<table>
<thead>
<tr>
<th>Title</th>
<th>Generalized balanced tournament designs with block size four</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Chee, Yeow Meng; Kiah, Han Mao; Wang, Chengmin</td>
</tr>
<tr>
<td>Date</td>
<td>2013</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10220/24017">http://hdl.handle.net/10220/24017</a></td>
</tr>
</tbody>
</table>

**Rights**

© 2013 The Authors (The Electronic Journal of Combinatorics). This paper was published in The Electronic Journal of Combinatorics and is made available as an electronic reprint (preprint) with permission of The Electronic Journal of Combinatorics. The paper can be found at the following official URL: [http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i2p51](http://www.combinatorics.org/ojs/index.php/eljc/article/view/v20i2p51). One print or electronic copy may be made for personal use only. Systematic or multiple reproduction, distribution to multiple locations via electronic or other means, duplication of any material in this paper for a fee or for commercial purposes, or modification of the content of the paper is prohibited and is subject to penalties under law.
Generalized Balanced Tournament Designs with Block Size Four

Yeow Meng Chee Han Mao Kiah Chengmin Wang
School of Physical and Mathematical Sciences Nanyang Technological University Singapore 637371 {YMChee,KIAH0001}@ntu.edu.sg
School of Science Jiangnan University Wuxi, China 214122 wcm@jiangnan.edu.cn

Submitted: Sep 16, 2012; Accepted: Jun 1, 2013; Published: Jun 7, 2013
Mathematics Subject Classifications: 05B05, 94B25

Abstract

In this paper, we remove the outstanding values $m$ for which the existence of a GBT(4, $m$) has not been decided previously. This leads to a complete solution to the existence problem regarding GBT(4, $m$)s.

Keywords: generalized balanced tournament design; holey generalized balanced tournament design; starter-adder

1 Introduction

A set system is a pair $\mathcal{G} = (X, \mathcal{B})$, where $X$ is a finite set of points and $\mathcal{B}$ is a collection of subsets of $X$. Elements of $\mathcal{B}$ are called blocks. The order of $\mathcal{G}$ is $|X|$, the number of points. Let $K$ be a set of positive integers. A set system $(X, \mathcal{B})$ is said to be $K$-uniform if $|B| \in K$ for all $B \in \mathcal{B}$. Let $(X, \mathcal{B})$ be a set system and $S \subseteq X$. A partial $\alpha$-parallel class over $X\setminus S$ of $(X, \mathcal{B})$ is a set of blocks $\mathcal{A} \subseteq \mathcal{B}$ such that each point of $X\setminus S$ occurs in exactly $\alpha$ blocks of $\mathcal{A}$, and each point of $S$ occurs in no block of $\mathcal{A}$. A partial $\alpha$-parallel class over $X$ is simply called an $\alpha$-parallel class. We adopt the convention that if $\alpha$ is not specified, then it is taken to be one, so that a parallel class refers to a 1-parallel class. A set system $(X, \mathcal{B})$ is said to be resolvable if $\mathcal{B}$ can be partitioned into parallel classes.

*Research of Y. M. Chee, H. M. Kiah, and C. Wang is supported in part by the Singapore National Research Foundation under Research Grant NRF-CRP2-2007-03. Research of C. Wang is also supported in part by NSFC under Grants No. 11271280 and 10801064.

†Corresponding author.
A balanced incomplete block design of order \(v\), block size \(k\), and index \(\lambda\), denoted by \((v, k, \lambda)\)-BIBD, is a \(\{k\}\)-uniform set system \((X, \mathcal{B})\) of order \(v\) such that every 2-subset of \(X\) is contained in precisely \(\lambda\) blocks of \(\mathcal{B}\). A resolvable \((km, k, k-1)\)-BIBD \((X, \mathcal{B})\) is called a generalized balanced tournament design (GBTD), or simply a GBTD\((k, m)\), if the \(m(km-1)\) blocks of \(\mathcal{B}\) are arranged in an \(m \times (km-1)\) array such that

(i) the set of blocks in each column is a parallel class, and

(ii) each point of \(X\) is contained in at most \(k\) cells of each row.

GBTDs were introduced by Lamken [3] and are useful in the construction of many combinatorial designs, including resolvable, near-resolvable, doubly resolvable, and doubly near-resolvable balanced incomplete block designs (see [2, 3]). More recently, GBTDs have also found applications in near constant-composition codes [12], and codes for power line communications [1].

Schellenberg et al. [8] showed that a GBTD\((2, m)\) exists for all positive integers \(m \neq 2\). Lamken [4] showed that a GBTD\((3, m)\) exists for all positive integers \(m \neq 2\). For \(k = 4\), Yin et al. [12] obtained the following.

**Theorem 1** (Yin et al. [12]). A GBTD\((4, m)\) exists for all positive integers \(m \geq 5\), except possibly when \(m \in \{28, 32, 33, 34, 37, 38, 39, 44\}\).

The purpose of this paper is to remove all the remaining eight possible exceptions in Theorem 1 and to show that a GBTD\((4, m)\) exists for \(m = 4\) but not for \(m \in \{2, 3\}\).

**Theorem 2.** For each \(m \in \{4, 28, 32, 33, 34, 37, 38, 39, 44\}\), a GBTD\((4, m)\) exists. For \(m = 2\) and 3, a GBTD\((4, m)\) does not exist.

A GBTD\((4, 1)\) exists trivially. Combining Theorem 1 and Theorem 2, the existence of GBTD\((4, m)\) is now completely determined.

**Theorem 3.** A GBTD\((4, m)\) exists if and only if \(m \geq 1\) and \(m \neq 2, 3\).

We first present a non-existence result.

**Proposition 1.1.** A GBTD\((k, 2)\) does not exist for all \(k \geq 2\).

**Proof:** Suppose \((X, \mathcal{B})\) is a \((2k, k, k-1)\)-BIBD whose blocks are organized into a \(2 \times (2k-1)\) array to form a GBTD\((k, 2)\). Since \((X, \mathcal{B})\) is a \((2k, k, k-1)\)-BIBD, each point in \(X\) appears in exactly \(2k - 1\) blocks, and hence each point must appear either \(k\) times in the first row and \(k - 1\) times in the second row, or vice versa.

Consider a point \(x \in X\) that appears \(k\) times in the first row and \(k - 1\) times in the second row. Let \(y \in X\) be distinct from \(x\). The cells in the first row can be classified as follows:

(i) Type-\(xy\): a cell that contains both \(x\) and \(y\).
(ii) Type-$x\bar{y}$: a cell that contains $x$ but not $y$.

(iii) Type-$\bar{x}y$: a cell that contains $y$ but not $x$.

(iv) Type-$\bar{x}\bar{y}$: a cell that contains neither $x$ nor $y$.

Let $\alpha$ and $\beta$ be the number of type-$xy$ cells and type-$\bar{x}y$ cells in the first row, respectively. Then we have the table

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$k - \alpha$</th>
<th>$\beta$</th>
<th>$k - 1 - \beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td># cells in first row</td>
<td>$\alpha$</td>
<td>$k - \alpha$</td>
<td>$\beta$</td>
<td>$k - 1 - \beta$</td>
</tr>
<tr>
<td># cells in second row</td>
<td>$k - 1 - \beta$</td>
<td>$\beta$</td>
<td>$k - \alpha$</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>

where the second row is obtained from the first by the following observation: if a cell is of type-$xy$ (respectively, type-$x\bar{y}$, type-$\bar{x}y$, type-$\bar{x}\bar{y}$) in the first row, then the cell in the second row of the corresponding column is of type-$\bar{x}\bar{y}$ (respectively, type-$\bar{x}y$, type-$x\bar{y}$, type-$xy$). On the other hand, the total number of type-$xy$ cells is $k - 1$, since $(X, \mathcal{B})$ is a BIBD of index $k - 1$. Hence, we have $\alpha + (k - 1 - \beta) = k - 1$, implying $\alpha = \beta$.

Considering the number of occurrences of $y$ in the first row and the number of occurrences of $y$ in the second row of the GBTD give the inequalities

\[
\alpha + \beta \leq k,
\]
\[
2k - 1 - \alpha - \beta \leq k,
\]

from which, and $\alpha = \beta$ shown earlier, follow that

\[
\alpha = \lfloor k/2 \rfloor.
\]

Table T1 can therefore be revised to

<table>
<thead>
<tr>
<th></th>
<th>$\lfloor k/2 \rfloor$</th>
<th>$\lfloor k/2 \rfloor$</th>
<th>$\lfloor k/2 \rfloor$</th>
<th>$\lfloor k/2 \rfloor - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td># cells in first row</td>
<td>$\lfloor k/2 \rfloor$</td>
<td>$\lfloor k/2 \rfloor$</td>
<td>$\lfloor k/2 \rfloor$</td>
<td>$\lfloor k/2 \rfloor - 1$</td>
</tr>
<tr>
<td># cells in second row</td>
<td>$\lfloor k/2 \rfloor - 1$</td>
<td>$\lfloor k/2 \rfloor$</td>
<td>$\lfloor k/2 \rfloor$</td>
<td>$\lfloor k/2 \rfloor$</td>
</tr>
</tbody>
</table>

Counting in two ways the number of elements in the set

\[
\{(y, C) : y \in X, y \neq x, \text{ and } C \text{ is a cell of type-} xy \text{ in the second row}\},
\]

gives

\[
(2k - 1)(\lfloor k/2 \rfloor - 1) = (k - 1)^2,
\]

which is a contradiction. \qed
2 Existence of GBTD($4,m$)s with $m = 3$ and 4

For a positive integer $n$, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. Let $\Sigma$ be a set of $q$ symbols. A $q$-ary code of length $n$ over $\Sigma$ is a subset $C \subseteq \Sigma^n$. Elements of $C$ are called codewords. The size of $C$ is the number of codewords in $C$. For $i \in [n]$, the $i$th coordinate of a codeword $u \in C$ is denoted $u_i$, so that $u = (u_1, u_2, \ldots, u_n)$.

The symbol weight of $u \in \Sigma^n$, denoted $\text{swt}(u)$, is the maximum frequency of appearance of a symbol in $u$, that is,

$$\text{swt}(u) = \max_{\sigma \in \Sigma} |\{u_i = \sigma : i \in [n]\}|.$$

A code has constant symbol weight $w$ if all of its codewords have symbol weight exactly $w$. The (Hamming) distance between $u, v \in \Sigma^n$ is $d_H(u, v) = |\{i \in [n] : u_i = v_i\}|$, the number of coordinates at which $u$ and $v$ differ. A code $C$ is said to have distance $d$ if $d_H(u, v) \geq d$ for all distinct $u, v \in C$. A $q$-ary code of length $n$, constant symbol weight $w$, and distance $d$ is referred to as an $(n, d, w)_q$-symbol weight code. An $(n, d, w)_q$-symbol weight code with maximum size is said to be optimal.

Chee et al. [1] established the following relation between a GBTD and a symbol weight code.

**Theorem 4** (Chee et al. [1]). A GBTD($k, m$) exists if and only if an optimal $(km - 1, k(m - 1), k)_m$-symbol weight code exists.

In view of Theorem 4, to prove the nonexistence of a GBTD($4, 3$), it suffices to show that there does not exist a ternary code of length 11, constant symbol weight four, and size 12, that is of equidistance eight. Consider the Gilbert graph $G = (V, E)$, where $V = \{u \in [3]^{11} : \text{swt}(u) = 4\}$ and two vertices $u, v \in V$ are adjacent in $G$ if and only if $d_H(u, v) = 8$. Then there exists a ternary code of length 11, constant symbol weight four, and size 12, that is of equidistance eight if and only if there exists a clique of size 12 in $G$. It is not hard to see that $G$ is vertex-transitive so that we can just search for a clique of size 11 in $G'$, the subgraph of $G$ induced by the set of vertices $\{v \in V : d_H(u, v) = 8\}$ for some fixed $u \in V$. This induced subgraph $G'$ has 8001 vertices and 7233060 edges. We solve this clique-finding problem using Cliquer, an implementation of Östergård’s clique-finding algorithm by Niskanen and Östergård [6]. The result is that the largest clique in $G'$ has size 10. Consequently, we have the following.

**Proposition 2.1.** There does not exist a GBTD($4, 3$).

There exists, however, a GBTD($4, 4$). Unfortunately, a GBTD($4, 4$) is too large to be found by the method of clique-finding above. Instead, to curb the search space, we assume the existence of some automorphisms acting on the GBTD($4, 4$) to be found. Let $(X, B)$ be a GBTD($4, 4$), where $X = \mathbb{Z}_4 \times \mathbb{Z}_4$. If $B \subseteq X$ and $x \in X$, $B + x$ denotes the set $\{b + x : b \in B\}$. If $A$ is an array over $X$ and $x \in X$, $A + x$ denotes the array obtained by adding $x$ to every entry of $A$. For succinctness, a point $(x, y) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ is sometimes written $x y$. 
The GBTD\((4, 4)\) we construct contains the \(4 \times 3\) subarray

\[
A_0 = \begin{bmatrix}
\{00, 02, 20, 22\} & \{11, 13, 31, 33\} & \{10, 12, 30, 32\} \\
\{01, 03, 21, 23\} & \{00, 02, 20, 22\} & \{11, 13, 31, 33\} \\
\{10, 12, 30, 32\} & \{01, 03, 21, 23\} & \{00, 02, 20, 22\} \\
\{11, 13, 31, 33\} & \{10, 12, 30, 32\} & \{01, 03, 21, 23\}
\end{bmatrix}
\]

The blocks in \(A_0\) contain exactly the 2-subsets \(\{ab, cd\} \subseteq X\), where \(a + c \equiv b + d \equiv 0 \mod 2\), each thrice. The remaining \(4 \times 12\) subarray of the GBTD\((4, 4)\) is built from a set of 12 base blocks \(S = \{B_{i,j} : i \in [3] \text{ and } 0 \leq j \leq 3\}\) as follows. Let

\[
A_1 = \begin{bmatrix}
B_{1,0} & B_{2,0} & B_{3,0} \\
B_{1,1} & B_{2,1} & B_{3,1} \\
B_{1,2} & B_{2,2} & B_{3,2} \\
B_{1,3} & B_{2,3} & B_{3,3}
\end{bmatrix}
\]

Then the \(4 \times 12\) subarray is given by

\[
\begin{bmatrix}
A_0 & A_1 & (0, 1) & A_1 + (0, 2) & A_1 + (0, 3)
\end{bmatrix}
\]

For

\[
\begin{bmatrix}
A_0 & A_1 & (0, 1) & A_1 + (0, 2) & A_1 + (0, 3)
\end{bmatrix}
\]

to be a GBTD\((4, 4)\), the following conditions are imposed:

(i) \(\bigcup_{j=0}^{3} B_{i,j} = Z_4 \times Z_4\), for \(i \in [3]\), so that every column is a parallel class.

(ii) For each \(j, 0 \leq j \leq 3\), each element of \(Z_4\) appears exactly three times as a first coordinate among the elements of \(\bigcup_{i=1}^{3} B_{i,j}\), so that every row contains each element of \(Z_4 \times Z_4\) at most three times.

(iii) Let \(k, l \in Z_4\) and define \(\Delta_{k,l}S\) to be the multiset \(\bigcup_{A \in S} \{x - y : (k, x), (l, y) \in A\}\). Then

\[
\Delta_{k,l}S = \begin{cases} 
\{1, 1, 1, 3, 3, 3\}, & \text{if } k = l \text{ or } k + l \equiv 0 \mod 2; \\
\{0, 0, 0, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3\}, & \text{otherwise}.
\end{cases}
\]

This ensures that every 2-subset of \(X\) not contained in any block in \(A_0\) is contained in exactly three blocks in \(A_1\), \(A_1 + (0, 1)\), \(A_1 + (0, 2)\), or \(A_1 + (0, 3)\).

A computer search found the following array \(A_1\) that satisfies all the conditions above.

\[
A_1 = \begin{bmatrix}
\{23, 22, 32, 11\} & \{10, 00, 21, 11\} & \{00, 01, 30, 33\} \\
\{20, 01, 30, 33\} & \{33, 02, 03, 12\} & \{10, 13, 22, 23\} \\
\{31, 00, 12, 21\} & \{01, 13, 20, 32\} & \{02, 11, 21, 32\} \\
\{02, 10, 13, 03\} & \{22, 23, 30, 31\} & \{03, 12, 20, 31\}
\end{bmatrix}
\]

Consequently, we have the following.

**Proposition 2.2.** There exists a GBTD\((4, 4)\).
3 Incomplete Holey GBTDs

Let \((X, \mathcal{B})\) be a set system, and let \(\mathcal{G}\) be a partition of \(X\) into subsets, called groups. The triple \((X, \mathcal{G}, \mathcal{B})\) is a group divisible design (GDD) of index \(\lambda\) when every 2-subset of \(X\) not contained in a group appears in exactly \(\lambda\) blocks, and \(|B \cap G| \leq 1\) for all \(B \in \mathcal{B}\) and \(G \in \mathcal{G}\). We denote a GDD \((X, \mathcal{G}, \mathcal{B})\) of index \(\lambda\) by \((K, \lambda)\)-GDD if \((X, \mathcal{B})\) is \(K\)-uniform. The type of a GDD \((X, \mathcal{G}, \mathcal{B})\) is the multiset \([^{|G| : G \in \mathcal{G}}\]. When more convenient, the exponential notation is used to describe the type of a GDD: a GDD of type \(g_1^{i_1}g_2^{i_2} \cdots g_s^{i_s}\) is a GDD where there are exactly \(i_i\) groups of size \(g_i\), \(i \in [s]\).

Suppose further \(\mathcal{G} = \{G_1, G_2, \ldots, G_s\}\) and \(\mathcal{H} = \{H_1, H_2, \ldots, H_s\}\) is a collection of subsets of \(X\) with the property \(H_i \subseteq G_i\), \(0 \leq i \leq s\). Let \(H = \bigcup_{i=1}^{s} H_i\). Then the quadruple \((X, \mathcal{G}, \mathcal{H}, \mathcal{B})\) is an incomplete group divisible design (IGDD) of index \(\lambda\) when every 2-subset of \(X\) not contained in a group or \(H\) appears in exactly \(\lambda\) blocks, and \(|B \cap G| \leq 1\) and \(|B \cap H| \leq 1\) for all \(B \in \mathcal{B}\) and \(G \in \mathcal{G}\). The type of an IGDD \((X, \{G_1, G_2, \ldots, G_s\}; \{H_1, H_2, \ldots, H_s\}, \mathcal{B}\) is the multiset \([^{|G_i|, |H_i| : 1 \leq i \leq s}\) and we use the exponential notation when more convenient.

Let \(k, g, u,\) and \(w\) be positive integers such that \(k \mid g\) and \(u \geq (k+1)w\). Let \(R_i = \{r \in \mathbb{Z} : ig/k \leq r \leq (i+1)g/k - 1\}\). An incomplete holey GBDT with block size \(k\) and type \(g^{(u,w)}\), denoted \(\text{IHGBTD}(k, g^{(u,w)})\), is a \((\{k\}, k-1)\)-IGDD \((X, \{G_0, G_1, \ldots, G_u\}; \mathcal{B})\) of type \((g,0)^{u-w}(g,g)^w\), whose blocks are arranged in a \((gu/k) \times (g-1)\) array \(A\), with rows and columns indexed by elements from the sets \(\{0,1, \ldots, gu/k-1\}\) and \(\{0,1, \ldots, g(u-1)-1\}\), respectively, such that the following properties are satisfied.

(i) The \((gw/k) \times (g-1)\) subarray whose rows are indexed by \(r \in R_i\), where \(u - w \leq i \leq u - 1\), and columns indexed by \(c\), where \(g(u-w) \leq c \leq g(u-1)-1\), is empty.

(ii) For each \(i, 0 \leq i \leq u - w - 1\), the blocks in row \(r \in R_i\) form a partial \(k\)-parallel class over \(X \setminus G_i\), and for each \(i, u - w \leq i \leq u - 1\), the blocks in row \(r \in R_i\) form a partial \(k\)-parallel class over \(X \setminus \left(\bigcup_{j=u-w}^{u-1} G_j\right)\).

(iii) For each \(j, 0 \leq j \leq g(u-w) - 1\), the blocks in column \(j\) form a parallel class, and for each \(j, g(u-w) \leq j \leq g(u-1)-1\), the blocks in column \(j\) form a partial parallel class over \(X \setminus \left(\bigcup_{i=u-w}^{u-1} G_j\right)\).

Each group acts as a hole of the design, since no block contains any 2-subset of a group. The design is also incomplete in the sense that the array \(A\) contains an empty \((gw/k) \times (g-1)\) subarray.

We note that an IHGBTD\((k, g^{(u-1)})\) is a holey GBDT, HGBTD\((k, g^u)\), as defined by Yin et al. [12]. The following was established by Yin et al. [12].

**Proposition 3.1** (Yin et al. [12]). If there exists an HGBTD\((k,k^u)\), then there exists a GBDT\((k,u)\).

Proposition 3.1 shows that a GBDT\((k,u)\) can be obtained from an HGBTD\((k,k^u)\). The next result shows how we can obtain an HGBTD\((k,g^u)\) (and, in particular, an HGBTD\((k,k^u)\)) from an IHGBTD\((k,g^{(u,w)})\) and an HGBTD\((k,g^w)\).
Proposition 3.2. If there exist an IHGBTD\((k,g^{(u,w)})\) and an HGBTD\((k,g^w)\), then there exists an HGBTD\((k,g^u)\).

Proof: When \(w = 1\), an HGBTD\((k,g^w)\) is empty and an IHGBTD\((k,g^{(u,w)})\) is just an HGBTD\((k,g^u)\). So assume \(w > 1\) and let \((X, G, B)\) be an IHGBTD\((k,g^{(u,w)})\) with \(G = \{G_0, G_1, \ldots, G_{u-1}\}\). Fill in the empty subarray of this IHGBTD with an HGBTD\((k,g^w)\), \((X', G', B')\), with \(G' = \{G_{u-w}, G_{u-w+1}, \ldots, G_{u-1}\}\) and \(X' = \bigcup_{i=u-w}^{u-1} G_i\). The resulting array is a HGBTD\((k,g^u)\), \((X, G, B \cup B')\).

\[\square\]

4 Starter-Adder Construction for IHGBTD

The starter-adder technique first used by Mullin and Nemeth [5] to construct Room squares (also a combinatorial array) has been useful in constructing many types of designs with orthogonality properties, including GBTDs (see [3, 7, 10, 11, 12]). Here, we extend the technique to the construction of IHGBTDs. Since only IHGBTD\((k,g^{(u,w)})\) with \(g = k\) are considered here, we describe our construction for this case.

Let \(\Gamma\) be an additive abelian group of order \(k(u-w)\) with \(u \geq (k+1)w\), and let \(\Gamma_0 \subseteq \Gamma\) be a subgroup of order \(k\). Fix a set, \(\Delta = \{\delta_0 = 0, \delta_1, \ldots, \delta_{u-w-1}\} \subseteq \Gamma\), of representatives for the cosets of \(\Gamma_0\) so that \(\Gamma_i = \Gamma_0 + \delta_i\), \(0 \leq i \leq u-w-1\), are the cosets of \(\Gamma_0\). Let \(H\) be a set of \(kw\) points such that \(H\) and \(\Gamma\) are disjoint. Further, let \(H\) be partitioned into \(w\) subsets \(H_0, H_1, \ldots, H_{w-1}\) of size \(k\) each.

We take \(X = \Gamma \cup H\) to be the point set of an IHGBTD\((k,k^{(u,w)})\). An intransitive starter for an IHGBTD\((k,k^{(u,w)})\), with groups \(\{G_0, G_1, \ldots, G_{u-1}\}\), where

\[G_i = \begin{cases} \Gamma_i, & \text{if } 0 \leq i \leq u-w-1; \\ H_{i-u+w}, & \text{if } u-w \leq i \leq u-1, \end{cases}\]

is defined as a quadruple \((X, S, R, C)\) satisfying the properties:

\[(i)\] \((X, S), (X, R), \text{ and } (X, C)\) are \(\{k\}\)-uniform set systems of size \(u-w, w, \text{ and } w-1\), respectively;

\[(ii)\] among the blocks in \(S\), \(kw\) of them intersects \(H\) in one point, that is, \(|\{B \in S : |B \cap H| = 1\}| = kw\);

\[(iii)\] blocks in \(R\) are each disjoint from \(H\);

\[(iv)\] blocks in \(C\) are each disjoint from \(H\), and \(\bigcup_{i=0}^{u-w-1}(\delta_i + C) = \Gamma\), for each \(C \in C\).

\[(v)\] \(S \cup R\) is a partition of \(X\);

\[(vi)\] the difference list from the base blocks of \(S \cup R \cup C\) contains every element of \(\Gamma \setminus \Gamma_0\) precisely \(k-1\) times, and no element in \(\Gamma_0\).
Suppose \( S = \{B_0, B_1, \ldots, B_{u-w-1}\} \). Then a corresponding *adder* \( \Omega(S) \) for \( S \) is a permutation \( \Omega(S) = (\omega_0, \omega_1, \ldots, \omega_{u-w-1}) \) of the \( u-w \) elements of the representative system \( \Delta \) satisfying the following property:

(vii) the multiset \( \bigcup_{i=0}^{u-w-1} (B_i + \omega_i) \) contains exactly \( k \) elements (not necessarily distinct) from \( \Gamma_j \) for \( 1 \leq j \leq u-w-1 \), and is disjoint from \( \Gamma_0 \). We remark that when \( B \in S \) and \( B \cap H = \{\infty\} \), or \( B = \{\infty, b_1, b_2, \ldots, b_{k-1}\} \), the block \( B + \gamma \) is defined to be \( \{\infty, b_1 + \gamma, b_2 + \gamma, \ldots, b_{k-1} + \gamma\} \) for \( \gamma \in \Gamma \).

The result below shows how to construct an IHGBTD from an intransitive starter and its corresponding adder.

**Proposition 4.1.** Let \( \Gamma \) be an additive abelian group of order \( k(u-w) \) with \( u \geq (k+1)w \) and \( \Gamma_0 \) be a subgroup of order \( k \). Define \( X \) and the groups \( G_i \) \( (0 \leq i \leq u-1) \) as above. If there exists an intransitive starter \( (X, S, R, C) \) with groups \( \{G_i : 0 \leq i \leq u-1\} \), a corresponding adder \( \Omega(S) \), then there exists an IHGBTD \( (k, k^{(u,w)}) \).

**Proof:** Retain the notations in the definition of intransitive starter and adder. Suppose

\[ A = \{A + \gamma : \gamma \in \Gamma, A \in S \cup R \cup C\}, \]

then \( (X, \{G_0, G_1, \ldots, G_{u-1}\}, \{\varnothing, \varnothing, \varnothing, H_0, \ldots, H_{w-1}\}, A) \) forms a \( (\{k\}, k-1) \)-IGDD of type \( (k, 0)^u \) \( (k, k)^w \) by Condition (vi) in the definition of intransitive starter. Therefore, it remains to arrange the blocks in a \( u \times k(u-w) \) array.

First, consider the blocks \( S \). Consider a \( (u-w) \times (u-w) \) array \( S \), whose rows and columns are indexed with the elements in \( \Delta \). Now identify the elements in \( \Delta \) with elements in the quotient group \( \Gamma/\Gamma_0 \), so that the binary operation \( + \) on \( \Delta \) is defined by the additive operation on \( \Gamma/\Gamma_0 \). In addition, for \( \delta \in \Delta \), denote the additive inverse of \( \delta \) by \( -\delta \). That is, \( \delta + (-\delta) = 0 \).

So, for \( 0 \leq i, j \leq u-w-1 \), we place the block \( B_i + \delta_j \) at the cell \( (\delta_j, -\delta_i, \delta_j) \) if \( \delta_i = \omega_i \). Note that this placement is well defined because \( \Omega(S) \) is a permutation of \( \Delta \). Let \( \Gamma_0 = \{\gamma_0 = 0, \gamma_1, \ldots, \gamma_{k-1}\} \). Form a \( (u-w) \times k(u-w) \) array \( \hat{S} \) from the square \( S \) by concatenating \( k \) squares \( D + \gamma_i \) where \( 0 \leq i \leq k-1 \) as follows.

\[ \hat{S} = \begin{array}{cccc} S & \gamma_1 & \cdots & \gamma_{k-1} \end{array} \]

Next, let \( R = \{R_1, R_2, \ldots, R_w\} \) and construct a \( w \times k(u-w) \) array \( \hat{R} \) in the following way:

\[ \hat{R} = \begin{array}{cccc} R & \gamma_1 & \cdots & \gamma_{k-1} \end{array} \]

where the \( w \times w \) subarray \( R \) is given by

\[ R = \begin{array}{cccc} R_1 & R_1 + \delta_1 & \cdots & R_1 + \delta_{u-w-1} \\
R_2 & R_2 + \delta_1 & \cdots & R_2 + \delta_{u-w-1} \\
\vdots & \vdots & \ddots & \vdots \\
R_w & R_w + \delta_1 & \cdots & R_w + \delta_{u-w-1} \end{array} \]
Similarly, let $\mathcal{C} = \{C_0, C_1, \ldots, C_{w-2}\}$, and we construct a $(u - w) \times k(w - 1)$ array $\hat{\mathcal{C}}$.

$$\hat{\mathcal{C}} = \begin{bmatrix} C_0 & C_1 & \cdots & C_{w-2} \end{bmatrix},$$

where each $(u - w) \times k$ subarray $C_i$, $0 \leq i \leq w - 2$, is given by

$$C_i = \begin{bmatrix} C_i & C_i + \gamma_1 & \cdots & C_i + \gamma_{k-1} \\ C_i + \delta_1 & C_i + \delta_1 + \gamma_1 & \cdots & C_i + \delta_1 + \gamma_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_i + \delta_{u-w-1} & C_i + \delta_{u-w-1} + \gamma_1 & \cdots & C_i + \delta_{u-w-1} + \gamma_{k-1} \end{bmatrix}.$$

Finally, let

$$A = \begin{bmatrix} \hat{S} & \hat{\mathcal{C}} \\ \hat{R} & \end{bmatrix},$$

and it is readily verified that the placement results in an IHGBTD$(k, k^{(u,w)})$.

\section{Proof of Theorem 1.2}

We first remove all the eight remaining values in Theorem 1.

\textbf{Lemma 5.} For $(u, w) \in \{(28, 5), (32, 5), (33, 6)\}$, an IHGBTD$(4, 4^{(u,w)})$ exists.

\textbf{Proof:} We apply Proposition 4.1 to construct the desired IHGBTDs. Take

$$\Gamma = \mathbb{Z}_{u-w} \times \mathbb{Z}_4,$$

$$\Gamma_0 = \{0\} \times \mathbb{Z}_4,$$

$$\Delta = \{(0, 0), (1, 0), \ldots, (u - w - 1, 0)\},$$

and

$$H = \bigcup_{i=0}^{w-1} H_i, \text{ where } H_i = \{\infty, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \text{ for } 0 \leq i \leq w - 1.$$ 

For each pair $(u, w) \in \{(28, 5), (32, 5), (33, 6)\}$, the desired intransitive starter and corresponding adder are displayed below. Here we write the element $(a, b)$ of $\Gamma$ as $a_b$ for succinctness.

When $(u, w) = (28, 5)$:

<table>
<thead>
<tr>
<th>$\mathcal{S}$</th>
<th>$\Omega(\mathcal{S})$</th>
<th>$\mathcal{S}$</th>
<th>$\Omega(\mathcal{S})$</th>
<th>$\Omega(\mathcal{S})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${a_1, 3_0, 7_0, 0_0}$</td>
<td>17_0</td>
<td>${5_0, 19_0, 12_1, 12}$</td>
<td>12_0</td>
<td>${18_0, 13_3, 16_3, 8_1}$</td>
</tr>
<tr>
<td>${\infty, 3_1, 12_2, 11_3}$</td>
<td>1_0</td>
<td>${\infty, 14_1, 6_0, 10_0}$</td>
<td>21_0</td>
<td>${\infty, 14_1, 9_1, 20_1}$</td>
</tr>
<tr>
<td>${3_0, 9_1, 10_1, 22_2}$</td>
<td>7_0</td>
<td>${3_0, 3_3, 1_3, 2_2}$</td>
<td>18_0</td>
<td>${\infty, 5_0, 15_1, 1_0}$</td>
</tr>
<tr>
<td>${\infty, 11_1, 6_3, 9_3}$</td>
<td>2_0</td>
<td>${\infty, 7_1, 14_0, 11_1, 0_1}$</td>
<td>10_0</td>
<td>${\infty, 8_0, 17_2, 21_2}$</td>
</tr>
<tr>
<td>${\infty, 4_3, 8_0, 21_0}$</td>
<td>6_0</td>
<td>${\infty, 10_1, 13_1, 19_2, 16_2}$</td>
<td>9_0</td>
<td>${\infty, 4_2, 21_3, 17_1}$</td>
</tr>
<tr>
<td>${\infty, 12_7, 5_2, 21_1}$</td>
<td>16_0</td>
<td>${\infty, 13_3, 5_1, 20_2, 11_2}$</td>
<td>4_0</td>
<td>${\infty, 14_2, 22_0, 23_0, 16_1}$</td>
</tr>
<tr>
<td>${\infty, 15_1, 18_3, 20_3, 2_0}$</td>
<td>0_0</td>
<td>${\infty, 16_3, 12_2, 21_2, 23}$</td>
<td>3_0</td>
<td>${\infty, 17_3, 7_1, 17_3}$</td>
</tr>
<tr>
<td>${\infty, 18_2, 9_0, 19_2}$</td>
<td>13_0</td>
<td>${\infty, 19_7, 8_3, 22_1}$</td>
<td>11_0</td>
<td></td>
</tr>
</tbody>
</table>
\[ C = \{18_0, 11_1, 5_3, 6_2\}, \{18_2, 8_3, 19_0, 6_1\}, \{14_3, 12_0, 3_2, 7_1\}, \{5_2, 7_1, 16_3, 11_0\}. \]
\[ \mathcal{R} = \{3_2, 18_2, 16_1, 10_2\}, \{8_2, 15_0, 20_0, 13_2\}, \{13_0, 9_2, 18_1, 15_3\}, \{6_1, 7_3, 14_2, 15_2\}, \{12_0, 10_0, 4_0, 11_0\}. \]

When \((u, w) = (32, 5)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{4_2, 17_2, 16_1, 22_2}</td>
<td>160</td>
<td>{3_1, 4_1, 10_9, 9_1}</td>
<td>110</td>
</tr>
<tr>
<td>{14_1, 6_0, 26_0, 3_0}</td>
<td>120</td>
<td>{\infty_0, 3_3, 24_2, 25_1}</td>
<td>7_0</td>
</tr>
<tr>
<td>{\infty_2, 0_1, 26_1, 20_2}</td>
<td>4_0</td>
<td>{\infty_3, 25_0, 15_0, 23_0}</td>
<td>15_0</td>
</tr>
<tr>
<td>{\infty_5, 5_0, 19_3, 12_1}</td>
<td>24_0</td>
<td>{\infty_6, 6_3, 14_3, 13_2}</td>
<td>1_0</td>
</tr>
<tr>
<td>{\infty_8, 0_2, 10_0, 19_0}</td>
<td>14_0</td>
<td>{\infty_9, 15_2, 18_2, 0_3}</td>
<td>2_0</td>
</tr>
<tr>
<td>{\infty_{11}, 12_3, 25_2, 11_3}</td>
<td>22_0</td>
<td>{\infty_{12}, 10_1, 21_3, 17_3}</td>
<td>18_0</td>
</tr>
<tr>
<td>{\infty_{14}, 20_0, 3_2, 16_3}</td>
<td>5_0</td>
<td>{\infty_{15}, 12_2, 21_1, 8_2}</td>
<td>9_0</td>
</tr>
<tr>
<td>{\infty_{17}, 1_1, 15_1, 17_1}</td>
<td>8_0</td>
<td>{\infty_{18}, 9_2, 16_2, 23_2}</td>
<td>13_0</td>
</tr>
</tbody>
</table>

When \((u, w) = (33, 6)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{22_0, 0_1, 23_0, 21_3}</td>
<td>13_0</td>
<td>{25_3, 4_3, 15_1, 20_1}</td>
<td>4_0</td>
</tr>
<tr>
<td>{\infty_{20}, 21_0, 3_0, 22_2}</td>
<td>18_0</td>
<td>{\infty_{21}, 0_0, 14_3, 10_1}</td>
<td>6_0</td>
</tr>
<tr>
<td>{\infty_{30}, 6_1, 23_2, 9_1}</td>
<td>23_0</td>
<td>{\infty_{31}, 4_0, 8_2, 14_2}</td>
<td>2_0</td>
</tr>
<tr>
<td>{\infty_{62}, 21_2, 24_2, 11_1}</td>
<td>9_0</td>
<td>{\infty_{63}, 5_0, 2_1, 25_1}</td>
<td>20_0</td>
</tr>
<tr>
<td>{\infty_{03}, 2_7, 19_2}</td>
<td>15_0</td>
<td>{\infty_{35}, 10_3, 16_0, 14_0}</td>
<td>24_0</td>
</tr>
<tr>
<td>{\infty_{12}, 7_0, 9_0, 26_1}</td>
<td>19_0</td>
<td>{\infty_{13}, 25_0, 7_1, 10_0}</td>
<td>21_0</td>
</tr>
<tr>
<td>{\infty_{15}, 42, 15_2, 13_3}</td>
<td>16_0</td>
<td>{\infty_{16}, 17_1, 20_0, 11_3}</td>
<td>5_0</td>
</tr>
<tr>
<td>{\infty_{18}, 26_2, 5_2, 17_2}</td>
<td>12_0</td>
<td>{\infty_{19}, 24_0, 13_1, 10_3}</td>
<td>1_0</td>
</tr>
<tr>
<td>{\infty_{21}, 3_2, 15_3, 24_5}</td>
<td>25_0</td>
<td>{\infty_{22}, 5_1, 18_5, 21_3}</td>
<td>10_0</td>
</tr>
</tbody>
</table>

\[ C = \{3_3, 10_1, 5_2, 15_0\}, \{8_3, 14_1, 9_2, 18_0\}, \{12_0, 10_3, 26_2, 5_1\}, \{21_2, 11_1, 23_0, 9_3\}, \{15_1, 9_2, 12_3, 3_0\}. \]
\[ \mathcal{R} = \{6_3, 2_0, 18_2, 19_0\}, \{8_3, 9_2, 3_1, 1_2\}, \{17_3, 3_5, 4_1, 22_3\}, \{19_3, 13_2, 6_2, 5_3\}, \{16_3, 23_1, 11_1, 19_1\}, \{20_3, 16_2, 8_1, 0_3\}. \]

\[ \square \]

**Lemma 6.** For \((u, w) \in \{(34, 6), (44, 8)\}\), an IHGBTD\(4, 4^{(u, w)}\) exists.

**Proof:** As with Lemma 5, we apply Proposition 4.1 to construct the desired IHGBTDs. Take
\[
\Gamma = \mathbb{Z}_{2(u-w)} \times \mathbb{Z}_2,
\]
\[ \Gamma_0 = \{0, u - w\} \times \mathbb{Z}_2, \]
\[ \Delta = \{(0, 0), (1, 0), \ldots, (u - w - 1, 0)\}, \]
and
\[ H = \bigcup_{i=0}^{w-1} H_i, \quad \text{where} \quad H_i = \{\infty_i, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \quad \text{for} \quad 0 \leq i \leq w - 1. \]
The desired intransitive starter and corresponding adder for \((u, w) \in \{(34, 6), (44, 8)\}\) are displayed below. Here we write the element \((a, b)\) of \(\Gamma\) as \(a_b\) for succinctness.

When \((u, w) = (34, 6)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{411, 160, 60, 150}</td>
<td>200</td>
<td>{360, 90, 33, 131}</td>
<td>160</td>
<td>{370, 180, 261, 41}</td>
</tr>
<tr>
<td>{161, 214, 40, 31}</td>
<td>3</td>
<td>{\infty, 20, 240, 42}</td>
<td>230</td>
<td>{\infty, 22, 300, 39}</td>
</tr>
<tr>
<td>{\infty, 214, 31, 11}</td>
<td>100</td>
<td>{\infty, 3, 480, 450, 80}</td>
<td>250</td>
<td>{\infty, 4, 251, 481, 14}</td>
</tr>
<tr>
<td>{\infty, 5, 310, 20}</td>
<td>120</td>
<td>{\infty, 6, 210, 44}</td>
<td>2</td>
<td>{\infty, 7, 40, 330, 52}</td>
</tr>
<tr>
<td>{\infty, 451, 21, 38}</td>
<td>180</td>
<td>{\infty, 9, 270, 280, 34}</td>
<td>170</td>
<td>{\infty, 10, 42, 351, 71}</td>
</tr>
<tr>
<td>{\infty, 11, 30, 220, 12}</td>
<td>190</td>
<td>{\infty, 12, 440, 350, 390}</td>
<td>140</td>
<td>{\infty, 13, 361, 70, 91}</td>
</tr>
<tr>
<td>{\infty, 11, 51, 51}</td>
<td>60</td>
<td>{\infty, 15, 530, 110, 51}</td>
<td>150</td>
<td>{\infty, 16, 500, 551, 10}</td>
</tr>
<tr>
<td>{\infty, 17, 52, 31, 171}</td>
<td>130</td>
<td>{\infty, 18, 550, 290, 25}</td>
<td>50</td>
<td>{\infty, 19, 71, 410}</td>
</tr>
<tr>
<td>{\infty, 20, 121, 31, 47}</td>
<td>80</td>
<td>{\infty, 21, 170, 27, 47}</td>
<td>210</td>
<td>{\infty, 22, 190, 230, 29}</td>
</tr>
<tr>
<td>{\infty, 23, 340, 40, 50}</td>
<td>260</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[\mathcal{C} = \{271, 100, 441, 51\}, \{351, 150, 500, 141\}, \{161, 511, 540, 27\},\]
\[\{241, 120, 370, 211\}, \{390, 21, 45, 50\} .\]

\[\mathcal{R} = \{130, 260, 380, 24\}, \{541, 231, 461, 491\}, \{10, 490, 181, 430\},\]
\[\{100, 2, 111, 540\}, \{460, 19, 43, 50\}, \{381, 320, 5, 0\} .\]

When \((u, w) = (44, 8)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>{320, 691, 361, 53}</td>
<td>200</td>
<td>{421, 65, 0, 43}</td>
<td>10</td>
<td>{391, 271, 451, 51}</td>
</tr>
<tr>
<td>{221, 390, 55, 33}</td>
<td>110</td>
<td>{\infty, 670, 40, 54}</td>
<td>220</td>
<td>{\infty, 230, 101, 341}</td>
</tr>
<tr>
<td>{\infty, 180, 671, 36}</td>
<td>280</td>
<td>{\infty, 25, 100, 28}</td>
<td>160</td>
<td>{\infty, 631, 60, 37}</td>
</tr>
<tr>
<td>{\infty, 16, 44, 20}</td>
<td>350</td>
<td>{\infty, 280, 50, 35}</td>
<td>100</td>
<td>{\infty, 430, 461, 32}</td>
</tr>
<tr>
<td>{\infty, 8, 690, 52}</td>
<td>130</td>
<td>{\infty, 371, 600, 71}</td>
<td>260</td>
<td>{\infty, 10, 701, 21, 24}</td>
</tr>
<tr>
<td>{\infty, 11, 710, 19}</td>
<td>320</td>
<td>{\infty, 11, 50, 19, 6}</td>
<td>250</td>
<td>{\infty, 13, 401, 47, 20}</td>
</tr>
<tr>
<td>{\infty, 14, 52, 400, 60}</td>
<td>240</td>
<td>{\infty, 15, 170, 60, 22}</td>
<td>0</td>
<td>{\infty, 16, 64, 541, 12}</td>
</tr>
<tr>
<td>{\infty, 17, 49, 9, 543}</td>
<td>40</td>
<td>{\infty, 18, 680, 01, 56}</td>
<td>150</td>
<td>{\infty, 19, 10, 212, 24}</td>
</tr>
<tr>
<td>{\infty, 20, 650, 681, 23}</td>
<td>20</td>
<td>{\infty, 22, 10, 181, 80}</td>
<td>310</td>
<td>{\infty, 22, 59, 17, 44}</td>
</tr>
<tr>
<td>{\infty, 23, 10, 70, 26}</td>
<td>120</td>
<td>{\infty, 24, 571, 11, 13}</td>
<td>210</td>
<td>{\infty, 25, 161, 5, 70}</td>
</tr>
<tr>
<td>{\infty, 26, 58, 40, 57}</td>
<td>50</td>
<td>{\infty, 27, 411, 131, 31}</td>
<td>190</td>
<td>{\infty, 28, 64, 560, 30}</td>
</tr>
<tr>
<td>{\infty, 29, 10, 4, 490}</td>
<td>60</td>
<td>{\infty, 30, 481, 58, 50}</td>
<td>330</td>
<td>{\infty, 31, 40, 49, 51}</td>
</tr>
</tbody>
</table>

\[\mathcal{C} = \{2, 31, 22, 690\}, \{281, 690, 19, 62\}, \{411, 401, 201, 59\},\]
\[\{570, 121, 40, 551\}, \{41, 21, 321, 80\}, \{71, 130, 141, 28\},\]
\[\{331, 210, 281, 520\} .\]

\[\mathcal{R} = \{601, 31, 250, 29\}, \{380, 340, 3, 24\}, \{550, 150, 62, 450\},\]
\[\{621, 610, 420, 290\}, \{541, 350, 300, 260\}, \{601, 11, 140, 381\},\]
\[\{141, 110, 310, 630\}, \{71, 330, 81, 410\} .\]

\[\square\]

**Lemma 7.** For each \((u, w) \in \{(37, 6), (38, 7), (39, 6)\}\), an IHGBTD\((4,4^{(u,w)})\) exists.
Proof: As with Lemma 5, we apply Proposition 4.1. Take

\[ \Gamma = \mathbb{Z}_{u-w} \times \mathbb{Z}_2 \times \mathbb{Z}_2, \]
\[ \Gamma_0 = \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \]
\[ \Delta = \{(0,0,0),(1,0,0),\ldots,(u-w-1,0,0)\}, \] and

\[ H = \bigcup_{i=0}^{w-1} H_i, \] where \( H_i = \{\infty_i, \infty_i + w, \infty_i + 2w, \infty_i + 3w\} \) for \( 0 \leq i \leq w - 1 \).

The desired intransitive starter and corresponding adder for \((u, w) \in \{(37, 6), (38, 7), (39, 6)\}\) are displayed below. Here we write the element \((a, b, c)\) of \(\Gamma\) as \(a_{bc}\) for succinctness. When \((u, w) = (37, 6)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(S)</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{00, 2500, 3000, 711}</td>
<td>3000</td>
<td>{2010, 1300, 2311, 2701}</td>
<td>2800</td>
</tr>
<tr>
<td>{2011, 1900, 900, 411}</td>
<td>1700</td>
<td>{2911, 2611, 2111, 201}</td>
<td>3000</td>
</tr>
<tr>
<td>{901, 2711, 410, 1611}</td>
<td>1100</td>
<td>{\infty, 2601, 2801, 500}</td>
<td>400</td>
</tr>
<tr>
<td>{\infty, 2400, 1111, 2301}</td>
<td>2400</td>
<td>{\infty, 2111, 510, 1800}</td>
<td>700</td>
</tr>
<tr>
<td>{\infty, 5801, 2501, 1511}</td>
<td>2500</td>
<td>{\infty, 601, 201, 710}</td>
<td>1400</td>
</tr>
<tr>
<td>{\infty, 801, 1211, 1910}</td>
<td>800</td>
<td>{\infty, 9, 3001, 2700, 811}</td>
<td>2700</td>
</tr>
<tr>
<td>{\infty, 411, 2211, 700}</td>
<td>2000</td>
<td>{\infty, 12, 2600, 601, 400}</td>
<td>1900</td>
</tr>
<tr>
<td>{\infty, 410, 1601, 2201}</td>
<td>1300</td>
<td>{\infty, 15, 401, 2900, 701}</td>
<td>1800</td>
</tr>
<tr>
<td>{\infty, 17, 1811, 1011, 1501}</td>
<td>100</td>
<td>{\infty, 18, 1710, 2310, 801}</td>
<td>2600</td>
</tr>
<tr>
<td>{\infty, 20, 310, 1801, 2400}</td>
<td>500</td>
<td>{\infty, 21, 3011, 2411, 1810}</td>
<td>900</td>
</tr>
<tr>
<td>{\infty, 23, 610, 1501, 2910}</td>
<td>600</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ C = \{3010, 1300, 711, 801\}, \quad \{701, 210, 2811, 1700\}, \quad \{611, 901, 1000, 130\}, \]
\[ \mathcal{R} = \{1400, 3000, 1310, 900\}, \quad \{910, 1610, 1500, 1100\}, \quad \{1000, 2510, 1710, 3010\}, \]
\[ \{2000, 501, 911, 1000\}, \quad \{2610, 1210, 1311, 1711\}, \quad \{1201, 1110, 100, 611\}. \]

When \((u, w) = (38, 7)\):

<table>
<thead>
<tr>
<th>(S)</th>
<th>(\Omega(S))</th>
<th>(S)</th>
<th>(\Omega(S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>{2800, 2900, 2211, 2700}</td>
<td>800</td>
<td>{2011, 2311, 1111, 511}</td>
<td>600</td>
</tr>
<tr>
<td>{\infty, 3001, 1300, 501}</td>
<td>300</td>
<td>{\infty, 1, 2801, 301, 2301}</td>
<td>2000</td>
</tr>
<tr>
<td>{\infty, 01, 411, 600}</td>
<td>1100</td>
<td>{\infty, 4, 2000, 900, 800}</td>
<td>2600</td>
</tr>
<tr>
<td>{\infty, 3000, 2901, 2101}</td>
<td>000</td>
<td>{\infty, 7, 2701, 1600, 1410}</td>
<td>1900</td>
</tr>
<tr>
<td>{\infty, 1100, 610, 1010}</td>
<td>2800</td>
<td>{\infty, 10, 1301, 2410, 120}</td>
<td>1400</td>
</tr>
<tr>
<td>{\infty, 12, 2110, 1230}</td>
<td>1600</td>
<td>{\infty, 13, 1100, 1500, 1411}</td>
<td>1800</td>
</tr>
<tr>
<td>{\infty, 15, 300, 2500, 1700}</td>
<td>2700</td>
<td>{\infty, 16, 1200, 2110, 2210}</td>
<td>2900</td>
</tr>
<tr>
<td>{\infty, 18, 000, 1911, 2010}</td>
<td>2300</td>
<td>{\infty, 19, 2400, 211, 401}</td>
<td>100</td>
</tr>
<tr>
<td>{\infty, 21, 2510, 701, 001}</td>
<td>1500</td>
<td>{\infty, 22, 1701, 2711, 700}</td>
<td>300</td>
</tr>
<tr>
<td>{\infty, 24, 010, 401, 1101}</td>
<td>500</td>
<td>{\infty, 25, 911, 1901, 2110}</td>
<td>400</td>
</tr>
<tr>
<td>{\infty, 27, 1401, 2501, 3010}</td>
<td>2400</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ C = \{1400, 2911, 2501, 3010\}, \quad \{2010, 911, 701, 500\}, \quad \{401, 2500, 2811, 1210\}, \]
\[ \{1300, 2410, 101, 2211\}, \quad \{710, 601, 201, 1000\}, \quad \{2401, 601, 401, 1611\}, \]
\[ \{811, 910, 1901, 1510\}, \quad \{2010, 711, 1310, 1711\}, \quad \{910, 1511, 601, 1000\}, \]
\[ \{2010, 1410, 2100, 2801\}, \quad \{2201, 1801, 1011, 1501\}, \quad \{311, 201, 1601, 2911\}, \]
\[ \{310, 2811, 1110, 611\}. \]
When \((u, w) = (39, 6)\):

\[
\begin{array}{cccccccc}
S & \Omega(S) & S & \Omega(S) & S & \Omega(S) & S \\
\{28_{10}, 29_{10}, 26_{10}, 20_{10}\} & 300 & \{24_{01}, 10_{11}, 9_{01}, 17_{00}\} & 1300 & \{30_{00}, 29_{00}, 6_{00}, 21_{01}\} & 000 \\
\{11_{01}, 30_{01}, 10_{00}, 7_{11}\} & 1000 & \{9_{11}, 26_{00}, 21_{00}, 20_{01}\} & 1100 & \{30_{00}, 32_{00}, 0_{01}, 8_{00}\} & 800 \\
\{22_{01}, 8_{10}, 18_{00}, 27_{01}\} & 900 & \{21_{10}, 30_{11}, 24_{01}, 4_{11}\} & 500 & \{32_{01}, 27_{10}, 18_{01}, 25_{00}\} & 2500 \\
\{\infty, 28_{01}, 16_{00}, 12_{11}\} & 3200 & \{\infty, 1, 01, 18_{10}, 16_{01}\} & 2000 & \{\infty, 2, 9_{10}, 6_{11}, 4_{01}\} & 300 \\
\{3_{3}, 15_{00}, 32_{10}, 6_{10}\} & 1900 & \{4_{01}, 32_{11}, 30_{10}, 1_{10}\} & 2700 & \{5_{0}, 29_{01}, 8_{11}, 34_{00}\} & 1600 \\
\{6_{01}, 26_{01}, 14_{11}, 23_{00}\} & 1800 & \{\infty, 7, 28_{00}, 13_{01}, 24_{10}\} & 1500 & \{\infty, 8, 24_{11}, 34_{01}, 13_{10}\} & 3400 \\
\{9_{0}, 27_{00}, 18_{11}, 12_{10}\} & 2800 & \{\infty, 10, 25_{11}, 13_{11}, 19_{11}\} & 2200 & \{\infty, 11, 5_{10}, 4_{00}, 0_{00}\} & 3000 \\
\{12_{0}, 7_{00}, 13_{00}, 19_{01}\} & 600 & \{13_{11}, 21_{10}, 16_{11}, 25_{01}\} & 2600 & \{14_{11}, 17_{01}, 7_{11}, 11_{0}\} & 700 \\
\{15_{11}, 15_{01}, 19_{10}, 2_{11}\} & 1700 & \{16_{1}, 22_{00}, 12_{00}, 10_{0}\} & 400 & \{17_{1}, 0_{10}, 14_{01}, 5_{00}\} & 100 \\
\{18_{1}, 15_{11}, 2_{01}, 14_{00}\} & 1200 & \{19_{1}, 4_{10}, 3_{01}, 23_{11}\} & 200 & \{20_{11}, 3_{10}, 16_{10}, 17_{10}\} & 1400 \\
\{21_{1}, 3_{11}, 19_{00}, 25_{10}\} & 2900 & \{22_{1}, 5_{11}, 11_{00}, 22_{11}\} & 2400 & \{23_{11}, 10_{20}, 22_{10}, 23_{01}\} & 2100 \\
\end{array}
\]

\(\mathcal{C} = \{1_{11}, 15_{10}, 23_{00}, 13_{01}\}, \{22_{11}, 4_{01}, 20_{00}, 27_{10}\}, \{12_{10}, 16_{11}, 8_{00}, 4_{01}\}, \{23_{11}, 12_{01}, 1_{00}, 9_{10}\}, \{20_{00}, 3_{00}, 23_{10}, 28_{11}\}\)

\(\mathcal{R} = \{20_{11}, 6_{01}, 28_{11}, 5_{01}\}, \{29_{11}, 12_{01}, 11_{11}, 31_{11}\}, \{31_{10}, 10_{01}, 15_{10}, 7_{10}\}, \{9_{00}, 27_{11}, 14_{10}, 20_{00}\}, \{23_{10}, 0_{11}, 20_{10}, 8_{01}\}, \{26_{11}, 1_{11}, 21_{11}, 17_{11}\}\)

\(\blacksquare\)

**Proof of Theorem 2:** We first construct a GBTD\((4, m)\) for any \(m \in N\), where \(N = \{28, 32, 33, 34, 37, 38, 39, 44\}\).

For each \(w \in \{5, 6, 7, 8\}\), an HGBTD\((4, 4^w)\) is given by Yin et al. [12]. For each \(m \in N\), apply Theorem 3.2, with IHGBTDs from Lemma 5, Lemma 6 and Lemma 7 and corresponding HGBTD\((4, 4^w)\)’s where \(w \in \{5, 6, 7, 8\}\) as ingredients, to produce the desired HGBTD\((4, 4^m)\). Hence, the desired GBTD\((4, m)\) follows from Proposition 3.1.

Combining Proposition 1.1, Proposition 2.1 and Proposition 2.2, we complete the proof.

\(\blacksquare\)

**Acknowledgement** We are grateful to the anonymous reviewers for their helpful comments.

**References**


guide, version 1.0, Tech. Report T48, Communications Laboratory, Helsinki University


nament designs, *Ars Combinatoria* 3 (1977), 303-318.

[9] N.V. Semakov, V.A. Zinov’ev, Equidistant q-ary codes with maximal distance and re-
solvable balanced incomplete block designs, *Problem Peredači Informacii*, 4 (1968),
3-10.

[10] J. Yan and J. Yin, Constructions of optimal GDRP($n, \lambda; v$) of type $\lambda^\mu v^{m-1}$, *Discrete

Codes Cryptogr.* 50 (2009), 61-76.