4$ appears exactly three times as a first coordinate among the elements of $\bigcup_{i=1}^{3} B_{i,j}$, so that every row contains each element of $Z_4 \times Z_4$ at most three times.

(iii) Let $k, l \in Z_4$ and define $\Delta_{k,l}S$ to be the multiset $\bigcup_{x \in S} \{x - y : (k, x), (l, y) \in A\}$. Then

\[
\Delta_{k,l}S = \begin{cases}
{1, 1, 1, 3, 3, 3}, & \text{if } k = l \text{ or } k + l \equiv 0 \mod 2; \\
{0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 3}, & \text{otherwise.}
\end{cases}
\]

This ensures that every 2-subset of $X$ not contained in any block in $A_0$ is contained in exactly three blocks in $A_1$, $A_1 + (0, 1)$, $A_1 + (0, 2)$, or $A_1 + (0, 3)$.

A computer search found the following array $A_1$ that satisfies all the conditions above.

\[
A_1 = \begin{bmatrix}
{23, 22, 32, 11} & {10, 00, 21, 11} & {00, 01, 30, 33} \\
{20, 01, 30, 33} & {33, 02, 03, 12} & {10, 13, 22, 23} \\
{31, 00, 12, 21} & {01, 13, 20, 32} & {02, 11, 21, 32} \\
{02, 10, 13, 03} & {22, 23, 30, 31} & {03, 12, 20, 31}
\end{bmatrix}
\]

Consequently, we have the following.

**Proposition 2.2.** There exists a GBTD$(4,4)$. 
3 Incomplete Holey GBTDs

Let \((X, \mathcal{B})\) be a set system, and let \(\mathcal{G}\) be a partition of \(X\) into subsets, called groups. The triple \((X, \mathcal{G}, \mathcal{B})\) is a group divisible design (GDD) of index \(\lambda\) when every 2-subset of \(X\) not contained in a group appears in exactly \(\lambda\) blocks, and \(|B \cap G| \leq 1\) for all \(B \in \mathcal{B}\) and \(G \in \mathcal{G}\). We denote a GDD \((X, \mathcal{G}, \mathcal{B})\) of index \(\lambda\) by \((K, \lambda)\)-GDD if \((X, \mathcal{B})\) is \(K\)-uniform. The type of a GDD \((X, \mathcal{G}, \mathcal{B})\) is the multiset \([|G| : G \in \mathcal{G}|\). When more convenient, the exponential notation is used to describe the type of a GDD: a GDD of type \(g_1^t_1 g_2^t_2 \cdots g_s^t_s\) is a GDD where there are exactly \(t_i\) groups of size \(g_i\), \(i \in [s]\).

Suppose further \(\mathcal{G} = \{G_1, G_2, \ldots, G_s\}\) and \(\mathcal{H} = \{H_1, H_2, \ldots, H_s\}\) is a collection of subsets of \(X\) with the property \(H_i \subseteq G_i, 0 \leq i \leq s\). Let \(H = \bigcup_{i=1}^s H_i\). Then the quadruple \((X, \mathcal{G}, \mathcal{H}, \mathcal{B})\) is an incomplete group divisible design (IGDD) of index \(\lambda\) when every 2-subset of \(X\) not contained in a group or \(H\) appears in exactly \(\lambda\) blocks, and \(|B \cap G| \leq 1\) and \(|B \cap H| \leq 1\) for all \(B \in \mathcal{B}\) and \(G \in \mathcal{G}\). The type of an IGDD \((X, \{G_1, G_2, \ldots, G_s\}, \{H_1, H_2, \ldots, H_s\}, \mathcal{B})\) is the multiset \([|G_i|, |H_i| : 1 \leq i \leq s]\) and we use the exponential notation when more convenient.

Let \(k, g, u, \) and \(w\) be positive integers such that \(k \mid g\) and \(u \geq (k+1)w\). Let \(R_i = \{r \in \mathbb{Z} : \frac{ig}{k} \leq r \leq \frac{(i+1)g}{k} - 1\}\). An incomplete holey GBTD with block size \(k\) and type \(g^{(u,w)}\), denoted IHGBTD \((k, g^{(u,w)})\), is a \((\{k\}, k-1)\)-IGDD \((X, \{G_0, G_1, \ldots, G_{u-1}\}, \{\emptyset, \ldots, \emptyset, G_{u-w}, \ldots, G_{u-1}\}, \mathcal{B})\) of type \((g,0)^{u-w}(g,g)^w\), whose blocks are arranged in a \((gu/k) \times g(u-1)\) array \(A\), with rows and columns indexed by elements from the sets \(\{0,1,\ldots, gu/k-1\}\) and \(\{0,1,\ldots, g(u-1)-1\}\), respectively, such that the following properties are satisfied.

(i) The \((g(w-1)\) subarray whose rows are indexed by \(r \in R_i\), where \(u-w \leq i \leq u-1\), and columns indexed by \(c\), where \(g(u-w) \leq c \leq g(u-1) - 1\), is empty.

(ii) For each \(i, 0 \leq i \leq u-w-1\), the blocks in row \(r \in R_i\) form a partial \(k\)-parallel class over \(X \setminus G_i\), and for each \(i, u-w \leq i \leq u-1\), the blocks in row \(r \in R_i\) form a partial \(k\)-parallel class over \(X \setminus \left(\bigcup_{j=u-w}^{u-1} G_j\right)\).

(iii) For each \(j, 0 \leq j \leq g(u-w) - 1\), the blocks in column \(j\) form a parallel class, and for each \(j, g(u-w) \leq j \leq g(u-1) - 1\), the blocks in column \(j\) form a partial parallel class over \(X \setminus \left(\bigcup_{i=u-w}^{u-1} G_j\right)\).

Each group acts as a hole of the design, since no block contains any 2-subset of a group. The design is also incomplete in the sense that the array \(A\) contains an empty \((gw/k) \times g(w-1)\) subarray.

We note that an IHGBTD \((k, g^{(u,1)})\) is a holey GBTD, HGBTD \((k, g^u)\), as defined by Yin et al. [12]. The following was established by Yin et al. [12].

**Proposition 3.1** (Yin et al. [12]). If there exists an HGBTD \((k, k^u)\), then there exists a GBTD \((k, u)\).

Proposition 3.1 shows that a GBTD \((k, u)\) can be obtained from an HGBTD \((k, k^u)\). The next result shows how we can obtain an HGBTD \((k, g^u)\) (and, in particular, an HGBTD \((k, k^u)\)) from an IHGBTD \((k, g^{(u,w)})\) and an HGBTD \((k, g^w)\).
Proposition 3.2. If there exist an IHGBTD($k, g^{(u,w)}$) and an HGBTD($k, g^w$), then there exists an HGBTD($k, g^u$).

Proof: When $w = 1$, an HGBTD($k, g^w$) is empty and an IHGBTD($k, g^{(u,w)}$) is just an HGBTD($k, g^u$). So assume $w > 1$ and let $(X, G, B)$ be an IHGBTD($k, g^{(u,w)}$) with $G = \{G_0, G_1, \ldots, G_{u-1}\}$. Fill in the empty subarray of this IHGBTD with an HGBTD($k, g^w$), $(X', G', B')$, with $G' = \{G_{u-w}, G_{u-w+1}, \ldots, G_{u-1}\}$ and $X' = \bigcup_{i=u-w}^{u-1} G_i$. The resulting array is a HGBTD($k, g^u$), $(X, G, B \cup B')$. □

4  Starter-Adder Construction for IHGBTD

The starter-adder technique first used by Mullin and Nemeth [5] to construct Room squares (also a combinatorial array) has been useful in constructing many types of designs with orthogonality properties, including GBTDs (see [3, 7, 10, 11, 12]). Here, we extend the technique to the construction of IHGBTDs. Since only IHGBTD($k, g^{(u,w)}$) with $g = k$ are considered here, we describe our construction for this case.

Let $\Gamma$ be an additive abelian group of order $k(u-w)$ with $u \geq (k+1)w$, and let $\Gamma_0 \subseteq \Gamma$ be a subgroup of order $k$. Fix a set, $\Delta = \{\delta_0 = 0, \delta_1, \ldots, \delta_{u-w-1}\} \subseteq \Gamma$, of representatives for the cosets of $\Gamma_0$ so that $\Gamma_i = \Gamma_0 + \delta_i$, $0 \leq i \leq u-w-1$, are the cosets of $\Gamma_0$. Let $H$ be a set of $kw$ points such that $H$ and $\Gamma$ are disjoint. Further, let $H$ be partitioned into $w$ subsets $H_0, H_1, \ldots, H_{w-1}$ of size $k$ each.

We take $X = \Gamma \bigcup H$ to be the point set of an IHGBTD($k, k^{(u,w)}$). An intransitive starter for an IHGBTD($k, k^{(u,w)}$), with groups $\{G_0, G_1, \ldots, G_{u-1}\}$, where

$$G_i = \begin{cases} \Gamma_i, & \text{if } 0 \leq i \leq u-w-1; \\ H_{i-u+w}, & \text{if } u-w \leq i \leq u-1, \end{cases}$$

is defined as a quadruple $(X, S, R, C)$ satisfying the properties:

(i) $(X, S)$, $(X, R)$, and $(X, C)$ are $\{k\}$-uniform set systems of size $u - w$, $w$, and $w - 1$, respectively;

(ii) among the blocks in $S$, $kw$ of them intersects $H$ in one point, that is, $|\{B \in S : |B \cap H| = 1\}| = kw$;

(iii) blocks in $R$ are each disjoint from $H$;

(iv) blocks in $C$ are each disjoint from $H$, and $\bigcup_{i=0}^{u-w-1}(\delta_i + C) = \Gamma$, for each $C \in C$.

(v) $S \cup R$ is a partition of $X$;

(vi) the difference list from the base blocks of $S \cup R \cup C$ contains every element of $\Gamma \setminus \Gamma_0$ precisely $k-1$ times, and no element in $\Gamma_0$. 

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Suppose \( S = \{B_0, B_1, \ldots, B_{u-w-1}\} \). Then a corresponding adder \( \Omega(S) \) for \( S \) is a permutation \( \Omega(S) = (\omega_0, \omega_1, \ldots, \omega_{u-w-1}) \) of the \( u-w \) elements of the representative system \( \Delta \) satisfying the following property:

(vii) the multiset \( (\bigcup_{i=0}^{u-w-1}(B_i + \omega_i)) \cup (\bigcup_{C \in C} C) \) contains exactly \( k \) elements (not necessarily distinct) from \( \Gamma_j \) for \( 1 \leq j \leq u-w-1 \), and is disjoint from \( \Gamma_0 \). We remark that when \( B \in S \) and \( B \cap H = \{\infty\} \), or \( B = \{\infty, b_1, b_2, \ldots, b_{k-1}\} \), the block \( B + \gamma \) is defined to be \( \{\infty, b_1 + \gamma, b_2 + \gamma, \ldots, b_{k-1} + \gamma\} \) for \( \gamma \in \Gamma \).

The result below shows how to construct an IHGBTD from an intransitive starter and its corresponding adder.

**Proposition 4.1.** Let \( \Gamma \) be an additive abelian group of order \( k(u-w) \) with \( u \geq (k+1)w \) and \( \Gamma_0 \) be a subgroup of order \( k \). Define \( X \) and the groups \( G_i \) \((0 \leq i \leq u-1)\) as above. If there exists an intransitive starter \((X, S, R, C)\) with groups \( \{G_i : 0 \leq i \leq u-1\} \), a corresponding adder \( \Omega(S) \), then there exists an IHGBTD\((k, k^{(u,w)})\).

**Proof:** Retain the notations in the definition of intransitive starter and adder. Suppose 

\[
A = \{A + \gamma : \gamma \in \Gamma, A \in S \cup R \cup C\},
\]

then \((X, \{G_0, G_1, \ldots, G_{u-1}\}, \{\varnothing, \varnothing, \varnothing, H_0, \ldots, H_{w-1}\}, A)\) forms a \((\{k\}, k-1)\)-IGDD of type \((k, 0)^u-(k, k)^w\) by Condition (vi) in the definition of intransitive starter. Therefore, it remains to arrange the blocks in a \( u \times k(u-1) \) array.

First, consider the blocks \( S \). Consider a \((u-w) \times (u-w)\) array \( S \), whose rows and columns are indexed with the elements in \( \Delta \). Now identify the elements in \( \Delta \) with elements in the quotient group \( \Gamma/\Gamma_0 \), so that the binary operation \(+\) on \( \Delta \) is defined by the additive operation on \( \Gamma/\Gamma_0 \). In addition, for \( \delta \in \Delta \), denote the additive inverse of \( \delta \) by \( -\delta \). That is, \( \delta + (-\delta) = 0 \).

So, for \( 0 \leq i, j \leq u-w-1 \), we place the block \( B_i + \delta_j \) at the cell \((\delta_j - \delta_i, \delta_j)\) if \( \delta_i = \omega_i \). Note that this placement is well defined because \( \Omega(S) \) is a permutation of \( \Delta \). Let \( \Gamma_0 = \{\gamma_0 = 0, \gamma_1, \ldots, \gamma_{k-1}\} \). Form a \((u-w) \times k(u-w)\) array \( \hat{S} \) from the square \( S \) by concatenating \( k \) squares \( D + \gamma_i \) where \( 0 \leq i \leq k-1 \) as follows.

\[
\hat{S} = S \quad S + \gamma_1 \quad \ldots \quad S + \gamma_{k-1}
\]

Next, let \( R = \{R_1, R_2, \ldots, R_w\} \) and construct a \( w \times k(u-w) \) array \( \hat{R} \) in the following way:

\[
\hat{R} = R \quad R + \gamma_1 \quad \ldots \quad R + \gamma_{k-1}
\]

where the \( w \times w \) subarray \( R \) is given by

\[
R = \begin{bmatrix}
R_1 & R_1 + \delta_1 & \cdots & R_1 + \delta_{u-w-1} \\
R_2 & R_2 + \delta_1 & \cdots & R_2 + \delta_{u-w-1} \\
\vdots & \vdots & \ddots & \vdots \\
R_w & R_w + \delta_1 & \cdots & R_w + \delta_{u-w-1}
\end{bmatrix}
\]
Similarly, let \( C = \{ C_0, C_1, \ldots, C_{w-2} \} \), and we construct a \((u - w) \times k(w - 1)\) array \( \hat{C} \):

\[
\hat{C} = \begin{bmatrix}
C_0 & C_1 & \cdots & C_{w-2}
\end{bmatrix},
\]

where each \((u - w) \times k\) subarray \( C_i \), \( 0 \leq i \leq w - 2 \), is given by

\[
C_i = \begin{bmatrix}
C_i & C_i + \gamma_1 & \cdots & C_i + \gamma_{k-1} \\
C_i + \delta_1 & C_i + \delta_1 + \gamma_1 & \cdots & C_i + \delta_1 + \gamma_{k-1} \\
\vdots & \vdots & \ddots & \vdots \\
C_i + \delta_{u-w-1} & C_i + \delta_{u-w-1} + \gamma_1 & \cdots & C_i + \delta_{u-w-1} + \gamma_{k-1}
\end{bmatrix},
\]

Finally, let

\[
A = \begin{bmatrix}
\hat{S} & \hat{C} \\
\hat{R}
\end{bmatrix},
\]

and it is readily verified that the placement results in an IHGBTD\((k, k^{(u,w)})\).

\[\square\]

5 Proof of Theorem 1.2

We first remove all the eight remaining values in Theorem 1.

Lemma 5. For \((u, w) \in \{(28,5), (32, 5), (33, 6)\}\), an IHGBTD\((4, 4^{(u,w)})\) exists.

Proof: We apply Proposition 4.1 to construct the desired IHGBTDs. Take

\[
\Gamma = \mathbb{Z}_{u-w} \times \mathbb{Z}_4,
\]

\[
\Gamma_0 = \{0\} \times \mathbb{Z}_4,
\]

\[
\Delta = \{(0,0), (1,0), \ldots, (u-w-1,0)\}, \text{ and}
\]

\[
H = \bigcup_{i=0}^{w-1} H_i, \text{ where } H_i = \{\infty, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w} \} \text{ for } 0 \leq i \leq w - 1.
\]

For each pair \((u, w) \in \{(28,5), (32, 5), (33, 6)\}\), the desired intransitive starter and corresponding adder are displayed below. Here we write the element \((a, b)\) of \( \Gamma \) as \( a_b \) for succinctness.

When \((u, w) = (28,5)\):

<table>
<thead>
<tr>
<th>( S )</th>
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<th>( S )</th>
<th>( \Omega(S) )</th>
<th>( S )</th>
<th>( \Omega(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {41, 30, 70, 0_0} )</td>
<td>170</td>
<td>( {50, 190, 121, 12} )</td>
<td>120</td>
<td>( {180, 133, 163, 81} )</td>
<td>190</td>
</tr>
<tr>
<td>( {\infty, 31, 122, 113} )</td>
<td>10</td>
<td>( {\infty, 143, 60, 104} )</td>
<td>210</td>
<td>( {\infty, 141, 91, 201} )</td>
<td>200</td>
</tr>
<tr>
<td>( {\infty, 19, 101, 222} )</td>
<td>70</td>
<td>( {\infty, 33, 13, 22} )</td>
<td>180</td>
<td>( {\infty, 5, 02, 151, 10} )</td>
<td>150</td>
</tr>
<tr>
<td>( {\infty, 11, 63, 93} )</td>
<td>20</td>
<td>( {\infty, 14, 111, 01} )</td>
<td>100</td>
<td>( {\infty, 8, 03, 172, 212} )</td>
<td>220</td>
</tr>
<tr>
<td>( {\infty, 43, 80, 210} )</td>
<td>60</td>
<td>( {\infty, 10, 131, 193, 162} )</td>
<td>90</td>
<td>( {\infty, 11, 42, 213, 171} )</td>
<td>50</td>
</tr>
<tr>
<td>( {\infty, 12, 170, 52, 211} )</td>
<td>160</td>
<td>( {\infty, 13, 51, 202, 112} )</td>
<td>40</td>
<td>( {\infty, 14, 220, 23, 160} )</td>
<td>140</td>
</tr>
<tr>
<td>( {\infty, 15, 183, 20, 210} )</td>
<td>30</td>
<td>( {\infty, 16, 123, 21, 223} )</td>
<td>30</td>
<td>( {\infty, 17, 53, 71, 173} )</td>
<td>80</td>
</tr>
<tr>
<td>( {\infty, 18, 62, 90, 192} )</td>
<td>130</td>
<td>( {\infty, 19, 72, 83, 221} )</td>
<td>110</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\( C = \{18_0, 11_1, 5_3, 6_2\}, \{18_2, 8_3, 19_0, 6_1\}, \{14_3, 12_0, 3_2, 7_1\}, \{5_2, 7_1, 16_3, 11_0\}. \)

\( R = \{3_2, 18_2, 16_1, 10_2\}, \{8_2, 15_0, 20_0, 13_3\}, \{13_0, 9_2, 18_1, 15_3\}, \{6_1, 7_3, 14_2, 15_2\}, \{12_0, 10_0, 4_0, 11_0\}. \)

When \((u, w) = (32, 5)\):

<table>
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<th>( \Omega(S) )</th>
<th>( S )</th>
<th>( \Omega(S) )</th>
<th>( \Omega(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{4_2, 17_2, 16_1, 22_3}</td>
<td>16 | 0 | 10 | 10 | 0</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>{14_1, 6_0, 26_0, 3_0}</td>
<td>12 | 0 | 10 | 10 | 0</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_2, 0_1, 26_1, 20_2}</td>
<td>4 | 0 | 10 | 10 | 0</td>
<td></td>
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</tr>
<tr>
<td>{\infty_5, 5_0, 19_3, 12_1}</td>
<td>24 | 0 | 10 | 10 | 0</td>
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<tr>
<td>{\infty_8, 0_2, 10_0, 19_0}</td>
<td>14 | 0 | 10 | 10 | 0</td>
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</tr>
<tr>
<td>{\infty_{11}, 12_3, 25_2, 11_3}</td>
<td>22 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{14}, 20_0, 3_2, 16_3}</td>
<td>5 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{17}, 1_1, 15_1, 17_1}</td>
<td>8 | 0 | 10 | 10 | 0</td>
<td></td>
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</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( S )</th>
<th>( \Omega(S) )</th>
<th>( S )</th>
<th>( \Omega(S) )</th>
<th>( \Omega(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2_2, 0_1, 23_0, 21_3}</td>
<td>13 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_3, 21_1, 3_0, 22_2}</td>
<td>18 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_6, 21_2, 24_2, 11_2}</td>
<td>9 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_9, 0_2, 7_2, 19_2}</td>
<td>15 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{12}, 7_0, 9_0, 26_1}</td>
<td>19 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{15}, 4_2, 15_2, 13_3}</td>
<td>16 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{18}, 26_2, 5_2, 17_2}</td>
<td>12 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{21}, 3_2, 15_3, 24_3}</td>
<td>25 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When \((u, w) = (33, 6)\):

<table>
<thead>
<tr>
<th>( S )</th>
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<th>( S )</th>
<th>( \Omega(S) )</th>
<th>( \Omega(S) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2_2, 0_1, 23_0, 21_3}</td>
<td>13 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_3, 21_1, 3_0, 22_2}</td>
<td>18 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_6, 21_2, 24_2, 11_2}</td>
<td>9 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_9, 0_2, 7_2, 19_2}</td>
<td>15 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{12}, 7_0, 9_0, 26_1}</td>
<td>19 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{15}, 4_2, 15_2, 13_3}</td>
<td>16 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{18}, 26_2, 5_2, 17_2}</td>
<td>12 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>{\infty_{21}, 3_2, 15_3, 24_3}</td>
<td>25 | 0 | 10 | 10 | 0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| \( C \) | \{3_3, 10_1, 5_2, 15_0\}, \{8_3, 14_1, 9_2, 18_0\}, \{12_0, 10_3, 26_2, 5_1\}, \{21_2, 11_1, 23_0, 9_3\}, \{15_1, 9_2, 12_3, 3_0\}. |
| \( R \) | \{3_3, 2_0, 18_2, 19_0\}, \{8_3, 9_2, 3_1, 12_2\}, \{17_3, 3_3, 4_1, 22_3\}, \{19_3, 13_2, 6_2, 5_3\}, \{16_3, 23_1, 11_1, 19_1\}, \{20_3, 16_2, 8_1, 0_3\}. |

**Lemma 6.** For \((u, w) \in \{(34, 6), (44, 8)\}, an IHGBTD\((4, 4^{(u,w)})\) exists.

**Proof:** As with Lemma 5, we apply Proposition 4.1 to construct the desired IHGBTDS. Take

\[
\begin{align*}
\Gamma &= \mathbb{Z}_{2(u-w)} \times \mathbb{Z}_2, \\
\Gamma_0 &= \{(0,0)\} \times \mathbb{Z}_2, \\
\Delta &= \{(0, 0), (1, 0), \ldots, (u-w-1, 0)\}, \text{ and} \\
H &= \bigcup_{i=0}^{w-1} H_i, \text{ where } H_i = \{\infty_i, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \text{ for } 0 \leq i \leq w - 1.
\end{align*}
\]
The desired intransitive starter and corresponding adder for \((u, w) \in \{(34, 6), (44, 8)\}\) are displayed below. Here we write the element \((a, b)\) of \(\Gamma\) as \(a_b\) for succinctness.

When \((u, w) = (34, 6)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{411, 160, 60, 150}</td>
<td>00</td>
<td>{360, 90, 331, 131}</td>
<td>{370, 180, 261, 41}</td>
</tr>
<tr>
<td>{161, 21, 40, 31}</td>
<td>11</td>
<td>{40, 20, 420}</td>
<td>{41, 22, 30, 39}</td>
</tr>
<tr>
<td>{\infty, 14, 31, 1}</td>
<td>10</td>
<td>{4, 450, 80}</td>
<td>{4, 251, 481, 141}</td>
</tr>
<tr>
<td>{\infty, 81, 30, 20}</td>
<td>12</td>
<td>{6, 61, 44}</td>
<td>{7, 40, 330, 52}</td>
</tr>
<tr>
<td>{\infty, 451, 21, 38}</td>
<td>17</td>
<td>{9, 270, 280, 34}</td>
<td>{10, 42, 351, 37}</td>
</tr>
<tr>
<td>{\infty, 30, 22, 12}</td>
<td>19</td>
<td>{440, 350, 39}</td>
<td>{13, 361, 70, 91}</td>
</tr>
<tr>
<td>{\infty, 14, 51, 51, 51}</td>
<td>15</td>
<td>{530, 110, 510}</td>
<td>{16, 500, 510, 101}</td>
</tr>
<tr>
<td>{\infty, 17, 52, 32, 171}</td>
<td>13</td>
<td>{550, 29, 250}</td>
<td>{19, 71, 410}</td>
</tr>
<tr>
<td>{\infty, 20, 12, 31, 470}</td>
<td>8</td>
<td>{21, 17, 27, 47}</td>
<td>{22, 19, 236, 290}</td>
</tr>
<tr>
<td>{\infty, 23, 34, 40, 50, 50}</td>
<td>26</td>
<td>{}</td>
<td>{}</td>
</tr>
</tbody>
</table>

\[ C = \{271, 100, 441, 510\}, \{351, 150, 500, 141\}, \{161, 511, 540, 270\}, \{241, 120, 370, 211\}, \{390, 21, 451, 50\}\]

\[ R = \{130, 260, 386, 241\}, \{541, 231, 461, 491\}, \{10, 490, 181, 430\}, \{10, 21, 111, 540\}, \{460, 191, 431, 50\}, \{381, 320, 51, 0\} \]

When \((u, w) = (44, 8)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{32, 69, 361, 53}</td>
<td>01</td>
<td>{42, 65, 0, 43}</td>
<td>{39, 27, 451, 51}</td>
</tr>
<tr>
<td>{221, 390, 551, 33}</td>
<td>22</td>
<td>{670, 40, 54}</td>
<td>{1, 250, 101, 34}</td>
</tr>
<tr>
<td>{\infty, 2, 671, 36}</td>
<td>28</td>
<td>{251, 100, 28}</td>
<td>{43, 631, 60, 37}</td>
</tr>
<tr>
<td>{\infty, 560, 44, 2}</td>
<td>35</td>
<td>{80, 501, 35}</td>
<td>{7, 430, 461, 32}</td>
</tr>
<tr>
<td>{\infty, 69, 52, 4}</td>
<td>13</td>
<td>{371, 600, 71}</td>
<td>{10, 70, 211, 241}</td>
</tr>
<tr>
<td>{\infty, 11, 71, 13, 47}</td>
<td>35</td>
<td>{9, 59, 19, 6}</td>
<td>{130, 49, 471, 20}</td>
</tr>
<tr>
<td>{\infty, 14, 52, 46, 60}</td>
<td>24</td>
<td>{150, 17, 600, 22}</td>
<td>{160, 640, 541, 12}</td>
</tr>
<tr>
<td>{\infty, 17, 490, 9, 53}</td>
<td>40</td>
<td>{680, 01, 56}</td>
<td>{190, 270, 121, 4}</td>
</tr>
<tr>
<td>{\infty, 20, 650, 681, 23}</td>
<td>20</td>
<td>{21, 17, 80}</td>
<td>{220, 59, 171, 441}</td>
</tr>
<tr>
<td>{\infty, 23, 1, 70, 26}</td>
<td>12</td>
<td>{24, 571, 111, 130}</td>
<td>{250, 161, 50, 70}</td>
</tr>
<tr>
<td>{\infty, 26, 58, 4, 57}</td>
<td>50</td>
<td>{27, 411, 13, 31}</td>
<td>{280, 641, 560, 30}</td>
</tr>
<tr>
<td>{\infty, 29, 10, 46, 21}</td>
<td>60</td>
<td>{30, 48, 58, 50}</td>
<td>{310, 400, 40, 51}</td>
</tr>
</tbody>
</table>

\[ C = \{2, 31, 220, 690\}, \{281, 690, 191, 620\}, \{411, 40, 201, 590\}, \{570, 12, 40, 551\}, \{410, 21, 321, 80\}, \{71, 130, 141, 280\}, \{331, 210, 281, 520\}\]

\[ R = \{690, 31, 250, 291\}, \{380, 340, 30, 24\}, \{550, 150, 620, 450\}, \{621, 610, 420, 290\}, \{510, 350, 300, 260\}, \{611, 11, 140, 381\}, \{141, 110, 310, 630\}, \{71, 330, 81, 410\} \]

\[ \square \]

Lemma 7. For each \((u, w) \in \{(37, 6), (38, 7), (39, 6)\}\), an \(IHGTD(4, 4^{(u,w)})\) exists.
Proof: As with Lemma 5, we apply Proposition 4.1. Take
\[ \Gamma = \mathbb{Z}_{u-w} \times \mathbb{Z}_2 \times \mathbb{Z}_2, \]
\[ \Gamma_0 = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \]
\[ \Delta = \{(0, 0, 0), (1, 0, 0), \ldots, (u - w - 1, 0, 0)\}, \]
and
\[ H = \bigcup_{i=0}^{w-1} H_i, \text{ where } H_i = \{\infty, \infty_i, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \text{ for } 0 \leq i \leq w - 1. \]

The desired intransitive starter and corresponding adder for \((u, w) \in \{(37, 6), (38, 7), (39, 6)\}\) are displayed below. Here we write the element \((a, b, c)\) of \(\Gamma\) as \(a_{bc}\) for succinctness.

When \((u, w) = (37, 6)\):

<table>
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<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
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<td>3000</td>
<td>{2010, 1300, 2311, 2701}</td>
<td>2800</td>
</tr>
<tr>
<td>{2011, 1900, 900, 411}</td>
<td>1700</td>
<td>{2911, 2611, 2110, 01}</td>
<td>3000</td>
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<tr>
<td>{901, 2711, 410, 1611}</td>
<td>1100</td>
<td>{0000, 2601, 2801, 500}</td>
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<tr>
<td>{62, 2100, 111, 2301}</td>
<td>2400</td>
<td>{2311, 2011, 510, 1800}</td>
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<td>{50, 2501, 1511}</td>
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<td>{8, 301, 1211, 1910}</td>
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<td>2700</td>
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<tr>
<td>{11, 411, 2211, 700}</td>
<td>2000</td>
<td>{122600, 601, 400}</td>
<td>1900</td>
</tr>
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<td>{14, 210, 1601, 2201}</td>
<td>1300</td>
<td>{15401, 2900, 500}</td>
<td>1800</td>
</tr>
<tr>
<td>{17, 1811, 101, 1510}</td>
<td>1000</td>
<td>{1817, 2013, 1010, 800}</td>
<td>2600</td>
</tr>
<tr>
<td>{20, 310, 1810, 2401}</td>
<td>500</td>
<td>{21301, 2411, 1810}</td>
<td>900</td>
</tr>
<tr>
<td>{236, 110, 1501, 2910}</td>
<td>600</td>
<td>{2600, 1511, 1000}</td>
<td>{610, 1911, 1000}</td>
</tr>
</tbody>
</table>

\[ C = \{3010, 1300, 711, 801\}, \{7011, 2010, 2811, 1700\}, \{6100, 9010, 1000, 1301\}. \]

\[ R = \{14000, 3000, 1310, 600\}, \{9010, 1100, 5010, 1100\}, \{10000, 25010, 1710, 3010\}, \{20000, 5011, 9011, 1000\}, \{26010, 1210, 1311, 1711\}, \{12011, 1010, 611\}. \]

When \((u, w) = (38, 7)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2800, 2900, 2211, 2700}</td>
<td>800</td>
<td>{2011, 2311, 1111, 501}</td>
<td>600</td>
</tr>
<tr>
<td>{80, 301, 1300, 501}</td>
<td>300</td>
<td>{192801, 301, 2301}</td>
<td>2000</td>
</tr>
<tr>
<td>{80, 301, 1111, 600}</td>
<td>100</td>
<td>{84001, 900, 800}</td>
<td>2600</td>
</tr>
<tr>
<td>{86, 2690, 2991, 2401}</td>
<td>000</td>
<td>{872791, 1600, 1810}</td>
<td>1900</td>
</tr>
<tr>
<td>{89, 301, 601, 1010}</td>
<td>2800</td>
<td>{81301, 2410, 2200}</td>
<td>1400</td>
</tr>
<tr>
<td>{12, 1100, 2310, 1210}</td>
<td>1600</td>
<td>{1311, 2100, 1411}</td>
<td>1800</td>
</tr>
<tr>
<td>{15, 300, 2500, 1700}</td>
<td>2700</td>
<td>{151600, 2611, 2210}</td>
<td>2900</td>
</tr>
<tr>
<td>{18, 000, 1911, 2010}</td>
<td>2000</td>
<td>{19, 2400, 2111, 401}</td>
<td>1000</td>
</tr>
<tr>
<td>{21, 2510, 701, 1101}</td>
<td>1500</td>
<td>{2517, 0010, 2110}</td>
<td>3000</td>
</tr>
<tr>
<td>{24, 010, 4011, 1101}</td>
<td>500</td>
<td>{251911, 1901, 2110}</td>
<td>4000</td>
</tr>
<tr>
<td>{27, 1401, 2501, 3010}</td>
<td>2400</td>
<td>{271401, 2001, 2110}</td>
<td>{271911, 1100, 611}.</td>
</tr>
</tbody>
</table>
When \((u, w) = (39, 6)\):

\[
\begin{array}{c|c|c|c|c}
S & \Omega(S) & S & \Omega(S) & S \\
\{2810, 2910, 2610, 200\} & 2300 & \{2401, 1011, 901, 1700\} & 1300 & \{3000, 2900, 600, 2101\} & 000 \\
\{1101, 3001, 1000, 711\} & 1000 & \{911, 2600, 2100, 2001\} & 1100 & \{3000, 3200, 001, 800\} & 800 \\
\{2201, 810, 1800, 2701\} & 900 & \{2110, 3011, 2400, 411\} & 500 & \{3201, 2710, 1801, 2500\} & 2500 \\
\{\infty, 2810, 1600, 1211\} & 3200 & \{\infty, 101, 1810, 161\} & 2000 & \{\infty, 910, 611, 401\} & 300 \\
\{\infty, 1500, 3210, 610\} & 1900 & \{\infty, 4211, 3010, 110\} & 2700 & \{\infty, 5209, 811, 3400\} & 1600 \\
\{\infty, 2601, 1411, 230\} & 1800 & \{\infty, 7208, 1301, 2410\} & 1500 & \{\infty, 8241, 3101, 1310\} & 3400 \\
\{\infty, 92700, 1811, 1210\} & 2800 & \{\infty, 102511, 1311, 1911\} & 2200 & \{\infty, 11510, 400, 000\} & 3000 \\
\{\infty, 12700, 1300, 1901\} & 600 & \{\infty, 132110, 1611, 2501\} & 2600 & \{\infty, 141701, 701, 1110\} & 700 \\
\{\infty, 151501, 1910, 211\} & 1700 & \{\infty, 162201, 1201, 1000\} & 400 & \{\infty, 17010, 1401, 500\} & 1000 \\
\{\infty, 181511, 201, 1400\} & 1200 & \{\infty, 191410, 301, 2311\} & 200 & \{\infty, 201310, 1610, 1710\} & 1400 \\
\{\infty, 21311, 1900, 2510\} & 2900 & \{\infty, 22511, 1100, 2211\} & 2400 & \{\infty, 231022, 2120, 2301\} & 2100 \\
\end{array}
\]

\[\mathcal{C} = \{1011, 1510, 2300, 1301\}, \{2211, 401, 2000, 2710\}, \{1210, 1611, 800, 401\}, \{2311, 1201, 100, 910\}, \{2000, 3001, 2310, 2811\}\]

\[\mathcal{R} = \{2011, 601, 2811, 501\}, \{2911, 1201, 1111, 3111\}, \{3110, 1001, 1510, 710\}, \{900, 2711, 1410, 2000\}, \{2310, 011, 2010, 801\}, \{2611, 111, 2111, 1711\}\]

\[\square\]

**Proof of Theorem 2**: We first construct a GBTD\((4, m)\) for any \(m \in N\), where \(N = \{28, 32, 33, 34, 37, 38, 39, 44\}\).

For each \(w \in \{5, 6, 7, 8\}\), an HGBTD\((4, 4w)\) is given by Yin et al. [12]. For each \(m \in N\), apply Theorem 3.2, with IHGBTDs from Lemma 5, Lemma 6 and Lemma 7 and corresponding HGBTD\((4, 4w)\)'s where \(w \in \{5, 6, 7, 8\}\) as ingredients, to produce the desired HGBTD\((4, 4m)\). Hence, the desired GBTD\((4, m)\) follows from Proposition 3.1.

Combining Proposition 1.1, Proposition 2.1 and Proposition 2.2, we complete the proof.

\[\square\]

**Acknowledgement** We are grateful to the anonymous reviewers for their helpful comments.

**References**


[10] J. Yan and J. Yin, Constructions of optimal GDRP($n, \lambda; v$) of type $\lambda^1\mu^{m-1}$, *Discrete Appl. Math.* 156 (2008), 2666-2678.
