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Adaptive Finite-time Consensus Control of a Group of Uncertain Nonlinear Mechanical Systems

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Abstract

In this paper we consider finite-time leaderless consensus control of multi-agent systems consisting of a group of nonlinear mechanical systems with parametric uncertainties. New adaptive finite time continuous distributed control algorithms are proposed for the multi-agent systems. It is shown that the states of the mechanical systems can reach a consensus within finite time under an undirected graph. Transient performances in terms of convergence rates and time are also analyzed. Finally simulation results illustrate and verify the effectiveness of the proposed schemes.

Key words: finite time; consensus; adaptive control; multi-agent system

1 Introduction

Distributed coordination for a group of dynamic agents has attracted many researchers in recent years, due to the broad applications of multi-agent systems in many areas, such as cooperative control of unmanned air vehicles (UAVs) in DeLima et al. (2009), formation control in Dong & Farrell (2009); Ren & Atkins (2007); Do (2008), flocking control in Do (2011); Olfati-Saber (2006), distributed sensor networks in Cortes & Bullo (2005), attitude alignment for a clusters of satellites in Trilaksono et al. (2011), congestion control in communication networks in Ignaciuk et al. (2011) etc. The consensus of multi-agent systems means the states of all the agents converge to a common value by invoking some control schemes for each agent in the group. Consensus control problem is studied intensively since the state synchronization is often a basic requirement in many group behaviors, such as flocking. Related works on consensus control include, but not limited to, Ren (2008); Olfati-Saber & Murray (2004); Hong et al. (2008); Yu & Xia (2012); Dong (2012); Ren & Beard (2005); Hong et al. (2006b).

In Ren & Beard (2005) the problem of static information consensus among multi-agents in the presence of limited and unreliable information exchange with dynamically changing topologies is considered, which shows that static consensus under dynamically changing interaction topologies can be achieved asymptotically if the directed interaction graphs have a spanning tree frequently enough. Asymptotical time-varying trajectory consensus is rather difficult if only parts of the agents in the group has access to the desired trajectory. In Hong et al. (2006b) multi-agent consensus problem with an active leader and variable interconnection topology is considered. To track the leader, a neighbor-based local controller together with a neighbor-based state observer is developed for each agent, while assuming leader’s acceleration is partially known to all the agents in the group. Similar idea also appears in Hong et al. (2008), where distributed observer is designed for each agent using local information while assuming leader’s acceleration is globally known. In Li et al. (2013) the unknown time-varying references can be compensated for by introducing discontinuous sign functions to cope with the unknown reference trajectories in generating the control signals, however chattering phenomenon of discontinuous input may affect the performance in practical implementation. In Dong (2011) consensus problem of multiple uncertain mechanical systems with a desired trajectory is considered. The desired trajectory is available to a portion of a group of mechanical systems. By assuming that the references are composed by a set of basic vector function, which are available to all the agents, distributed adaptive control laws are proposed such that the state of each system...
asymptotically converges to the desired trajectory with information interchange between agents. In Mei et al. (2011) adaptive distributed leader-following control for a group of uncertain Euler-Lagrange systems is considered, where only a subset of the followers have access to the leader and the followers have only local interactions.

An important performance indicator for the consensus problem is the convergence rate. Most of the existing consensus control schemes for multi-agent systems so far achieve asymptotic convergence, namely the convergence rate at best is exponential, which means it needs infinite time for the tracking errors to converge to the origin. It is shown in Olfati-Saber & Murray (2004) that the second smallest eigenvalue of the interaction Laplacian matrix quantifies the rate of convergence. To get better convergence rate, several researchers try to find better interaction graph such as to get a larger ‘second smallest eigenvalue’. Kim & Mesbahi (2006) consider the problem of finding the best vertex positional configuration so that the ‘second smallest eigenvalue’ of the associated interaction graph is maximized. In Xiao & Boyd (2006) by using semi-definite convex programming the weights among the agents are designed such that convergence rate is increased. However, all these efforts are only to choose proper interaction graphs, but not to find control schemes to achieve higher performance. In practice, it is often required that the consensus be reached in finite time. Thus several researchers invoke the finite-time control schemes to guarantee that the consensus can be reached within finite time. In addition, finite-time control schemes poses better disturbance rejection properties as shown in Bhat & Bernstein (2000). In Xiao et al. (2009) a finite-time formation control framework for multi-agent systems of first-order dynamics is developed. In Wang & Xiao (2010) a continuous finite time control scheme is developed for the state consensus problems for multi-agent systems of first-order dynamics. Li et al. (2011) address the finite-time consensus problem for leaderless and leader-follower multi-agent systems of second-order double integrator dynamics. In Zhang & Yang (2013) the finite-time consensus tracking problem with one leader and the finite-time containment control problem with multiple leaders are considered.

Nonetheless, uncertainties do exist in practical systems. Adaptive control is one of the most effective ways to cope with parametric uncertainties by employing on-line parameter estimators. Great deal of works has been done for adaptive control of linear and nonlinear systems with unknown parameters, see for example Krstic et al. (1995). It should be emphasized that for adaptive control systems it is very difficult to establish exponential convergence mainly due to the use of online parameter estimators that gives rise to a highly nonlinear closed loop system, which hinders the potential applications of adaptive control. Therefore ensuring finite-time convergence will be more interesting and significant for adaptive control systems. However, achieving adaptive finite-time control is rather challenging. Certain key techniques employed in existing adaptive control literature cannot be applied. For example, the Barbalat’s lemma normally adopted for analyzing asymptotic convergence cannot be applied to the analysis of finite-time convergence. On the other hand, the finite-time convergence analysis tools adopted in systems without parametric uncertainties such as those in Bhat & Bernstein (1998) and Li et al. (2011) cannot be applied to adaptive finite time control directly. In Hong et al. (2006b) finite-time stabilization control for a class of single nonlinear system with parametric uncertainties is investigated with a backstepping-like recursive control scheme, but the convergence time is not expressed explicitly. Furthermore, the existing finite-time consensus control schemes mainly focus on first-order integrator as in Wang & Xiao (2010) or second-order double-integrator as in Li et al. (2011). Note that adaptive consensus control for uncertain multi-agent nonlinear systems with an explicitly established convergence-time is still unavailable, to the best of the authors’ knowledge.

In this paper, we address such an issue for a group of general nonlinear mechanical systems with parametric uncertainties. New continuous adaptive distributed finite-time controllers are proposed for each agent in the group of leaderless consensus control. In order to make the stability analysis method in Qian & Lin (2001a) applicable, we need to design suitable online parameter estimators to guarantee that the position errors and the virtual control errors converge to a pre-defined compact set within finite time, which is quite challenging. An adaptive distributed controller is designed such that all the positions of the agents converge to a consensus state with a configuration within finite-time. By assuming that only some of the agents have access to the static rendezvous location, the proposed adaptive distributed controllers ensure that all the agents converge to the rendezvous location with a configuration in finite time. We also establish the transient performance for the tracking errors and the virtual control errors.

The remainder of the paper is organized as follows. The problem of this paper is formulated and some useful preliminaries are presented in Section 2. In Section 3, distributed adaptive finite-time consensus design schemes are proposed and analyzed. Simulation results are given in Section 4 to validate the theoretical results. Finally, we conclude the paper in Section 5.

2 Problem Formulation

In this paper, let \(1 = [1, ..., 1]^T\) and \(0 = [0, ..., 0]^T\). If \(P\) is a positive definite matrix, then let \(\lambda_{\max}(P)\) and \(\lambda_{\min}(P)\) denote its maximum and minimum eigenvalues, respectively. For a vector \(\| \cdot \|\) denotes a standard Euclidean norm. For a matrix \(\| \cdot \|\) denotes a standard column sum norm.

2.1 Graph Theory

In this paper, the communications among the \(n\) agents are represented by a graph \(G \equiv (V, E)\) where \(V = \{1, ..., n\}\) denotes the set of indexes (or vertices) corresponding to each agent, \(E \subseteq V \times V\) is the set of edges between two distinct agents. If \(G\) is undirected, then the edge \((i, j)\) also means \((j, i)\) which indicates that agents \(i\) and \(j\) can obtain state and structure information from each other. In this case, agent \(j\) is called a neighbor of agent \(i\), and vice versa. We denote the set of neighbors for agent \(i\) as \(N_i \equiv \{j \in V: (i, j) \in E\}\). Note that self edges \((i, i)\) is not allowed, thus \((i, i) \notin E\) and \(i \notin N_i\). \(G\) is connected means that there is an undirected sequence of edges between every pair of distinct agents Ren & Cao (2010). The connectivity matrix \(A = [a_{ij}] \in \mathbb{R}^{n \times n}\) is defined that \(a_{ij} \neq 0\) if \((i, j) \in E\), and \(a_{ij} = 0\) if \((i, j) \notin E\). Clearly the diagonal
elements $a_{ii} = 0$ and for an undirected graph $A$ is symmetric. We introduce an in-degree matrix $\Delta$ such that $\Delta = \text{diag}(\Delta_i) \in \mathbb{R}^{n \times n}$ with $\Delta_i = \sum_{j \in V_i} a_{ij}$ being the $i$th row sum of $A$. Then, the Laplacian matrix of $\mathcal{G}$ is defined as $\mathcal{L} = \Delta - A$.

2.2 Preliminaries

To investigate the finite-time stability, some basic concepts and definitions are firstly introduced.

**Definition 1** Bhat & Bernstein (1998). Consider a dynamic system

$$\dot{x} = f(x, t), \quad f(0, t) = 0, \quad x \in U_0 \subset \mathbb{R}^n$$  \hspace{1cm} (1)

where $f : U_0 \times \mathbb{R}^+ \to \mathbb{R}^n$ is continuous on an open neighborhood $U_0$ of the origin $x = 0$. The equilibrium $x = 0$ of the system is (locally) finite-time stable if it is Lyapunov stable and for any initial condition $x(t_0) = x_0 \in U$ where $U \subset U_0$, if there is a settling time $T > t_0$, such that every solution $x(t; t_0, x_0)$ of system (1) satisfies $x(t; t_0, x_0) \in U \setminus \{0\}$ for $t \in [t_0, T)$, and

$$\lim_{t \to T} x(t; t_0, x_0) = 0, \quad x(t; t_0, x_0) = 0 \forall t > T$$

If $U = \mathbb{R}^n$, then the origin $x = 0$ is a globally finite-time stable equilibrium.

The following lemmas are useful for establishing system stability.

**Lemma 1** Bhat & Bernstein (1998). Suppose there is a $C^1$ positive definite Lyapunov function $V(x, t)$ defined on $U \times \mathbb{R}^+$ where $U \subset U_0$ is the neighborhood of the origin, and there are positive real constants $c > 0$ and $0 < \alpha < 1$, such that $V(x, t) + cV'(x, t)$ is negative semidefinite on $U$. Then $V(x, t)$ is locally in finite-time convergent with a settling time

$$T \leq \frac{V(0, t)}{c(1 - \alpha)}$$

for any given initial condition $x(t_0)$ in the neighborhood of the origin in $U$.

**Lemma 2** Qian & Lin (2001a). If $0 < p = p_1/p_2 \leq 1$, where $p_1 > 0$ and $p_2 > 0$ are positive odd integers, then $|x^p - y^p| \leq 2^{1-p} |x - y|^p$.

**Lemma 3** Hardy et al. (1952). For $x_i \in \mathbb{R}$, $i = 1, ..., n$, $0 < p \leq 1$, then

$$\left( \sum_{i=1}^{n} |x_i| \right)^p \leq \sum_{i=1}^{n} |x_i|^p \leq n^{1-p} \left( \sum_{i=1}^{n} |x_i| \right)^p$$

**Lemma 4** Qian & Lin (2001a). Let $c$ and $d$ be positive constants and $\gamma(x, y) > 0$ is a real value function. Then

$$|x|^c |y|^d \leq \frac{c \gamma(x, y)}{c + d} |x|^{c + d} + \frac{d \gamma(x, y)}{c + d} |y|^{c + d}$$

**Lemma 5** Olfati-Saber & Murray (2004). For a connected undirected graph $\mathcal{G}$, the Laplacian matrix $\mathcal{L}$ has the following property: (1) $\mathcal{L}$ is semi-definite. (2) 0 is a simple eigenvalue of $\mathcal{L}$ and 1 is the associated eigenvector. (3) Assuming the eigenvalue of $\mathcal{L}$ is denoted as $\lambda_0, \lambda_2, ..., \lambda_n$, satisfying $0 \leq \lambda_2 \leq ... \leq \lambda_n$, then the second smallest eigenvalue $\lambda_2 > 0$. Furthermore, if $\mathbf{1}^T \mathbf{x} = 0$, then $\mathbf{x}^T \mathcal{L} \mathbf{x} \geq \lambda_2 \mathbf{x}^T \mathbf{x}$.

Consider $y = x^p$ where $p$ is a positive integer and $q$ is a positive odd integer. If we ignore all the complex roots, then obviously $y = \text{sign}(x)|x|^q$ if $p$ is an odd integer; otherwise $y = |x|^q$.

2.3 System Model

We consider a class of multiple mechanical nonlinear systems

$$M_i \ddot{q}_i + C_i(q_i, \dot{q}_i) \dot{q}_i + D_i(q_i, \dot{q}_i) = \tau_i$$  \hspace{1cm} (2)

where $q_i = (q_{i,1}, ..., q_{i,m})^T \in \mathbb{R}^n$, $i = 1, ..., n$ is the state of the $i$th system, $\tau_i \in \mathbb{R}^m$ is the control input vector, $M_i \in \mathbb{R}^{m \times m}$ is an inertia matrix, $C_i(q_i, \dot{q}_i)$ is the centripetal and coriolis matrix and $D_i(q_i, \dot{q}_i)$ denotes the friction terms. Denote $v_i = \dot{q}_i$. The following assumptions are needed for the design of finite-time adaptive consensus controllers.

**Assumption 1** The graph $\mathcal{G}$ is connected.

**Assumption 2** $M_i = \text{diag}(f_{i1}, ..., f_{im})$ where $f_{ik}$, $k = 1, ..., m$ are unknown positive constants. $(\bar{C_i}(q_i, v_i) v_i + \bar{D}_i(q_i, v_i) = \bar{G}_i(q_i, v_i) \theta_i$, where $\theta_i \in \mathbb{R}^n$ denotes the vector of unknown parameters and $\bar{G}_i(q_i, v_i) \in \mathbb{R}^{m \times l}$ does not contain unknown parameters. Furthermore, $\|\bar{G}_i(q_i, v_i)\|_1 \leq \rho \|v_i\|$, where $\rho$ is a known positive constant and $\rho(f)$ is a polynomial with no constant term, i.e. $P(0) = 0$. $f_{ik}, k = 1, ..., m$ and $\theta_i$ are in known compact set.

**Remark 1**

- For a typical mechanical system, $q_i$ appears in the centripetal force and Coriolis force. For a constant $v_i$, there exists a constant $p$ such that $|\bar{G}_i(q_i, v_i)| < p$. Taking the two-link robot manipulator as an example, $q_i$ is the joint-variable vector, consisting of joint angles $\theta_i$ and a joint offset $d_i$. The joint angles appear in $\bar{G}_i(q_i, v_i)$ in terms of $\sin(\theta_i)$ or $\cos(\theta_i)$, and a joint offset $d_i$ is always bounded.

- Typical examples of mechanical system satisfying Assumption 2 include Newton-Euler rigid body and three-link cylindrical robot manipulator, as described in Lewis et al. (2004).

3 Control Design and System Analysis

System (2) can be written as

$$\ddot{q}_i + \bar{C}_i(q_i) v_i + \bar{D}_i(q_i) v_i = \tau_i$$  \hspace{1cm} (3)
where \( v_i = [v_{i1}, ..., v_{im}]^T, \ i \in \mathcal{V} \) denotes the velocity. The designed control law should ensure that all the agents reach consensus in finite time without additional information. A distributed adaptive control law is designed for each agent such that the positions of all the agents converge to a consensus location based on its neighbor’s information. The controller is designed using a ‘backstepping-like’ procedure. In the first step virtual control \( v_i^* \) for \( v_i, i = 1, ..., n \) is designed such that the position error converges to zero in finite-time. In the second step adaptive controllers and parameter estimators are designed such that \( v_i \) converge to \( v_i^* \) in finite-time. These finite time convergence results are summarized in Theorem 1. The transient performance of the close-loop adaptive control systems is established and presented in Theorem 2, by showing that the position errors and the virtual control errors converge faster than an exponential rate.

### 3.1 Controller Design

- **Step 1.** Let \( x_j = [q_{ij} - \delta_{ij}, ..., q_{nj} - \delta_{nj}]^T, x = [x_1^T, ..., x_m^T]^T, j = 1, ..., m, \) where \( \delta_{kj}, k = 1, ..., n \) are constants denoting the final consensus configuration such that \( q_{ij} - q_{kj} = \delta_{ij} - \delta_{kj}, \ i, k \in \mathcal{V} \). Define a Lyapunov function

\[
V_i(t) = \frac{1}{2} x^T (I_m \otimes L)x
= \frac{1}{4} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k \in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj})^2
\]

Taking the derivative of \( V_i \) yields

\[
\dot{V}_i = \dot{x}^T (I_m \otimes L)x
= \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k \in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}) \dot{q}_{ij}
\]

Let

\[
e_{ij} = \sum_{k \in N_i} a_{ik}(q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj})
\]

Then

\[
\dot{V}_i = \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}v_{ij}
\]

Denoting \( v_{ij}^* \) as the virtual control of \( v_{ij} \), we have

\[
\dot{V}_i = \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}v_{ij}^* + \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^*)
\]

By choosing the virtual control as

\[
v_{ij}^* = -c_1 e_{ij}^{2\sigma^{2} - 1}
\]

where \( \sigma \geq 2 \) is a positive integer and \( c_1 \) is a positive constant to be designed, we get

\[
\dot{V}_i = \sum_{j=1}^{m} \sum_{i=1}^{n} -c_1 e_{ij}^{2\sigma^{2} - 1} + \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^*)
\leq -c_1 \left( \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}^{2\sigma^{2}} \right) + \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}(v_{ij} - v_{ij}^*)
\]

(7)

Let \( e_j = [e_{1j}, ..., e_{nj}]^T \), then \( e_j = Lx_j \). Thus \( e_j^T e_j = x_j^T L^T L x_j = x_j^T L^2 x_j \). Since \( L \) is a diagonalizable symmetric semi-positive definite matrix, then it is easy to prove that \( L^{1/2} \) is also a symmetric semi-positive definite matrix, and \( L^{1/2} L^{1/2} = I \). Let \( w = L^{1/2} \mathbf{1} \), we get \( w^T w = (L^{1/2} \mathbf{1}^T (L^{1/2} \mathbf{1}) = I^T \mathbf{1} \mathbf{1} = 0 \). Thus \( w = 0 \), which yields \( w^T x_j = 0 \), i.e. \( 1^T L^{1/2} x_j = 0 \). Then according to Lemma 5

\[
e_{ij}^T e_j = (L^{1/2} x_j)^T L(L^{1/2} x_j) \geq \lambda_2 x_j^T L x_j.
\]

(8)

Let \( e = [e_1^T, ..., e_m^T]^T \), then we get

\[
\sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}^2 \geq e^T e \geq \lambda_2 x^T (I_m \otimes L)x = 2\lambda_2 V_i
\]

(9)

- **Step 2.** Let \( g = \frac{2\sigma^{2} - 1}{\sigma} \). Invoke the adding a power integrator technique as in Qian & Lin (2001b) by defining another Lyapunov function

\[
W_i = \frac{1}{2^{1-g}} \sum_{j=1}^{m} f_{ij} \frac{v_{ij}}{v_{ij}^*} (s^{1/g} - v_{ij}^{1/g})^{2-g} ds
\]

and a new variable

\[
\xi_{ij} = v_{ij}^{1/g} - v_{ij}^{1/g}
\]

(11)

where \( f_{ij} \) is defined in Assumption 2. From Qian & Lin (2001a), \( W_i \) is positive semi-definite and \( C^1 \). Taking the time derivative of \( W_i \), we get

\[
\dot{W}_i = \frac{1}{2^{1-g}} \sum_{j=1}^{m} f_{ij} \xi_{ij}^{2-g} b_{ij}
- (2 - g)f_{ij} \frac{v_{ij}}{v_{ij}^*} (s^{1/g} - v_{ij}^{1/g})^{1-g} ds \sum_{k \in N_i} \frac{\partial v_{ij}^{1/g}}{\partial q_{kj}} \xi_{kj}
\]

where \( \bar{N}_i = N_i \cup \{i\} \). Since

\[
\left| \frac{v_{ij}}{v_{ij}^*} (s^{1/g} - v_{ij}^{1/g})^{1-g} ds \right| \leq |v_{ij} - v_{ij}^*| |\xi_{ij}|^{1-g}
= \left| (v_{ij}^{1/g} - v_{ij}^{1/g})^{g} \right| |\xi_{ij}|^{1-g} \leq 2^{1-g} |\xi_{ij}|^{1-g} = 2^{1-g} (|\xi_{ij}|)
\]

(12)
Define a new Lyapunov function 

$$\nu_i \leq \frac{m}{g} \sum_{j=1}^{m} f_{ij} \xi_{ij} \dot{\theta}_{ij} + (2 - g) \sum_{j=1}^{m} f_{ij} |\xi_{ij}| \left| \frac{\partial \nu_{ij}}{\partial \theta_{ij}} \right| |v_{kj}|$$

$$= \frac{1}{2^{1-g}} \xi^T \left[-C_i(q_i, v_i) \dot{v}_i + D_i v_i + \tau_i \right]$$

$$+ (2 - g) \sum_{j=1}^{m} f_{ij} |\xi_{ij}| \left| \frac{\partial \nu_{ij}}{\partial \theta_{ij}} \right| |v_{kj}|$$

where 

$$\xi_i = [\xi_{i1}, ..., \xi_{im}]^T.$$  

(13)

Let \( \tilde{\theta}_i, i \in V, j = 1, ..., m \) denote the estimate of \( \theta_i \) and define \( \bar{\theta}_i = \tilde{\theta}_i - \theta_i \). Then

$$\dot{W}_i \leq \frac{1}{2^{1-g}} \xi^T \left(-G_i(q_i, v_i) \bar{\theta}_i + \tau_i \right) + \frac{1}{2^{1-g}} \xi^T G_i(q_i, v_i) \bar{\theta}_i$$

$$+ (2 - g) c_1^{1/g} \sum_{j=1}^{m} f_{ij} |\xi_{ij}| \left( \mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right)$$

where \( \mu = \max_{v_i \in V} \left\{ \sum_{j \in N_i} a_{ij} \right\} \) and \( \eta = \max_{v_i, j \in V} \{a_{ij}\} \). From Lemma 2 and Lemma 4, we have

$$\sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij} (v_{ij} - v'_{ij}) \leq \sum_{j=1}^{m} \sum_{i=1}^{n} 2^{1-g} |e_{ij}| |\xi_{ij}|^{1/g}$$

$$\leq \frac{2^{1-g}}{1+g} \sum_{j=1}^{m} \sum_{i=1}^{n} (c_{1+g} + g^{1+g} \xi_{ij})$$

Define a new Lyapunov function

$$V_2 = V_1 + \sum_{i=1}^{n} W_i$$  

(14)

Then

$$\dot{V}_2 \leq -c_1 \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{e_{ij}}{\xi^T} + \frac{2^{1-g}}{1+g} \sum_{j=1}^{m} \sum_{i=1}^{n} \left( e_{1+g} + g^{1+g} \xi_{ij} \right)$$

$$+ \frac{1}{2^{1-g}} \sum_{i=1}^{n} \xi_i^T \left(-G_i(q_i, v_i) \bar{\theta}_i + \tau_i \right) + \frac{1}{2^{1-g}} \sum_{i=1}^{n} \xi_i^T G_i(q_i, v_i) \bar{\theta}_i$$

$$+ (2 - g) c_1^{1/g} \sum_{i=1}^{m} f_{ij} |\xi_{ij}| \left( \mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right)$$

(15)

Based on Lemmas 2-4 and (11),

$$(2 - g) c_1^{1/g} \sum_{i=1}^{m} f_{ij} |\xi_{ij}| \left( \mu |v_{ij}| + \eta \sum_{k \in N_i} |v_{kj}| \right)$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{m} \left( f_{ij} \mu + \frac{1}{1+g} \sum_{i=1}^{n} \xi_i^T \left(-G_i(q_i, v_i) \bar{\theta}_i + \tau_i \right) + \frac{1}{2^{1-g}} \sum_{i=1}^{n} \xi_i^T G_i(q_i, v_i) \bar{\theta}_i$$

where \( n_i = \dim N_i \). Since \( F_i^{-1} = \text{diag}(1/f_{i1}, ..., 1/f_{im}) \), the torque is designated as

$$\tau_i = G(q_i, v_i) \bar{\theta}_i - c_2 \xi_{ij}^T$$  

(16)

where \( \xi_i = [\xi_{i1}^{2g-1}, ..., \xi_{im}^{2g-1}]^T \) and \( c_2 \) is a positive constant to be chosen. From (15) and (16), we get

$$V_2 \leq -k_1 \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}^{2+g} - k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^{2+g}$$

$$+ \frac{1}{2^{1-g}} \sum_{i=1}^{n} \xi_i^T \left(-G_i(q_i, v_i) \bar{\theta}_i + \tau_i \right)$$

(17)

where

$$k_1 = c_1 - \frac{2^{1-g}}{1+g} \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}^{2+g}$$

$$k_2 = c_2 - \frac{2^{1-g}}{1+g} \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^{2+g} - (2 - g) c_1^{1/g} \left( \sum_{i=1}^{n} \xi_i^T \left(-G_i(q_i, v_i) \bar{\theta}_i + \tau_i \right) \right)$$

(18)

with \( f_{\text{max}} \geq \max_{i=1, ..., m} (f_{ij}, ..., f_{ij}) \) being a positive constant known from Assumption 2. Thus we can find \( c_1 \) and \( c_2 \) such that \( k_1 > 0 \) and \( k_2 > 0 \).

Consider the following Lyapunov function

$$V_3 = V_2 + \frac{1}{2} \sum_{i=1}^{n} \bar{\theta}_i^T \Gamma_i \bar{\theta}_i.$$  

(19)

where \( \Gamma_i \) is a positive diagonal matrix. Then

$$\dot{V}_3 \leq -k_1 \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}^{2+g} - k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^{2+g}$$

$$+ \sum_{i=1}^{n} \bar{\theta}_i^T \Gamma_i^{-1} \left( \bar{\theta}_i + \frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \xi_i \right).$$

(20)

The parameter update law for \( \bar{\theta}_i \) is

$$\dot{\bar{\theta}}_i = \text{Proj} \left( \beta_i(k), \bar{\theta}_i \right), \quad k = 1, ..., l$$

(20)

where \( \beta_i(k) \) is the \( k \)th element of \( \beta_i = -\frac{1}{2^{1-g}} \Gamma_i G_i(q_i, v_i)^T \xi_i \).

The operator \( \text{Proj}(\cdot, \cdot) \) is a Lipschitz continuous projection al-
algorithm in Pomet & Praly (1992) which is defined as follows:

\[
\text{Proj}(a, b) = \begin{cases} 
  a & \text{if } \mu(b) \leq 0 \\
  a & \text{if } \mu(b) \geq 0 \text{ and } \mu'(b)a \leq 0 \\
  (1 - \mu(b))a & \text{if } \mu(b) > 0 \text{ and } \mu'(b)a > 0
\end{cases}
\] (21)

where \(\mu(b) = \frac{b^2 - b_{M}^2}{\epsilon^2 + 2kb_M}, \mu'(b) = \frac{\partial \mu(b)}{b}, \epsilon \) is an arbitrarily small positive constant, \(b_M\) is a positive constant satisfying \(|b| < b_M\).

Then from (17), (19), (20) and based on the property of the parameter projection in Krstic et al. (1995), we get

\[
V_3 \leq -k_1 \sum_{j=1}^{m} \sum_{i=1}^{n} \varepsilon_{ij}^2 \leq k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^2
\] (22)

3.2 Stability Analysis

We are now at the position to state our first result in the following theorem.

Theorem 1 Consider the undirected leaderless multi-agent uncertain systems (2), under the control of the distributed adaptive controllers (16) with parameter estimators in (20). If Assumptions 1 and 2 are satisfied, then the positions of the group of mechanical systems will reach consensus with specified configurations \(\delta_{ij}\) in finite time \(T\) satisfying

\[
T \leq \frac{g_1}{k_s(1 - \kappa)} + V_3(t_0) - g_1 \frac{d_{e_3}}{d_{e_3}}
\] (23)

where \(\kappa, k_s, g_1\) and \(d_{e_3}\) are computable.

Proof: From (22) we get \(V_3 \leq 0\). Therefore \(e_{ij}\) and \(\bar{\theta}_i\) are bounded and it can be easily checked that \(\lim_{t \to \infty} e_{ij} = 0\). Since we cannot use \(V_3 + C_2e_{ij}^2 \leq 0\) to establish the finite-time stability of \(e_{ij}\) and \(\xi_{ij}\) due to the existence of the terms \(\theta_i^2 \bar{\theta}_i\), we will prove that there exist two positive constants \(k_e\) and \(0 < \kappa < 1\) such that \(V_2 + k_eV_3^2 \leq 0\) which enable us to show that the systems are finite-time stable.

By using the parameter projection operation and based on Assumption 2, we know there exists a positive constant \(S_1\) such that \(\|\bar{\theta}_i\| \leq S_1\). Then from (10) and (12) we have

\[
W_i \leq \sum_{j=1}^{m} f_{ij} \xi_{ij}^2 \leq f_{max} \sum_{j=1}^{m} \xi_{ij}^2
\] (24)

Thus from (9) we obtain

\[
V_2 \leq \frac{1}{2k_2} \sum_{i=1}^{n} \sum_{j=1}^{m} \varepsilon_{ij}^2 + f_{max} \sum_{i=1}^{n} \sum_{j=1}^{m} \xi_{ij}^2
\] (25)

Suppose \(P(\cdot)\) in Assumption 2 is

\[
P(x) = p_1x + p_2x^2 + \ldots + p_hx^h
\] (26)

where \(h \geq 2\) is an integer and \(p_k, k = 1, \ldots, h\) are positive constants. From (11)

\[
|v_{ij}| \leq |\xi_{ij}| + |v^*_{ij}|^{1/2} \leq |\xi_{ij}| + |c_1|e_{ij}^q
\] (27)

Define a compact set

\[
\Xi = \{\xi_{ij}, e_{ij} : r_{ij} < |\xi_{ij}|, d_{ij} < \xi_{ij} = 0\}
\] (28)

in the neighborhood of \(e_{ij} = 0, \xi_{ij} = 0\) where \(r_{ij}\) and \(d_{ij}\) are positive constants chosen to be

\[
r_{ij} = 1 - \frac{1}{2c_1} \frac{1}{\sqrt{\frac{1}{m} - 1}}
\]

(29)

From (27) it is easy to check that in this set we have \(|v_i| \leq 1\). Thus from Assumption 2 we get \(\|G_i(q_i, v_i)\|_1 \leq ph_{\max}\|v_i\|\) where \(p_{\max} = \max(p_1, \ldots, p_n)\). Furthermore, \(\|G_i(q_i, v_i)(j, k)\| < ph_{\max}\|v_i\|, k = 1, \ldots, l\) since \(|v_i| \leq 1\). Then from (13), (17) and (27)

\[
\frac{1}{2^q - 1} \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_{ij}^2 \leq \frac{1}{2^q - 1} \sum_{i=1}^{m} \alpha_i \xi_{ij}^2 \leq \frac{1}{2c_1} \frac{1}{\sqrt{\frac{1}{m} - 1}}
\] (30)

where \(\xi_{ij}\) and \(\|G_i(q_i, v_i)\|\) are taken element-wisely and \(k_3\) and \(k_4\) are

\[
k_3 = \frac{\max ph_{\max}}{2^q - 1}, k_4 = \frac{\max ph_{\max}}{2^q - 1}(1 + \frac{2 - q}{2c_1})
\] (31)

in which \(\max \) is defined as \(\max(S_1, \ldots, S_n)\). Thus from (17) we get

\[
V_2 \leq \sum_{j=1}^{m} \sum_{j=1}^{m} \varepsilon_{ij}^2 - \sum_{j=1}^{m} \sum_{j=1}^{m} \xi_{ij}^2 - k_3 \sum_{j=1}^{m} \sum_{j=1}^{m} \xi_{ij}^2
\] (32)

Now we will establish the finite-time stability in two cases. In the first case we will prove that \(e_{ij}\) and \(\xi_{ij}\) will converge to zero in finite-time if \(e_{ij}(t_0) \in \Xi\) and \(\xi_{ij}(t_0) \in \Xi\). In the second case we will show that \(e_{ij}\) and \(\xi_{ij}\) will converge to \(\Xi\) within finite-time and thus converge to zero in finite-time if \(e_{ij}(t_0) \notin \Xi\) and/or \(\xi_{ij}(t_0) \notin \Xi\).

- Case 1: The initial conditions of \(e_{ij}\) and \(\xi_{ij}\) satisfy

\[
e_{ij}(t_0) \in \Xi, \xi_{ij}(t_0) \in \Xi
\]
From (18) and (31) we can choose the design parameters $c_1$ and $c_2$ such that $k_1 > 2k_3$ and $k_2 > 2k_4$. Since $|e_{ij}| < 1$ and $|\xi_{ij}| < 1$, it is not hard to see that

$$\sum_{j=1}^{m} \frac{k_4e_{ij}^2}{2} + \sum_{j=1}^{m} \sum_{i=1}^{n} \frac{k_4\xi_{ij}^2}{2} < 0$$

if we choose $c_1$ and $c_2$ in this way. Thus we have

$$\dot{V}_2 \leq -k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^2 - k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^2$$

(33)

From (25) and Lemma 3 we get $V_2^\prime \leq \frac{1}{2\sigma} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{d_{ij}}{g_{ij}}$ where $k_2 < \frac{1}{g_{ij}}$, $\frac{k_2}{f_{min}}$ is a positive constant. Since $< g < 1$, then based on Lemma 1, $V_2$ will converge to zero in a finite time $T_1$ that satisfies

$$T_1 \leq \frac{V_2(t_0)}{k_2(1-\kappa)}$$

(35)

- Case 2: The initial values of $e_{ij}$ and/or $\xi_{ij}$ are outside the set $\Xi$. For the period of time when $e_{ij}$ and/or $\xi_{ij}$ are outside of the set $\Xi$, from (22), we know

$$\dot{V}_3 \leq -v_{d_3}$$

(36)

where $v_{d_3} \geq \min \left( \frac{n \frac{m}{i=1} \frac{n}{j=1} \frac{e_{ij}}{d_{ij}} \right)$ is a positive constant.

To determine the time $T_2$ that $e_{ij}$ and $\xi_{ij}$ reach the set $\Xi$ from the initials, we have to calculate a $\vartheta_1$ such that $V_3 \geq \vartheta_1$ when $e_{ij}(t) \notin \Xi$ and/or $\xi_{ij}(t) \notin \Xi$. Then $T_2$ will not be longer than $(V_3(t_0) - \vartheta_1)/v_{d_3}$. From (14) and (19), such a $\vartheta_1$ is calculated as

$$\vartheta_1 = \min \left( \left\{ V_3|_{e_{ij}=r_{ij}, \xi_{ij}=0}: \sum_{i=1}^{n} \sum_{j=1}^{m} W_i \right\} \right)$$

(37)

From (4) we know

$$V_2|_{e_{ij}=r_{ij}, \xi_{ij}=0} = V_1|_{e_{ij}=r_{ij}} \geq \frac{\zeta}{4\eta^2} \sum_{j=1}^{m} \sum_{i=1}^{n} \eta^2 (q_{ij} - q_{k_j} - \delta_{ij} + \delta_{k_j})^2$$

$$\geq \frac{\zeta}{4\eta^2 n_{max}} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k \in N_i} \left[ \sum_{k \in N_i} \alpha_{ik} (q_{ij} - q_{k_j} - \delta_{ij} + \delta_{k_j})^2 \right]$$

$$= \frac{\zeta}{4\eta^2 n_{max}} \sum_{j=1}^{m} \sum_{i=1}^{n} \sum_{k \in N_i} \frac{q_{ij}}{d_{ij}}$$

(38)

where $\zeta = \min_{i,j \in V} \{ a_{ij} \}$, $n_{max} = \max \{ n_1, ..., n_N \}$, $\eta = \max_{i,j \in V} \{ a_{ij} \}$, $n_i = \dim (N_i)$. It can be checked that

$$\int \left( \frac{1}{2} - \frac{1}{2}\left| \frac{d_{ij} + \eta_{ij}}{\eta} \right|^2 \eta \right) ds \geq \frac{2\sigma - 1}{2 + \frac{1}{2}\left| \frac{d_{ij} + \eta_{ij}}{\eta} \right|^2 (4\sigma + 2)} |v_{ij} - \vartheta_{ij}|^2$$

$$\geq \frac{2\sigma - 1}{2 + \frac{1}{2}\left| \frac{d_{ij} + \eta_{ij}}{\eta} \right|^2 (4\sigma + 2)} (d_{ij} + \eta_{ij})^2 - \frac{4\zeta^2}{\eta^2} = s_{ij}$$

(39)

This is because for a fixed $\xi_{ij}$, $v_{ij} - \vartheta_{ij}$ decreases monotonically as $v_{ij}$ increases. Thus from (10)

$$\left( \sum_{i=1}^{n} \sum_{j=1}^{m} W_i \right) \leq \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} s_{ij}}{f_{min}} \geq \frac{\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} s_{ij}}{f_{min}}$$

where $f_{min} \leq \min_{i,j \in V} f_{ij}$ is a positive constant known from Assumption 2.

So with $\vartheta_1$, $e_{ij}$ and $\xi_{ij}$ will reach the set within a time $T_2$ which satisfies

$$T_2 \leq \frac{V_2(t_0) - \vartheta_1}{d_{e_3}}$$

(40)

After reaching the set, $e_{ij}$ and $\xi_{ij}$ will stay in this set forever because $V_2 < 0$ in this set, and it will take a time less than $3 = \frac{1}{k_2(1-\kappa)}$ to arrive at the origin. Thus in this case the system will be finite time stable with $T$ satisfying (23). In summary, the closed-loop system is globally finite-time convergent. This completes the proof. □

Remark 2 From (25) and (36) we know $k_2$ and $d_{e_3}$ will be larger if we choose larger controller parameters $c_1$ and $c_2$. Since $\kappa$ is a constant, thus from (18) and (22) we know (23) can be adjusted arbitrarily small, but it will result in larger control effort.

Remark 3 From the proof of Theorem 1 we can see that local finite-time stability and global asymptotical stability yield global finite-time stability.

3.3 Transient Performance Analysis

As mentioned in the introduction, for adaptive control of nonlinear systems it is very difficult to establish exponential convergence rate. Nevertheless we explore the transient performance of the resulting closed-loop adaptive control system here. The following theorem is established.

Theorem 2 Tracking errors $e_{ij}$ and virtual control errors $\xi_{ij}$ converge to the origin with a speed faster than an exponential rate when $e_{ij} \in \Xi$ and $\xi_{ij} \in \Xi$.

Proof: Based on (34) which holds when $(e_{ij}, \xi_{ij}) \in \Xi$, we get $V_2 \leq -k_2 V_2^2 \leq 0$. Thus $V_2 \leq V_2(t_0)$, and we can find a positive
From (38) we have a constant $k_p$ such that
\[ V_2 + k_p V_2 \leq 0 \]
(40)
where $k_p = k_p^{1/(2n)} V_2(t_0)^{1/(2n)}$. Therefore
\[ V_2(t) \leq V_2(t_0)e^{-k_p \epsilon t} \]
(41)
From (37) we get $V_2 \geq \frac{\zeta}{4n^2 \eta_{\max}} \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}^2$. Thus the tracking errors satisfy
\[ \|e(t)\|^2 = \sum_{j=1}^{m} \sum_{i=1}^{n} e_{ij}(t)^2 \leq \frac{4n^2 \eta_{\max}}{\zeta} V_2(t_0) e^{-k_p \epsilon t} \]
(42)
From (38) we have
\[ \sum_{j=1}^{m} \sum_{i=1}^{n} \left| e_{ij} - e_{ij}' \right|^4 \leq \vartheta_0 V_2(t_0) e^{-k_p \epsilon t} \]
(43)
where $\vartheta_0 = \frac{2(1 - 2 - \vartheta_0) \sigma + 2}{2\sigma - 1}$. Based on (6), (42) and (43), we get
\[ |v_{ij}| \leq |v_{ij} - v_{ij}'| + |v_{ij}'| \leq \vartheta_1 e^{-k_1 \epsilon t} + \vartheta_2 e^{-k_2 \epsilon t} \]
(44)
where $\vartheta_1 = (\vartheta_0 V_2(t_0))^{\frac{2\sigma - 1}{\sigma - 1}}$, $k_1 = k_p^{2\sigma - 1/2} - 1/2$, $k_2 = k_p g$, $\vartheta_2 = c_1 \left( \frac{4n^2 \eta_{\max}}{\zeta} V_2(t_0) \right)^{\frac{\sigma}{2}}$. From (44) we get $|v_{ij}| \leq \vartheta_1 + \vartheta_2$, and $|v_{ij}'| < \vartheta_2$. According to the differential mean value theorem, there exists a $\psi$ with $|\psi| < \vartheta_1 + \vartheta_2$, such that $\left| v_{ij}^{1/g} - v_{ij}'^{1/g} \right| = \frac{1}{g} |\psi|^{1/g} |v_{ij} - v_{ij}'|$. Since function $y = |x|^{1/g}$ is monotonously decreasing or increasing when $x < 0$ or $x \geq 0$ respectively, thus there exists a positive constant $k_3$, such that
\[ |v_{ij}^{1/g} - v_{ij}'^{1/g}| < k_3 |v_{ij} - v_{ij}'| \]
(45)
where $k_3 = \frac{1}{g} (\vartheta_1 + \vartheta_2)^{1+\frac{1}{g}}$. Then from Lemma 3, (43) and (45) we obtain
\[ \|\xi(t)\|^2 \leq \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}(t)^2 \leq \sum_{j=1}^{m} \sum_{i=1}^{n} k_3^2 |v_{ij} - v_{ij}'|^2 \leq k_3^2 \sqrt{mn} \left( \sum_{j=1}^{m} \sum_{i=1}^{n} |v_{ij} - v_{ij}'|^4 \right)^{\frac{1}{4}} \leq k_3^2 \sqrt{mn} \left( \sum_{j=1}^{m} \sum_{i=1}^{n} |v_{ij} - v_{ij}'|^{4+\frac{2}{2}} \right)^{\frac{1}{4+\frac{2}{2}}} \leq k_3 e^{-k_p \epsilon t} \]
(46)
where $k_3 = k_3^2 \sqrt{mn} \left( \vartheta_0 V_2(t_0) \right)^{\frac{4+\frac{2}{2}}{4+\frac{2}{2}}}$. This completes the proof. 

Remark 4

- From (42) and (46) we know the tracking errors converge with exponential rate at least and will converge to the origin within finite time if $(v_{ij}, \xi_{ij}) \in \Xi$.

- If the upper bound of $\zeta$ is unknown, then from (22) we can still choose the control parameters $c_1$ and $c_2$ by incorporating $|\dot{v}| \leq V_2(t_0) \sum_{j=1}^{m} \sum_{i=1}^{n} \xi_{ij}^2 + \vartheta_1^2 + \vartheta_2^2$. This will ensure that $v_{ij}$ and $\xi_{ij}$ converge to $\Xi$ and then converge to the origin within finite time. However this is not a global results since the initial conditions of system states must be incorporated into the controllers.

In order to consider the rendezvous problem, we now define the following set of error variables
\[ z_{ij} = \sum_{k \in N_i} a_{ik} (q_{ij} - q_{kj} - \delta_{ij} + \delta_{kj}) + b_i (q_{ij} - \delta_{ij} - Y_j) \]
(47)
where $Y = \{Y_1, \ldots, Y_m\}$ is a rendezvous location which is only available to part of the agents. Let $z_j = [z_{ij}, \ldots, z_{nj}]^T$, then we have
\[ z_j = (L + B) (z_j - Y_j) \]
(48)
where $z_j = [z_{ij}, \ldots, z_{nj}]^T$, $z_j = [z_{ij} - \delta_{ij}, \ldots, q_{nj} - \delta_{nj}]^T$ and $Y_j = [Y_{ij}, \ldots, Y_{nj}]^T \in R^n$. Define a Lyapunov function
\[ V_4 = \frac{1}{2} z^T (L + B)^{-1} z \]
(49)
where $z = [z_1^T, \ldots, z_m^T]^T$. Taking the derivative of $V_4$ yields
\[ V_4 = z^T v = \sum_{i=1}^{m} \sum_{j=1}^{m} z_{ij} v_{ij} \]
(50)
where $v = [v_{ij}, \ldots, v_{nj}]^T$ with $v_{ij} = [v_{ij}, \ldots, v_{nj}]$, $j = 1, \ldots, m$. Choose the virtual control of $v_{ij}$ as
\[ v_{ij}^* = -c_3 z_{ij} \]
(50)
Then we get

$$
\dot{V}_4 = -c_3 \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij}^{1+\gamma} + \sum_{i=1}^{n} \sum_{j=1}^{m} (v_{ij} - \hat{v}_{ij})
$$

Also from (49) we know

$$
z^T z = \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij}^2 \geq \frac{2}{\lambda_{\text{max}} \left(I_m \otimes (L + B)\right)} \dot{V}_4
$$

(51)

Define a new Lyapunov function

$$
V_5 = V_4 + \sum_{i=1}^{n} \Delta_i
$$

(52)

where

$$
\Delta_i = \frac{1}{2 - \gamma} \sum_{j=1}^{m} f_{ij} \int_{v_{ij}}^{v_{ij}^*} (s^{1/\gamma} - v_{ij}^{1/\gamma})^{2-\gamma} ds
$$

(53)

and design the controller $\tau_i$ as

$$
\tau_i = -G_i(q_i, v_i) \hat{\theta}_i - c_2 \chi_i^*
$$

(54)

where $\chi_{ij} = v_{ij}^{1/\gamma} - v_{ij}^{1/\gamma}$ and $\chi_i^* = [\chi_{i1}^{2g-1}, ..., \chi_{im}^{2g-1}]$. Then

$$
\dot{V}_5 \leq -k_1 \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij}^{4\gamma-1} - k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \chi_{ij}^{4\gamma-1}
$$

$$
+ \frac{1}{2 - \gamma} \sum_{i=1}^{n} \chi_i^T G_i(q_i, v_i) \hat{\theta}_i
$$

(55)

Thus by defining

$$
V_6 = V_5 + \frac{1}{2} \sum_{i=1}^{n} \bar{\theta}_i^T \Gamma^{-1} \bar{\theta}_i
$$

(56)

and choosing the parameter update law

$$
\dot{\bar{\theta}}_i = \text{Proj} \left( \gamma_i(k), \bar{\theta}_i \right), \quad k = 1, ..., l
$$

(57)

where $\gamma_i = - \frac{1}{2 - \gamma} \Gamma_i G_i(q_i, v_i)^T \chi_i$ and $\gamma_i(k)$ is the $k$th element of $\gamma_i$, we get

$$
\dot{V}_6 \leq -k_1 \sum_{j=1}^{m} \sum_{i=1}^{n} z_{ij}^{4\gamma-1} - k_2 \sum_{j=1}^{m} \sum_{i=1}^{n} \chi_{ij}^{4\gamma-1}
$$

(58)

Then from (58) the following corollary is established.

**Corollary 1** Consider multi-agent systems (2) under the control of distributed adaptive controllers (50) and (54) with parameter estimators (57). If Assumptions 1 and 2 are satisfied, then all the positions of the agents converge to $Y$ with specified configurations $\delta_{ij}$ within finite time $T$ satisfying

$$
T \leq \frac{\bar{q}_i^{1-n}}{k_{\gamma}(1 - \kappa)} + \frac{V_6(t_0) - \bar{q}_2}{d_{\theta_0}}.
$$

(59)

**Proof**: Based on (58) and following the similar procedure from (24) to (39), the conclusion can be established.

4 Simulation Illustrations

In this section we use four cylindrical robot arms chosen from section 3.2 in Lewis et al. (2004) to demonstrate the effectiveness of our proposed finite-time control schemes. The robot parameters are: $\hat{M} = [m_1, 0; 0, m_2], \hat{C} = [\cos(q_1), c_1 q_2; c_2 q_2, \sin(q_2)]$, $D = [d_1, 0; 0, d_2]$ with $q = [q_1, q_2]^T$, $m_1 = m_2 = 5$, $c_1 = c_2 = 2$, $d_1 = d_2 = 3$. The control parameters are chosen as $c_1 = 5$, $c_2 = 13$, $c_{11} = c_{31} = 3$, $c_{12} = c_{22} = c_{32} = 10$, $\Gamma_1 = \Gamma = I_4$. The initial values of the estimates are selected as 60% of their true values, respectively. The connection graph is shown in Fig. 1. The initial positions of the robots are at $(20, 54)$, $(4, 20)$, $(44, 1)$ and $(-2, -29)$ respectively. Fig. 2 shows the positions of the robots. For rendezvous seeking, Fig. 1 is also the connection graph and only robot 1 has access to the rendezvous point, which is given to be $(43, 65)$. The initial positions are at $(0, 4)$, $(4, 20)$, $(14, 1)$ and $(12, -29)$. The results on positions of the robots are shown in Fig. 3.

To see the effects of control parameters on the convergence time, we consider two group of parameters: $c_1 = 20, c_2 = 5$ and $c_1 = 5, c_2 = 5$, respectively. In Fig. 4 and Fig. 5, $\|e(t)\|$ and the torque of Robot 1 are presented. These results confirm our discussions in Remark 2.

We also make a comparison on convergence time between our proposed scheme and a typical non-finite-time based adaptive scheme that was proposed in Mei et al. (2011), where the controller is given as

$$
v_i^* = -c_1 e_i, \quad \tau_i = -c_2 (v_i - v_i^*) + Y_i(q_i, v_i) \hat{\Theta}_i,
$$

$$
\hat{\Theta}_i = -\Lambda_i Y_i(q_i, v_i)^T e_i
$$

(60)

where $v_i^*$ is the virtual control for the velocity $v_i$, $\hat{\Theta}_i$ is the estimate of unknown system parameter $\Theta_i$, $e_i$ is the position error defined similarly to (5). Both controllers are applied to the same group of robots with the same unknown parameters mentioned above for leaderless consensus. Fig. 6 shows the comparison results with the same controller parameters $c_1 = 5$ and $c_2 = 5$, the same initial positions and the same initial parameter estimates. It can be observed that $\|e(t)\|$ converge to 0.0001 in 0.06s with our proposed controllers whereas it takes 0.25s to converge to 0.0024 when the controller in Mei et al. (2011) is used.

5 Conclusion

In this paper we investigate the finite-time consensus control for a group of nonlinear mechanical systems with parametric
Fig. 1. Undirected graph topology for leaderless consensus and rendezvous seeking.

Fig. 2. Positions of four mechanical robots in leaderless consensus

Fig. 3. Positions of four mechanical robots in location seeking uncertainties. New continuous distributed controllers are proposed for the multi-agent systems. Firstly for the leaderless multi-agent systems, it is shown that the states of the mechanical systems can reach a consensus within finite time. We also prove that with our control schemes all the systems can reach a static rendezvous location in finite time under an directed graph when only part of the agents have accesses to the rendezvous location. Transient performances in terms of convergence rates and time are also analyzed and established.

Fig. 4. Convergence rates with different control parameters for the case of leaderless consensus. Black solid line: $c_1 = 20$, $c_2 = 5$. Green dash line: $c_1 = 5$, $c_2 = 5$.

Fig. 5. Robot 1’s torque with different control parameters for the leaderless consensus. Black lines: $c_1 = 20$, $c_2 = 5$. Green lines: $c_1 = 5$, $c_2 = 5$.

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References

Fig. 6. Convergence time and rates with different controllers for leaderless consensus. Green dash line: non-finite-time controllers in Mei et al. (2011); Black solid line: finite-time controllers


