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Mean Square Containment Control of Multi-agent Systems with Transmission Noises

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Abstract The containment control problem for multi-agent systems with transmission noises is considered. Due to the existence of the noises, a decaying gain function is introduced to attenuate the noises. Both dynamically switching topologies and randomly switching topologies are considered. Sufficient conditions on the communication graph and control gain function are derived to guarantee the mean square containment. Some numerical examples are provided to verify the results.

Key words Containment control, transmission noise, multi-agent system, random graph, mean square

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In the last decade, consensus control was widely studied by many researchers, see, e.g., [1–4]. Based on the existence of leader, consensus control can be classified into two categories, leaderless consensus and leader-following consensus^[5]. In the leader-following case, there is an autonomous system which is considered as the leader agent. All the rest agents are connected to the leader directly or indirectly and their states are required to converge to that of the leader by locally exchanging information with neighbors based on some consensus protocols. See [3, 6–8] and the references therein.

The containment control problem where a collection of agents are to be driven to a certain compact set^[9–10], which is different from the single leader case, has recently attracted much interest. The containment control may be considered as a leader-following problem with multiple leaders. The leaders may be connected to form a certain geometric formation. All the follower agents are to be controlled to track the convex hull of the leaders^[11]. One motivation of this problem is to ensure that a collection of autonomous robots do not venture into hazardous areas. As such, some virtual/actual leaders are introduced to guide the followers to move around safe areas. Other motivations include distributed sensor localization^[12]. In [13], necessary and sufficient conditions were provided for the containment control based on both continuous and sampled-data control protocols. The followers can converge to the convex hull formed by the leaders for both stationary and dynamic leaders cases. For the dynamically switching topology case, the containment control problem were studied in [14] under the assumption that the communication graphs for the leaders and the followers are both undirected. For the case of switching directed graphs, necessary and sufficient conditions on the graph were provided in [15] to guarantee set input-to-state stability and set integral input-to-state stability based on a nonlinear protocol. In [16], the communication topology was described by a continuous-time irreducible Markov chain. In the case where the leaders are stationary, a necessary and sufficient condition is provided to guarantee that all the followers almost surely asymptot-

ically achieve containment tracking. For the case of dynamically moving leaders, the containment tracking error is worked out. The containment control problem for multi-agent systems with general linear dynamics was considered in [17–18]. In [17], the leaders were assumed to be driven by bounded unknown inputs. When the linear model of each agent further contains uncertainties, the containment control problem was studied in [19].

In this paper, we consider the containment control problem in the presence of transmission noises and under both dynamically switching topologies and randomly switching topologies. In [20], multi-agent systems with a fixed communication topology were considered. However, the square summable condition on the gain function was needed. The extension of the results to dynamically switching topologies and randomly switching topologies cases is not trivial. The techniques in [20] cannot be used to address the problem in this paper. Motivated by [5], the conditions on the gain function can be weakened. Moreover, for the dynamically switching topologies case, the graph does not need to contain a united spanning tree at each time instant if we slightly change the conditions on the gain function.

Before closing this section, some remarks on notation are given as follows. Given an arbitrary matrix T , we use $T^{i,j}$ to denote its (i, j) -th element. $\lambda_i[T]$ is the i -th eigenvalue of matrix T . If all the eigenvalues of T are real numbers, we denote $\lambda_{\min}[T] = \min_i \lambda_i[T]$ and $\lambda_{\max}[T] = \max_i \lambda_i[T]$. $\mathbf{1}_n$ stands for a column vector with dimension n and every element being 1. $\mathbf{0}$ is a zero vector or matrix with appropriate dimension. e_i is the column vector with the i -th element equal to 1 and all the rest elements being zero. $\|\cdot\|_2$ denotes the Euclidean norm. \mathbf{R} (or \mathbf{R}^+) denotes the set of real numbers (or positive real numbers) and $\mathbf{R}^+ = \mathbf{R}^+ \cup \{0\}$. \mathbf{R}^n and $\mathbf{R}^{n \times m}$ denote the sets of n dimensional real vectors and $n \times m$ real matrices, respectively. \mathbf{Z}^+ denotes the set of all positive integers and $\bar{\mathbf{Z}}^+ = \mathbf{Z}^+ \cup \{0\}$. $\Pi_{i,j}^M$ is the transition matrix of $M(k)$ defined as

$$\Pi_{i,j}^M = \begin{cases} M(i)M(i-1)\cdots M(j), & i \geq j \\ I, & i < j \end{cases}$$

We denote $E(X)$ the mathematical expectation of random variable X . A row vector is called stochastic vector if it is nonnegative and the sum is equal to 1. A matrix is called row stochastic matrix if each row of it is a stochastic vector.

Let $Z = \{z_1, \dots, z_k\}$ be a finite set in \mathbf{R} . The convex hull of Z , denoted by $co(Z)$, is defined as $co(Z) = \{\sum_{i=1}^k \lambda_i z_i \mid \sum_{i=1}^k \lambda_i = 1, \lambda_i \geq 0\}$. $dist(x, S)$ means the distance from $x \in \mathbf{R}^n$ to the set $S \subseteq \mathbf{R}^n$ in the sense of

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Euclidean norm, i.e.,

$$\text{dist}(x, \mathcal{S}) = \inf_{y \in \mathcal{S}} \|x - y\|_2$$

1 Preliminaries in graph theory

In this paper, the multi-agent systems are modelled based on graph theory. Some basics of graph theory will be first recalled in this section.

A directed graph, denoted by \mathcal{G} , with vertex set $\mathcal{V}(\mathcal{G})$ and edge set $\mathcal{E}(\mathcal{G}) \subseteq \mathcal{V} \times \mathcal{V}$ is often used to model the communications among agents. For simplicity, we define $\mathcal{V} = \{1, 2, \dots, n\}$, where n is the number of vertices. For any pair $(i, j) \in \mathcal{E}(\mathcal{G})$, i is called parent vertex whose information is transmitted to agent j . The set of neighbors of vertex i is denoted by $\mathcal{N}_i = \{j \mid j \in \mathcal{V}, (j, i) \in \mathcal{E}(\mathcal{G})\}$. We denote by $a_{i,j} \geq 0$ the weighting on the edge (j, i) . When $j \notin \mathcal{N}_i$, which means there is no information flow from vertex j to vertex i , it has $a_{i,j} = 0$. $a_{i,j} \neq 0$ indicates that there exists an edge from vertex j to vertex i , i.e. $j \in \mathcal{N}_i$. There is a path from vertex i to vertex j if there exists a sequence $l_1, \dots, l_p \in \mathcal{V}$ satisfying $(l_1, l_2), \dots, (l_{p-1}, l_p) \in \mathcal{E}(\mathcal{G})$ where $l_1 = i$ and $l_p = j$. Given a graph \mathcal{G} , it contains a spanning tree if there exists one vertex i such that for any other vertex j , there is a path from i to j . Moreover, if \mathcal{N}_i is empty, vertex i is called the leader and the corresponding graph is a leader-following graph. A graph is balanced if the in-degree $\text{deg}_{\text{in}}(i) = \sum_{j \in \mathcal{V} \setminus \{i\}} a_{i,j}$ and the out-degree $\text{deg}_{\text{out}}(i) = \sum_{j \in \mathcal{V} \setminus \{i\}} a_{j,i}$ are equal for all $i \in \mathcal{V}$. Given a graph \mathcal{G} with $\mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G})$ and a vertex subset $\mathcal{V}_s \subset \mathcal{V}(\mathcal{G})$, a subgraph induced by \mathcal{V}_s is a graph with vertex set \mathcal{V}_s and edge set $\mathcal{E}(\mathcal{G}) \cap (\mathcal{V}_s \times \mathcal{V}_s)$.

When we consider time-varying graphs, the notion of union graph needs to be introduced. Given a sequence of graphs, the union graph is a graph whose vertex and edge sets are respectively the union of the vertex and edge sets of these graphs. For example, we denote $\mathcal{G}(k, h)$ the union graph of $\{\mathcal{G}(i), k \leq i < k + h\}$, then we have

$$\mathcal{V}(\mathcal{G}(k, h)) = \bigcup_{i=k}^{k+h-1} \mathcal{V}(\mathcal{G}(i)), \quad \mathcal{E}(\mathcal{G}(k, h)) = \bigcup_{i=k}^{k+h-1} \mathcal{E}(\mathcal{G}(i))$$

We divide the vertex set $\mathcal{V}(\mathcal{G})$ of a graph \mathcal{G} into two groups $\mathcal{V}_l, \mathcal{V}_f$ satisfying $\mathcal{V}_l \cup \mathcal{V}_f = \mathcal{V}$, $\mathcal{V}_l \cap \mathcal{V}_f = \emptyset$. If the divided vertex sets further satisfy $\forall i \in \mathcal{V}_l, \mathcal{N}_i \cap \mathcal{V}_f = \emptyset$ and $\forall j \in \mathcal{V}_f, \exists i \in \mathcal{V}_l$ such that there is a path from i to j , then we say the graph contains a united spanning tree, where \mathcal{V}_l and \mathcal{V}_f are called leader set and follower set, respectively. $\{\mathcal{G}(i), k \leq i < k + h\}$ jointly contains a united spanning tree if the union graph $\mathcal{G}(k, h)$ contains a united spanning tree. $\{\mathcal{G}(i), i \in \mathbf{Z}^+\}$ uniformly jointly contains a united spanning tree if there exists $T \in \mathbf{Z}^+$ such that $\forall k \in \mathbf{Z}^+, \{\mathcal{G}(i), k \leq i < k + T\}$ jointly contains a united spanning tree.

2 Problem statement

In this section, we shall formulate the containment tracking problem to be studied for multi-agent systems with multiple leaders. The communication graph of all the agents is described by \mathcal{G} . The system to be considered consists of n autonomous agents, labeled from 1 to n , i.e. $\mathcal{V}(\mathcal{G}) = \{1, \dots, n\}$. The agent i is assumed to have the following dynamics

$$x_i(k+1) = x_i(k) + u_i(k), \quad i \in \mathcal{V} \quad (1)$$

where $x_i \in \mathbf{R}$ and $u_i \in \mathbf{R}$ are the state and input of agent i , respectively. It should be noted that the results to be

developed later can be extended to the vector value state case by introducing Kronecker product which, however, is omitted here for simplicity. Without loss of generality, it is assumed that all the follower agents are labeled by $1, \dots, n_f$ which form the follower label set \mathcal{V}_f defined in the last section. All the leaders are assumed to be stationary, i.e. $u_i(k) \equiv 0, i \in \mathcal{V}_l$, where $\mathcal{V}_l = \mathcal{V} \setminus \mathcal{V}_f$. In networked systems, additive noises often exist in transmission. Hence, we denote the received state value of j by i as

$$y_{i,j}(k) = x_j(k) + w_{i,j}(k), \quad j \in \mathcal{N}_i, i \in \mathcal{V} \quad (2)$$

The noises $w_{i,j}(k)$ represents the unreliable information exchanges in the network. We have the following assumption:

Assumption 1. The noises $\{w_{i,j}(k), k \in \mathbf{Z}^+, i \in \mathcal{V}, j \in \mathcal{N}_i\}$ are white noises with zero mean and uniformly bounded covariances.

Our objective is to design $u_i(k)$, which is a function of $\{x_i(k), y_{i,j}(k): j \in \mathcal{N}_i\}, i \in \mathcal{V}_f$ such that the states of all the followers asymptotically converge into the convex hull formed by the states of the leaders in the stochastic sense. The following definition characterizes the asymptotic behavior of the followers^[2, 21]:

Definition 1 (Mean square containment tracking). The agents are said to reach containment tracking if $\forall i \in \mathcal{V}_f$,

$$\lim_{k \rightarrow \infty} \text{Edist}^2(x_i(k), \text{co}\{\mathcal{V}_l\}) = 0$$

In the definition and hereafter, we slightly abuse the notation by writing $\text{co}\{x_j, j \in \mathcal{V}_l\}$ as $\text{co}\{\mathcal{V}_l\}$.

3 Dynamically switching topologies case

In this section, we consider the multi-agent system (1) and (2) with dynamically switching topologies. The control protocol for the i -th follower is proposed as

$$u_i(k) = c(k) \sum_{j \in \mathcal{V}} a_{i,j}(k) (y_{i,j}(k) - x_i(k)), \quad i \in \mathcal{V}_f \quad (3)$$

where $a_{i,j} \in \mathcal{C} \cup \{0\}$ and \mathcal{C} is a compact set on \mathbf{R}^+ , $c(k) \in \bar{\mathbf{R}}^+$ is a control gain function^[2, 5, 22]. The following assumption needs to be introduced:

Assumption 2. The subgraph of \mathcal{G} induced by \mathcal{V}_f is balanced at each time instant.

In the following theorem, we shall provide sufficient conditions for the switching communication graphs and control gain $c(k)$ to guarantee the containment tracking objective.

Theorem 1. Consider the multi-agent system (1) and (2) with interaction protocol (3). Under Assumptions 1 and 2, the mean square containment tracking is achieved if the communication graph contains a united spanning tree at each time instant and the control gain function satisfies the persistence condition:

$$\sum_{k=0}^{\infty} c(k) = +\infty, \quad \lim_{k \rightarrow \infty} c(k) = 0 \quad (4)$$

Proof. Define $X_f = [x_1 \ x_2 \ \dots \ x_{n_f}]^T$ and $X_l = [x_{n_f+1} \ \dots \ x_n]^T$. Then by substituting (3) into (1) and (2), we can write the closed loop system into the following compact form

$$\begin{bmatrix} X_f(k+1) \\ X_l(k+1) \end{bmatrix} = (I - c(k)L(k)) \begin{bmatrix} X_f(k) \\ X_l(k) \end{bmatrix} + c(k) \begin{bmatrix} W_f(k) \\ W_l(k) \end{bmatrix} \quad (5)$$

where $W_f(k) = [w_1 \cdots w_{n_f}]^T$, $w_i = \sum_{j=1}^n a_{i,j}(k)w_{i,j}(k)$, $W_l(k) = 0$. According to Assumption 1, $\{W_f(k), k \in \mathbf{Z}^+\}$ are white noises with uniformly bounded covariances denoted by Q . $L(k)$ is called Laplacian matrix^[23] which can be written in the following form

$$L(k) = \left[\begin{array}{c|c} A(k) & B(k) \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right] \in \mathbf{R}^{n \times n} \quad (6)$$

where $A(k) \in \mathbf{R}^{n_f \times n_f}$, $B(k) \in \mathbf{R}^{n_f \times n_l}$ and

$$A^{i,j}(k) = \begin{cases} \sum_{s=1}^n a_{i,s}(k), & i = j \\ -a_{i,j}(k), & i \neq j \end{cases}, \quad B^{i,j}(k) = -a_{i,n_f+j}(k)$$

It can be verified that

$$A(k)\mathbf{1}_{n_f} + B(k)\mathbf{1}_{n_l} = \mathbf{0} \quad (7)$$

From (5) we know that $X_l(k) \equiv X_l(0)$, $\forall k \in \mathbf{Z}^+$ and

$$X_f(k+1) = (I - c(k)A(k))X_f(k) - c(k)B(k)X_l + c(k)W_f(k) \quad (8)$$

According to (8), we have

$$\begin{aligned} X_f(k+1) &= \Pi_{k,0}^{I-cA} X_f(0) - \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j)B(j)X_l + \\ &\quad \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j)W_f(j) \end{aligned} \quad (9)$$

In the following, we need to prove that the first and third terms on the right-hand side of (9) converge to zero in the mean square sense and $-\sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j)B(j)$ converges to a row stochastic matrix.

Define

$$P = \left[\begin{array}{c|c} I & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1}_{n_l}^T \end{array} \right] \in \mathbf{R}^{(n_f+1) \times n} \quad (10)$$

Then we have

$$L_p(k) = PL(k)P^T = \left[\begin{array}{c|c} A(k) & B(k)\mathbf{1}_{n_l} \\ \hline \mathbf{0} & 0 \end{array} \right] \in \mathbf{R}^{(n_f+1) \times (n_f+1)} \quad (11)$$

Note that $L_p(k)$ is a valid Laplacian matrix corresponding to a leader-following graph \mathcal{G}_p ^[24]. According to the definition of united spanning tree in Section 1, it can be checked that \mathcal{G}_p contains a spanning tree if and only if \mathcal{G} contains a united spanning tree. Therefore, by Lemma 1 in [5], all the eigenvalues of $L_p(k)$ are with positive real components except one which is equal to zero. In view of the form of $L_p(k)$ in (11), we have that all the eigenvalues of $A(k)$ are on the open right-hand plane. Denote

$$\hat{A}(k) = A(k) + A^T(k) \quad (12)$$

Since the subgraph induced by the follower set \mathcal{V}_f is always balanced, it can be verified that

$$\hat{L}(k) = \left[\begin{array}{c|c} \hat{A}(k) & B(k)\mathbf{1}_{n_l} \\ \hline \mathbf{0} & 0 \end{array} \right]$$

is also a Laplacian matrix corresponding to a leader-following graph $\hat{\mathcal{G}}$. In comparing \mathcal{G}_p with $\hat{\mathcal{G}}$, the latter has no less edges than the former one, which yields that all the eigenvalues of $\hat{A}(k)$ are on the open right-hand plane. Since $\hat{A}(k)$ is a symmetric matrix, we have $\lambda_i[\hat{A}(k)] \in \mathbf{R}^+$,

$i = 1, \dots, n_f$. Thanks to the compactness of \mathcal{C} , it follows that $\lambda_i[\hat{A}(k)]$ is uniformly lower bounded away from zero. Denote

$$0 < \inf_{k \in \mathbf{Z}^+} \lambda_{\min}[\hat{A}(k)] = \lambda \leq \bar{\lambda} = \sup_{k \in \mathbf{Z}^+} \lambda_{\max}[\hat{A}(k)] < \infty$$

Due to that $\lim_{k \rightarrow \infty} c(k) = 0$, there must exist a time instant k_1 such that $\forall k \geq k_1$, $c(k) < 1/\bar{\lambda}$. Then we have $\forall k \geq k_1$,

$$\|I - c(k)\hat{A}(k)\|_2 \leq 1 - c(k)\lambda < 1$$

On the other hand, due to the compactness of \mathcal{C} and $\lim_{k \rightarrow \infty} c(k) = 0$, there exists a time instant k_2 such that $\sup_{k \geq k_2} \{c(k)\|A(k)\|_2^2\} < \lambda$. Then for any $k \geq k_3 = \max\{k_1, k_2\}$,

$$\begin{aligned} \|I - c(k)A(k)\|_2^2 &= \\ \lambda_{\max} \left[(I - c(k)A(k))^T (I - c(k)A(k)) \right] &= \\ \lambda_{\max} \left[I - c(k)\hat{A}(k) \right] + c^2(k)\|A(k)\|_2^2 &\leq \\ 1 - (c(k)\lambda - c^2(k)\|A(k)\|_2^2) &\leq \\ 1 - \varepsilon c(k) &\leq 1 \end{aligned} \quad (13)$$

where $\varepsilon = \lambda - \sup_{k \geq k_2} \{c(k)\|A(k)\|_2^2\} > 0$.

Next, we introduce the following system

$$p(k+1) = (I - c(k)A(k))p(k), \quad k \geq k_3$$

with $p(k_3)$ a finite constant vector. Then we have

$$p(k+1) = \Pi_{k,k_3}^{I-cA} p(k_3)$$

and

$$\begin{aligned} \|p(k+1)\|_2^2 &= \|\Pi_{k,k_3}^{I-cA} p(k_3)\|_2^2 \leq \\ \|p(k_3)\|_2^2 \|\Pi_{k,k_3}^{I-cA}\|_2^2 &\leq \\ \|p(k_3)\|_2^2 \prod_{i=k_3}^k \|I - c(i)A(i)\|_2^2 &\leq \\ \|p(k_3)\|_2^2 \prod_{i=k_3}^k (1 - \varepsilon c(i)) &\leq \\ \|p(k_3)\|_2^2 e^{-\sum_{i=k_3}^k \varepsilon c(i)} \end{aligned}$$

According to the definition of Euclidean norm, there is

$$\begin{aligned} \|\Pi_{k,k_3}^{I-cA}\|_2 &= \sup_{\|p(k_3)\|_2 \neq 0} \frac{\|\Pi_{k,k_3}^{I-cA} p(k_3)\|_2}{\|p(k_3)\|_2} = \\ \sup_{\|p(k_3)\|_2 \neq 0} \frac{\|p(k+1)\|_2}{\|p(k_3)\|_2} &\leq \\ \sqrt{e^{-\sum_{i=k_3}^k \varepsilon c(i)}} \end{aligned}$$

which together with $\sum_{k=0}^{\infty} c(k) = +\infty$ yields that $\lim_{k \rightarrow \infty} \|\Pi_{k,k_3}^{I-cA}\|_2 = 0$. Since k_3 is finite and $\|A(k)\|_2$ is uniformly bounded for all $k \geq 0$, it follows that

$$\lim_{k \rightarrow \infty} \|\Pi_{k,0}^{I-cA}\|_2 \leq \|\Pi_{k_3,0}^{I-cA}\|_2 \lim_{k \rightarrow \infty} \|\Pi_{k,k_3}^{I-cA}\|_2 = 0 \quad (14)$$

which implies that $\lim_{k \rightarrow \infty} \Pi_{k,0}^{I-cA} = 0$.

By considering (7), we have

$$\begin{aligned} -\sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j) \mathbf{1}_{n_l} &= \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) A(j) \mathbf{1}_{n_f} = \\ &= \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} \mathbf{1}_{n_f} - \sum_{j=0}^k \Pi_{k,j}^{I-cA} \mathbf{1}_{n_f} = \\ &= (I - \Pi_{k,0}^{I-cA}) \mathbf{1}_{n_f} \end{aligned}$$

which together with (14) yields that each row sum of $-\sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j) \mathbf{1}_{n_l}$ converges to 1. Since $I - c(i)A(i)$ and $-c(i)B(i)$, $i \in \bar{\mathbf{Z}}^+$ are non-negative matrices, we can see that $-\sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j)$ converges to a row stochastic matrix as $k \rightarrow \infty$.

Denote

$$S(k) = \sum_{j=0}^{k-1} \left(\Pi_{k-1,j+1}^{I-cA} \right)^T \Pi_{k-1,j+1}^{I-cA} c^2(j)$$

It follows that

$$S(k+1) = (I - c(k)A^T(k))S(k)(I - c(k)A(k)) + c^2(k)$$

and $\forall k \geq k_3$

$$\begin{aligned} \|S(k+1)\|_2 &= \|(I - c(k)A^T(k))S(k)(I - c(k)A(k)) + \\ &+ c^2(k)\|_2 \leq \\ &\|(I - c(k)A(k))\|_2^2 \|S(k)\|_2 + c^2(k) \leq \\ &(1 - \varepsilon c(k)) \|S(k)\|_2 + c^2(k) \end{aligned} \quad (15)$$

According to Lemma 6 in [5], (15) implies that

$$\lim_{k \rightarrow \infty} \|S(k)\|_2 \leq \lim_{k \rightarrow \infty} \frac{c^2(k)}{\varepsilon c(k)} = 0$$

In light of (9), we have

$$\begin{aligned} \lim_{k \rightarrow \infty} \mathbb{E} \left\| X_f(k+1) + \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j) X_l \right\|_2^2 &= \\ \lim_{k \rightarrow \infty} \mathbb{E} \left\| \Pi_{k,0}^{I-cA} X_f(0) + \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) W_f(j) \right\|_2^2 &\leq \\ 2 \lim_{k \rightarrow \infty} \left\| \Pi_{k,0}^{I-cA} \right\|_2^2 \|X_f(0)\|_2^2 + 2 \lim_{k \rightarrow \infty} \|S(k)\|_2 \operatorname{tr}(Q) &= 0 \end{aligned} \quad (16)$$

Then $\forall i \in \mathcal{V}_f$, it follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} \operatorname{Edist}^2(x_i(k), \operatorname{co}\{\mathcal{V}_i\}) &\leq \\ 2 \lim_{k \rightarrow \infty} \operatorname{Edist}^2 \left(x_i(k), -e_i \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j) X_l \right) &+ \\ 2 \lim_{k \rightarrow \infty} \operatorname{Edist}^2 \left(-e_i \sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j) X_l, \operatorname{co}\{\mathcal{V}_i\} \right) &= 0 \end{aligned}$$

The first term in the inequality is zero due to (5). The second term is equal to zero since the matrix

$-\sum_{j=0}^k \Pi_{k,j+1}^{I-cA} c(j) B(j)$ converges to a row stochastic matrix. \square

Remark 1. Theorem 1 shows that the containment tracking can be achieved asymptotically as long as the communication graph contains a united spanning tree. This result coincides with that of the fixed topology case^[20]. However, in the dynamically changing topologies case in this paper, the condition for the feedback gain $c(k)$ is weaker than the one in [20] since $\lim_{k \rightarrow \infty} c(k) = 0$ may not guarantee that $\sum_{k=0}^{\infty} c^2(k) < +\infty$.

In the leader-following problem, the communication graph may not contain a spanning tree at every time instant. However, the consensus can still be guaranteed if the graph uniformly jointly contains a spanning tree. In the following, we extend the result of the leader-following problem in [5] to containment tracking. It will be shown that the conditions are similar to those in [5]. Before providing the result, the following condition is introduced:

$$0 < \inf_{k \in \mathbf{Z}^+} \frac{c(k)}{c(k+1)} \leq \sup_{k \in \mathbf{Z}^+} \frac{c(k)}{c(k+1)} < +\infty \quad (17)$$

Then we have the following theorem.

Theorem 2. Consider the multi-agent system (1) and (2) with interaction protocol (3). Under Assumptions 1 and 2, the mean square containment tracking is achieved if the communication graph uniformly jointly contains a united spanning tree and the control gain function $c(k)$ satisfies the persistence condition (4) as well as (17).

Proof. Following the same line in the proof of Theorem 1, we can write the closed-loop system in the form of (5) with $L(k)$ defined in (6). Since the graph may not contain a united spanning tree, the eigenvalues of matrix $A(k)$ in $L(k)$ may not be on the open right-hand plane, which will cause $\Pi_{k,0}^{I-cA}$ not to converge. In the following, we shall prove that the jointly united spanning tree can still guarantee the convergence.

Given any positive integer $h \in \mathbf{Z}^+$, according to (8) we have

$$\begin{aligned} X_f((k+1)h) &= \Pi_{(k+1)h-1, kh}^{I-cA} X_f(kh) + \\ &+ \sum_{j=kh}^{(k+1)h-1} \Pi_{(k+1)h-1, j+1}^{I-cA} [-c(j)B(j)X_l + c(j)W_f(j)] \end{aligned} \quad (18)$$

Since $c(k)$ satisfies (17), there exist constants $C_{0,h}, C_{1,h} \in \mathbf{R}^+$ such that $C_{0,h}c(kh) \leq c(kh+i) \leq C_{1,h}c((k+1)h)$, $i = 0, \dots, h-1$. Denote $\bar{C} = \sup_{k \in \mathbf{Z}^+} c(k)$. In view of (4) one has that $0 < \bar{C} < +\infty$. Then we have

$$\begin{aligned} \left\| \Pi_{(k+1)h-1, kh}^{I-cA} \right\|_2^2 &= \left\| \Pi_{(k+1)h-1, kh}^{I-cA} (\Pi_{(k+1)h-1, kh}^{I-cA})^T \right\|_2^2 \leq \\ \left\| I - \sum_{i=kh}^{(k+1)h-1} c(i) \hat{A}(i) \right\|_2^2 &+ \|M(k, h)\|_2^2 \end{aligned} \quad (19)$$

where \hat{A} is as defined in (12),

$$\begin{aligned} M(k, h) &= \Pi_{(k+1)h-1, kh}^{I-cA} (\Pi_{(k+1)h-1, kh}^{I-cA})^T - I + \\ &+ \sum_{i=kh}^{(k+1)h-1} c(i) \hat{A}(i) \end{aligned}$$

By complex calculation, we can find that

$$\|M(k, h)\|_2 \leq m(h)c^2(kh) \quad (20)$$

where

$$m(h) = \frac{C_{1,h}^2}{C^2} \left[(1 - \bar{C} \sup_{k \in \mathbf{Z}^+} \|A(k)\|_2)^{2h} - 1 + 2h\bar{C} \sup_{k \in \mathbf{Z}^+} \|A(k)\|_2 \right] \in \mathbf{R}^+ \quad (21)$$

Since the communication graph uniformly jointly contains a united spanning tree, there must exist a constant $T \in \mathbf{Z}^+$ such that the graph corresponding to $\sum_{i=kT}^{(k+1)T-1} L(i)$ contains a united spanning tree. Similar to the proof of Theorem 1, we have that $\lambda_i[\sum_{i=kT}^{(k+1)T-1} c(i)\hat{A}(i)] \in \mathbf{R}^+$, $i \in \mathcal{V}_f$. Then we denote

$$\lambda_1 = \inf_{k \in \mathbf{Z}^+} \lambda_{\min} \left[\sum_{i=kT}^{(k+1)T-1} \hat{A}(i) \right] > 0$$

$$\bar{\lambda}_1 = \sup_{k \in \mathbf{Z}^+} \lambda_{\max} \left[\sum_{i=kT}^{(k+1)T-1} \hat{A}(i) \right] < \infty$$

Define a time instant k_4 such that $\forall k \geq k_4$, $c(k) \leq 1/C_{0,h}\lambda_1$. Then we have $\forall k \geq k_4$,

$$\left\| I - \sum_{i=kT}^{(k+1)T-1} c(i)\hat{A}(i) \right\|_2 \leq \left\| I - C_{0,h}c(kT) \sum_{i=kT}^{(k+1)T-1} \hat{A}(i) \right\|_2 \leq 1 - C_{0,h}c(kT)\lambda_1 \quad (22)$$

There exists a time instant k_5 such that $\forall k \geq k_5$, $\sup_{k \geq k_5} c(kT) < C_{0,h}\lambda_1/m(T)$. Then by considering (19) \sim (22), for any $k \geq k_6 = \max\{k_4, k_5\}$, we have

$$\left\| \Pi_{(k+1)T-1, kT}^{I-cA} \right\|_2^2 \leq 1 - \varepsilon_1 c(kT) < 1 \quad (23)$$

where $\varepsilon_1 = C_{0,h}\lambda_1 - m(T) \sup_{k \geq k_5} c(kT) > 0$. By defining $\phi(k, T) = \Pi_{(k+1)T-1, kT}^{I-cA}$, it can be proved that $\forall k \geq k_6$,

$$\left\| \Pi_{k, k_6}^{\phi(\cdot, T)} \right\|_2^2 \leq e^{-\varepsilon_1 \sum_{i=k_6}^k c(ih)}$$

By considering (4) and (17), we have

$$\sum_{k=0}^{\infty} c(kT) \geq \frac{1}{C_{1,h}T} \sum_{k=0}^{\infty} c(k) = +\infty$$

which implies that

$$\lim_{k \rightarrow \infty} \left\| \Pi_{k,0}^{\phi(\cdot, T)} \right\|_2^2 = \lim_{k \rightarrow \infty} \left\| \Pi_{k,0}^{I-cA} \right\|_2^2 = 0 \quad (24)$$

According to (18), we have

$$X_f((k+1)T) = \Pi_{k,0}^{\phi(\cdot, T)} X_f(0) + \sum_{i=0}^k \Pi_{k, i+1}^{\phi(\cdot, T)} \times$$

$$\sum_{j=ih}^{(i+1)h-1} \Pi_{(i+1)h-1, j+1}^{I-cA} [-c(j)B(j)X_l + c(j)W_f(j)]$$

Denote

$$F(k, T) = \sum_{i=0}^k \Pi_{k, i+1}^{\phi(\cdot, T)} \sum_{j=ih}^{(i+1)h-1} \Pi_{(i+1)h-1, j+1}^{I-cA} c(j)B(j)$$

Note that

$$-F(k, T)\mathbf{1}_{n_f} = \sum_{i=0}^k \Pi_{k, i+1}^{\phi(\cdot, T)} \sum_{j=iT}^{(i+1)T-1} \Pi_{(i+1)T-1, j+1}^{I-cA} c(j)A(j)\mathbf{1}_{n_f} = \sum_{i=0}^k \Pi_{k, i+1}^{\phi(\cdot, T)} \sum_{j=iT}^{(i+1)T-1} \left[\Pi_{(i+1)T-1, j+1}^{I-cA} - \Pi_{(i+1)T-1, j}^{I-cA} \right] \mathbf{1}_{n_f} = \sum_{i=0}^k \Pi_{k, i+1}^{\phi(\cdot, T)} (I - \phi(i, T)) \mathbf{1}_{n_f} = \sum_{i=0}^k \left[\Pi_{k, i+1}^{\phi(\cdot, T)} - \Pi_{k, i}^{\phi(\cdot, T)} \right] \mathbf{1}_{n_f} = (I - \Pi_{k,0}^{\phi(\cdot, T)}) \mathbf{1}_{n_f}$$

which together with (24) yields that $-F(k, T)$ converges to a row stochastic matrix as $k \rightarrow \infty$. Next, define

$$S_1(k) = \sum_{i=0}^k \left(\Pi_{k, i+1}^{\phi(\cdot, T)} \right)^T \Pi_{k, i+1}^{\phi(\cdot, T)} c^2(iT)$$

Similar to the proof of Theorem 1, we have

$$S_1(k+1) = \left(\Pi_{k, i+1}^{\phi(\cdot, T)} \right)^T S_1(k) \Pi_{k, i+1}^{\phi(\cdot, T)} + c^2(kT) \leq (1 - \varepsilon_1 c(kT)) S_1(k) + c^2(kT)$$

and

$$\lim_{k \rightarrow \infty} \|S_1(k)\|_2 \leq \lim_{k \rightarrow \infty} \frac{c^2(kT)}{\varepsilon_1 c(kT)} = 0$$

Then we arrive at

$$\lim_{k \rightarrow \infty} \mathbb{E} \|X_f((k+1)T) + F(k, T)X_l\|_2^2 =$$

$$\lim_{k \rightarrow \infty} \mathbb{E} \left\| \Pi_{k,0}^{\phi(\cdot, T)} X_f(0) + \right.$$

$$\left. \sum_{i=0}^k \Pi_{k, i+1}^{\phi(\cdot, T)} \sum_{j=iT}^{(i+1)T-1} \Pi_{(i+1)T-1, j+1}^{I-cA} c(j)W_f(j) \right\|_2^2 \leq$$

$$2 \lim_{k \rightarrow \infty} \left\| \Pi_{k, i+1}^{\phi(\cdot, T)} \right\|_2^2 \|X_f(0)\|_2^2 +$$

$$2T \lim_{k \rightarrow \infty} \|S_1(k)\|_2 \text{tr}(Q) = 0 \quad (25)$$

The last inequality is due to that $\{W_f(k), k \in \mathbf{Z}^+\}$ is a white noise process and $\|\Pi_{(i+1)T-1, j+1}^{I-cA}\|_2 < 1$. From (25),

we know that $\mathbb{E}\|X_f(kT)\|_2$ is uniformly bounded. Then according to (18) we have

$$\begin{aligned} & \mathbb{E}\|X_f(kT+i) - X_f(kT)\|_2^2 = \\ & \mathbb{E}\left\|\left(\Pi_{kT+i-1, kT}^{I-cA} - I\right)X_f(kT) - \right. \\ & \left. \sum_{j=kT}^{kT+i-1} \Pi_{kT+i-1, j+1}^{I-cA} c(j)B(j)X_l + \right. \\ & \left. \sum_{j=kT}^{kT+i-1} \Pi_{kT+i-1, j+1}^{I-cA} c(j)W_f(j)\right\|_2^2 \leq \\ & 2\left\|\left(\Pi_{kT+i-1, kT}^{I-cA} - I\right)X_f(kT)\right\|_2^2 + \\ & 2\left\|\sum_{j=kT}^{kT+i-1} \Pi_{kT+i-1, j+1}^{I-cA} c(j)B(j)X_l\right\|_2^2 + \\ & \mathbb{E}\left\|\sum_{j=kT}^{kT+i-1} \Pi_{kT+i-1, j+1}^{I-cA} c(j)W_f(j)\right\|_2^2 \leq \\ & 2\varepsilon_1 c(kT) \sup_{k \in \bar{\mathcal{Z}}^+} \|X_f(kT)\|_2^2 + \\ & 2(i-1)^2 C_{1,h}^2 c^2(kT) \sup_{k \in \bar{\mathcal{Z}}^+} \|B(k)\|_2^2 \|X(l)\|_2^2 + \\ & (i-1)C_{0,h} c^2(kT) \text{tr}(Q) \end{aligned}$$

which implies that $\lim_{k \rightarrow \infty} \mathbb{E}\|X_f(kT+i) - X_f(kT)\|_2^2 = 0$. Then $\forall i \in \mathcal{V}_f$, it follows that

$$\begin{aligned} & \lim_{k \rightarrow \infty} \text{Edist}^2(x_i(k), \text{co}\{\mathcal{V}_l\}) \leq \\ & 3 \lim_{k \rightarrow \infty} \text{Edist}^2\left(x_i(k), x_i\left(\left\lfloor \frac{k}{T} \right\rfloor T\right)\right) + \\ & 3 \lim_{k \rightarrow \infty} \text{Edist}^2\left(x_i\left(\left\lfloor \frac{k}{T} \right\rfloor T\right), -F\left(\left\lfloor \frac{k}{T} \right\rfloor, T\right)X_l\right) + \\ & 3 \lim_{k \rightarrow \infty} \text{Edist}^2\left(-F\left(\left\lfloor \frac{k}{T} \right\rfloor, T\right)X_l, \text{co}\{\mathcal{V}_l\}\right) = 0 \end{aligned}$$

□

4 Randomly switching topologies case

In some multi-agent systems, random packet dropouts cause the communication link randomly breaking and the topology becomes randomly switching among a set of topologies. In this section, we consider the containment control under randomly switching communication topologies. Protocol (3) is still applied with $a_{i,j}(k)$ being random variable. We have the following assumption:

Assumption 3. The weighting functions $a_{i,j}(k)$, $i, j \in \mathcal{V}$ are randomly distributed over $\mathcal{C} \cup \{0\}$. Moreover, the weighting functions are i.i.d. and independent of communication noises $w_{i,j}(k)$.

We introduce the mean Laplacian matrix $\bar{L} = \mathbb{E}(L(k))$ with $L(k)$ defined in (5). It is clear that \bar{L} is a valid Laplacian matrix. The graph $\bar{\mathcal{G}}$ corresponding to Laplacian matrix \bar{L} is called mean graph. Then we have the following result.

Theorem 3. Consider the multi-agent system (1) and (2) with interaction protocol (3). Under Assumptions 1 and 3, the mean square containment tracking is achieved if the mean graph $\bar{\mathcal{G}}$ contains a united spanning tree and the control gain function satisfies the persistence condition (4).

Proof. We write the multi-agent system in the compact form (5) with Laplacian matrix $L(k)$ a random matrix. According to Assumption 3 we note that $L(k)$ is independent of $W_f(k)$. We partition the Laplacian matrix and its mean into the following form

$$L(k) = \begin{bmatrix} A(k) & B(k) \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \bar{L} = \begin{bmatrix} \bar{A} & \bar{B} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbf{R}^{n \times n} \quad (26)$$

where $A(k)$, $\bar{A} \in \mathbf{R}^{n_f \times n_f}$, $B(k)$, $\bar{B} \in \mathbf{R}^{n_f \times n_l}$ satisfying

$$A(k)\mathbf{1}_{n_f} + B(k)\mathbf{1}_{n_l} = \mathbf{0}, \quad \bar{A}\mathbf{1}_{n_f} + \bar{B}\mathbf{1}_{n_l} = \mathbf{0} \quad (27)$$

We denote $\tilde{A}(k) = A(k) - \bar{A}$, $\tilde{B}(k) = B(k) - \bar{B}$. Then according to (5), we have

$$\begin{aligned} X_f(k+1) &= (I - c(k)\bar{A} - c(k)\tilde{A}(k))X_f(k) - \\ & c(k)(\bar{B} + \tilde{B}(k))X_l + c(k)W_f(k) \end{aligned} \quad (28)$$

Denote

$$\bar{L}_p = P\bar{L}P' = \begin{bmatrix} \bar{A} & \bar{B}\mathbf{1}_{n_l} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbf{R}^{(n_f+1) \times (n_f+1)}$$

Then it can be found that \bar{L}_p is a valid Laplacian matrix corresponding to a leader-following graph $\bar{\mathcal{G}}_p$. That graph $\bar{\mathcal{G}}$ contains a united spanning tree implies that $\bar{\mathcal{G}}_p$ contains a spanning tree and all the eigenvalues of \bar{A} are on the open right-hand half plane. Since \bar{A} is a bounded matrix and $c(k)$ converges to zero, there must exist a time \bar{k}_1 such that $\forall i \in \mathcal{V}_f$,

$$\sup_{k \geq \bar{k}_1} \max_{i \in \mathcal{V}_f} |1 - c(k)\lambda_i[\bar{A}]| = \rho < 1$$

We write $\bar{A} = T\Lambda T^{-1}$, where $\Lambda = \text{diag}\{\lambda_1[\bar{A}], \dots, \lambda_{n_f}[\bar{A}]\}$, T is the transformation matrix of \bar{A} . Then we have

$$\begin{aligned} \left\|\Pi_{k,0}^{I-c\bar{A}}\right\|_2 &\leq \left\|T\Pi_{k,\bar{k}_1}^{I-c\Lambda}T^{-1}\right\|_2 \left\|\Pi_{\bar{k}_1-1,0}^{I-c\bar{A}}\right\|_2 \leq \\ & \|T\|_2 \|T^{-1}\|_2 \left\|\Pi_{\bar{k}_1-1,0}^{I-c\bar{A}}\right\|_2 \left\|\Pi_{k,\bar{k}_1}^{I-c\bar{A}}\right\|_2 \leq \\ & \|T\|_2 \|T^{-1}\|_2 \left\|\Pi_{\bar{k}_1-1,0}^{I-c\bar{A}}\right\|_2 \rho^{k-\bar{k}_1} \end{aligned}$$

which implies that $\lim_{k \rightarrow \infty} \left\|\Pi_{k,0}^{I-c\bar{A}}\right\|_2 = 0$.

Since all the eigenvalues of \bar{A} are non-zero, $\bar{A}^{-1}\bar{B}$ is a bounded matrix. According to (27) we have that $-\bar{A}^{-1}\bar{B}\mathbf{1}_{n_l} = \mathbf{1}_{n_f}$. It is noted that \bar{A} is an M -matrix, which leads to that \bar{A}^{-1} is nonnegative. Since $-\bar{B}$ is also nonnegative, we have that $-\bar{A}^{-1}\bar{B}$ is a row stochastic matrix.

We define $\tilde{X}_f(k) = X_f(k) + \bar{A}^{-1}\bar{B}X_l$. It follows that

$$\begin{aligned} \tilde{X}_f(k+1) &= (I - c(k)\bar{A} - c(k)\tilde{A}(k))\tilde{X}_f(k) - \\ & c(k)\tilde{B}(k)X_l + c(k)\tilde{A}(k)\bar{A}^{-1}\bar{B}X_l + \\ & c(k)W_f(k) \end{aligned} \quad (29)$$

Since $(\tilde{A}(k), \tilde{B}(k))$ are i.i.d. with respect to k , $(\tilde{A}(k), \tilde{B}(k))$ and $\tilde{X}_f(k)$ are independent, and

$$\mathbb{E}(\tilde{X}_f(k+1)) = (I - c(k)\tilde{A})\tilde{\mathbb{E}}(X_f(k)) \quad (30)$$

which implies that $\lim_{k \rightarrow \infty} \mathbb{E}(X_f(k)) = 0$.

Define $\hat{A} = \bar{A} + \bar{A}^T$. Following the same line of arguments as that of Theorem 1, we know that $\lambda_i[\hat{A}] \in \mathbf{R}^+$, $i = 1, \dots, n_f$. Denote

$$M_1 = \lambda_{\max} \left[\bar{A}^T \bar{A} + \sup_{k \in \mathbf{Z}^+} \mathbb{E}(\tilde{A}^T(k)\tilde{A}(k)) \right] < +\infty$$

Then there exists a time \bar{k}_2 such that $\forall k \geq \bar{k}_2$,

$$c(k) < \min \left\{ \frac{2}{\lambda_{\min}[\hat{A}]}, \frac{\lambda_{\min}[\hat{A}]}{2M_1} \right\} \quad (31)$$

By defining $V(k) = \mathbb{E}[\tilde{X}_f^T(k)\tilde{X}_f(k)]$, according to (29), we have $\forall k \geq \bar{k}_2$,

$$\begin{aligned} V(k+1) &= \mathbb{E}[\tilde{X}_f^T(k)(I - c(k)A(k))^T(I - c(k)A(k))\tilde{X}_f(k)] + \\ &\quad c^2(k)X_l^T \mathbb{E}(\tilde{B}^T(k)\tilde{B}(k))X_l + \\ &\quad c^2(k)X_l^T \bar{B}^T(\bar{A}^T)^{-1} \mathbb{E}(\tilde{A}^T(k)\tilde{A}(k))\bar{A}^{-1}\bar{B}X_l + \\ &\quad c^2(k)\mathbb{E}(W_f^T(k)W_f(k)) - \\ &\quad 2c^2(k)X_l^T \mathbb{E}(\tilde{B}^T(k)\tilde{A}(k))\bar{A}^{-1}\bar{B}X_l + \\ &\quad 2c^2(k)\mathbb{E}(\tilde{X}_f^T)\mathbb{E}(\tilde{A}^T(k)\tilde{B}(k))X_l - \\ &\quad 2c^2(k)\mathbb{E}(\tilde{X}_f^T)\mathbb{E}(\tilde{A}^T(k)\tilde{A}(k))\bar{A}^{-1}\bar{B}X_l \leq \\ &\quad (1 - \lambda_{\min}[\hat{A}]c(k) + M_1c^2(k))V(k) + M_2c^2(k) \end{aligned} \quad (32)$$

where

$$\begin{aligned} M_2 &= \|X_l\|_2^2 \sup_{k \geq 0} \left\{ \|\mathbb{E}(\tilde{B}^T(k)\tilde{B}(k))\|_2 + \right. \\ &\quad \left. \|\bar{B}^T(\bar{A}^T)^{-1} \mathbb{E}(\tilde{A}^T(k)\tilde{A}(k))\bar{A}^{-1}\bar{B}\|_2 \right\} + \text{tr}(Q) + \\ &\quad 2\|X_l\|_2^2 \sup_{k \geq 0} \left\{ \mathbb{E}(\tilde{B}^T(k)\tilde{A}(k))\bar{A}^{-1}\bar{B} \right\} + \\ &\quad 2\|X_l\|_2 \sup_{k \geq 0} \|\mathbb{E}(X_f)\|_2 \left\{ \sup_{k \geq 0} \|\mathbb{E}(\tilde{A}^T(k)\tilde{B}(k))\|_2 + \right. \\ &\quad \left. \sup_{k \geq 0} \|\mathbb{E}(\tilde{A}^T(k)\tilde{A}(k))\bar{A}^{-1}\bar{B}\|_2 \right\} < +\infty \end{aligned}$$

In light of (31), (32) results in that

$$\lim_{k \rightarrow \infty} V(k) \leq \lim_{k \rightarrow \infty} \frac{2M_2c^2(k)}{\lambda_{\min}[\hat{A}]c(k)} = 0 \quad (33)$$

Then $\forall i \in \mathcal{V}_f$, it follows that

$$\begin{aligned} &\lim_{k \rightarrow \infty} \text{Edist}^2(x_i(k), \text{co}\{\mathcal{V}_l\}) \leq \\ &2 \lim_{k \rightarrow \infty} \text{Edist}^2(x_i(k), -e_i \bar{A}^{-1} \bar{B} X_l) + \\ &2 \lim_{k \rightarrow \infty} \text{Edist}^2(-e_i \bar{A}^{-1} \bar{B} X_l, \text{co}\{\mathcal{V}_l\}) \leq \\ &2 \lim_{k \rightarrow \infty} V(k) = 0 \end{aligned}$$

The second term in the first inequality is equal to zero since matrix $-\bar{A}^{-1}\bar{B}$ is a row stochastic matrix. \square

5 Numerical examples

In this section, we shall verify our algorithms by some numerical examples. We consider 8 agents with 5 followers labeled 1 to 5 and 3 leaders labeled 6 to 8. The state of each agent is associated with an (x, y) coordinate. The initial position of each agent is given below:

$$\begin{aligned} x_1(0) &= 1.5, & y_1(0) &= 1, & x_2(0) &= -0.5, & y_2(0) &= -0.5 \\ x_3(0) &= 0, & y_3(0) &= -0.5, & x_4(0) &= -1, & y_4(0) &= 1 \\ x_5(0) &= 1.5, & y_5(0) &= 0, & x_6(0) &= -1, & y_6(0) &= 0 \\ x_7(0) &= 0, & y_7(0) &= 1, & x_8(0) &= 1, & y_8(0) &= 0 \end{aligned}$$

The communication noises are assumed to be Gaussian with zero mean and covariance 0.01. The control gain function $c(k)$ is chosen as $1/(k+3)$. It is clear that $c(k)$ satisfies the persistent condition (4).

Now we first consider the time-varying communication graph case. Two graphs are shown in Figs. 1 and 2. The corresponding matrices A and B for both graphs are

$$\begin{aligned} A_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}, & B_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, & B_2 &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

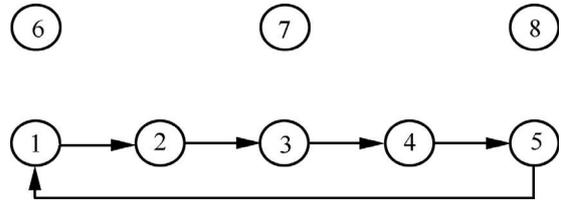


Fig. 1 Communication graph \mathcal{G}_1

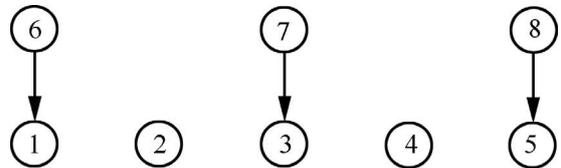


Fig. 2 Communication graph \mathcal{G}_2

We assume that the communication graph is given as follows

$$\mathcal{G}(k) = \begin{cases} \mathcal{G}_1, & k \text{ is odd} \\ \mathcal{G}_2, & k \text{ is even} \end{cases}$$

It can be checked that neither \mathcal{G}_1 nor \mathcal{G}_2 contains a united spanning tree. However, according to Theorem 1, the containment tracking can be achieved by applying protocol (3). Fig. 3 shows the state trajectories of the followers. Each follower starts with “o” and ends with “*”. It can be found that all the followers are finally contained in the triangle formed by the tree leaders. Next, we consider the randomly switching topologies case. We assume that the communication topology randomly switches from \mathcal{G}_1 and \mathcal{G}_2 with the same probability. It is noted that the mean graph contains a united spanning tree. Fig. 4 shows the state trajectories of the followers.

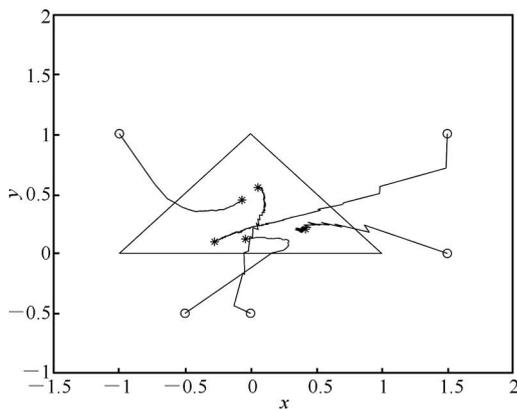


Fig. 3 State trajectories of the followers under dynamically switching topologies

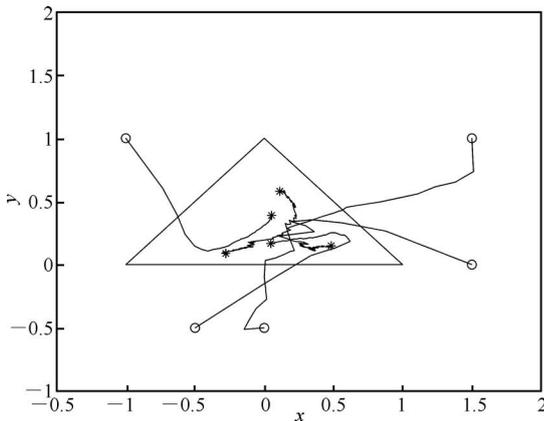


Fig. 4 State trajectories of the followers under randomly switching topologies

6 Conclusion

In this paper, the containment control problem for multi-agent systems with noises in transmission channels has been studied. Both dynamically switching topologies and randomly switching topologies have been considered. A decaying gain function has been introduced to attenuate the noises. Sufficient conditions on the gain function and the communication graphs have been provided to guarantee the containment tracking in the mean square sense. The results

can be extended to multi-agent systems with continuous dynamics. Two numerical examples have been given to demonstrate the results.

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