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INDEPENDENCE TEST FOR HIGH DIMENSIONAL DATA BASED ON REGULARIZED CANONICAL CORRELATION COEFFICIENTS

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This paper proposes a new statistic to test independence between two high dimensional random vectors $X: p_1 \times 1$ and $Y: p_2 \times 1$. The proposed statistic is based on the sum of regularized sample canonical correlation coefficients of $X$ and $Y$. The asymptotic distribution of the statistic under the null hypothesis is established as a corollary of general central limit theorems (CLT) for the linear statistics of classical and regularized sample canonical correlation coefficients when $p_1$ and $p_2$ are both comparable to the sample size $n$. As applications of the developed independence test, various types of dependent structures, such as factor models, ARCH models and a general uncorrelated but dependent case, etc., are investigated by simulations. As an empirical application, cross-sectional dependence of daily stock returns of companies between different sections in the New York Stock Exchange (NYSE) is detected by the proposed test.

1. Introduction. A prominent feature of data collection nowadays is that the number of variables is comparable with the sample size. This type of data poses great challenges because traditional multivariate approaches do not necessarily work, which were established for the case of the sample size $n$ tending to infinity and the dimension $p$ remaining fixed (see [1]). There have been a substantial body of research work dealing with high dimensional data, for example, [2, 4, 7, 9, 10, 12], etc.

The importance of the independence assumption for inference arises in many aspects of multivariate analysis. For example, it is often the case in multivariate analysis that a number of variables can be rationally classified into several mutually exclusive categories. When variables can be grouped in such a way, a natural question is whether there is any significant relationship between the groups of variables. In other words, can we claim that the groups are mutually independent so that further statistical analysis such as classification and testing hypothesis of equality of mean vectors and covariance matrices could be conducted? When the dimension $p$ is fixed, [20] used the likelihood ratio statistic to test independence

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for $k$ sets of normal distributed random variables and one may also refer to Chapter 12 of [1] regarding to this point. Relying on the asymptotic theory of sample canonical correlation coefficients, this paper proposes a new statistic to test independence between two high dimensional random vectors.

Specifically, the aim is to test the hypothesis

$$H_0 : \mathbf{x} \text{ and } \mathbf{y} \text{ are independent; \quad against}$$

$$H_1 : \mathbf{x} \text{ and } \mathbf{y} \text{ are dependent,}$$

where $\mathbf{x} = (x_1, \ldots, x_{p_1})^T$ and $\mathbf{y} = (y_1, \ldots, y_{p_2})^T$. Without loss of generality, suppose that $p_1 \leq p_2$.

It is well known that canonical correlation analysis (CCA) deals with the correlation structure between two random vectors (see Chapter 12 of [1]). Draw $n$ independent and identically distributed (i.i.d.) observations from these two random vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, and group them into $p_1 \times n$ random matrix $\mathbf{X} = (X_{ij})_{p_1 \times n}$ and $p_2 \times n$ random matrix $\mathbf{Y} = (Y_{ij})_{p_2 \times n}$, respectively. CCA seeks the linear combinations $\mathbf{a}^T \mathbf{x}$ and $\mathbf{b}^T \mathbf{y}$ that are most highly correlated, that is, to maximize

$$\gamma = \text{Corr}(\mathbf{a}^T \mathbf{x}, \mathbf{b}^T \mathbf{y}) = \frac{\mathbf{a}^T \Sigma_{xy} \mathbf{b}}{\sqrt{\mathbf{a}^T \Sigma_{xx} \mathbf{a}} \sqrt{\mathbf{b}^T \Sigma_{yy} \mathbf{b}}},$$

where $\Sigma_{xx}$ and $\Sigma_{yy}$ are the population covariance matrices for $\mathbf{x}$ and $\mathbf{y}$, respectively, and $\Sigma_{xy}$ is the population covariance matrix between $\mathbf{x}$ and $\mathbf{y}$. After finding the maximal correlation $r_1$ and associated vectors $\mathbf{a}_1$ and $\mathbf{b}_1$, CCA continues to seek a second linear combination $\mathbf{a}_2^T \mathbf{x}$ and $\mathbf{b}_2^T \mathbf{y}$ that has the maximal correlation among all linear combinations uncorrelated with $\mathbf{a}_1^T \mathbf{x}$ and $\mathbf{b}_1^T \mathbf{y}$. This procedure can be iterated and successive canonical correlation coefficients $\gamma_1, \ldots, \gamma_{p_1}$ can be found.

It turns out that the population canonical correlation coefficients $\gamma_1, \ldots, \gamma_{p_1}$ can be recast as the roots of the determinant equation

$$\det(\Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{xy}^T - \gamma^2 \Sigma_{xx}) = 0.$$  

Regarding this point, one may refer to page 284 of [15]. The roots of the determinant equation above go under many names, because they figure equally in discriminant analysis, canonical correlation analysis and invariant tests of linear hypotheses in the multivariate analysis of variance.

Traditionally, sample covariance matrices $\hat{\Sigma}_{xx}$, $\hat{\Sigma}_{xy}$ and $\hat{\Sigma}_{yy}$ are used to replace the corresponding population covariance matrices to solve the nonnegative roots $\rho_1, \rho_2, \ldots, \rho_{p_1}$ to the determinant equation

$$\det(\hat{\Sigma}_{xy} \hat{\Sigma}_{yy}^{-1} \hat{\Sigma}_{xy}^T - \rho^2 \hat{\Sigma}_{xx}) = 0.$$
where
\[
\hat{\Sigma}_{xx} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})^{T}, \quad \hat{\Sigma}_{xy} = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})^{T}, \\
\hat{\Sigma}_{yy} = \frac{1}{n} \sum_{i=1}^{n} (y_i - \bar{y})(y_i - \bar{y})^{T}, \quad \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i.
\]

However, it is inappropriate to use these types of sample covariance matrices to replace population covariance matrices to test (1.1) in some cases. We demonstrate such an example in Section 6.3.

Therefore, in this paper we instead consider the nonnegative roots \(r_1, r_2, \ldots, r_{p_1}\) of an alternative determinant equation as follows:
\[
\text{det}(A_{xy}A_{yy}^{-1}A_{xy}^{T} - r^2A_{xx}) = 0,
\]
where
\[
A_{xx} = \frac{1}{n} XX^{T}, \quad A_{yy} = \frac{1}{n} YY^{T}, \quad A_{xy} = \frac{1}{n} XY^{T}.
\]

We also call \(A_{xx}, A_{yy}\) and \(A_{xy}\) sample covariance matrices, as in the random matrix community. However, whichever sample covariance matrices are used they are not consistent estimators of population covariance matrices, which is called the “curse of dimensionality,” when the dimensions \(p_1\) and \(p_2\) are both comparable to the sample size \(n\). As a consequence, it is conceivable that the classical likelihood ratio statistic (see [20] and [1]) does not work well in the high dimensional case (in fact, it is not well defined and we will discuss this point in the later section).

Moreover, from (1.4), when \(p_1 < n, p_2 < n\), one can see that \(r^2_1, r^2_2, \ldots, r^2_{p_1}\) are the eigenvalues of the matrix
\[
S_{xy} = A_{xx}^{-1}A_{xy}A_{yy}^{-1}A_{xy}^{T}.
\]
Evidently, \(A_{xx}^{-1}\) and \(A_{yy}^{-1}\) do not exist when \(p_1 > n\) and \(p_2 > n\). For this reason, we also consider the eigenvalues of the regularized matrix
\[
T_{xy} = A_{rx}^{-1}A_{xy}A_{yy}^{-1}A_{xy}^{T},
\]
where \(A_{rx}^{-1} = \left(\frac{1}{n} XX^{T} + tI_{p_1}\right)^{-1}\), \(t\) is a positive constant number and \(I_{p_1}\) is a \(p_1 \times p_1\) identity matrix, and \(A_{yy}^{-1}\) denotes the Moore–Penrose pseudoinverse matrix of \(A_{yy}\). Hence, \(T_{xy}\) is well defined even in the case of \(p_1, p_2 \geq n\). Moreover, \(T_{xy}\) reduces to \(S_{xy}\) when \(p_1, p_2\) are both smaller than \(n\) and \(t = 0\).

We now look at CCA from another perspective. The original random vectors \(x\) and \(y\) can be transformed into new random vectors \(\xi\) and \(\eta\) as
\[
\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
such that
\[
\begin{pmatrix}
A' & 0 \\
0 & B'
\end{pmatrix}
\begin{pmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{yx} & \Sigma_{yy}
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & B
\end{pmatrix} =
\begin{pmatrix}
I_{p_1} & P \\
P' & I_{p_2}
\end{pmatrix},
\]
where \( P = (P_1, 0) \), \( P_1 = \text{diag}(\gamma_1, \ldots, \gamma_{p_1}) \) and
\[
A = \Sigma_{xx}^{-1/2}Q_1, 
B = \Sigma_{yy}^{-1/2}Q_2,
\]
with \( Q_1 : p_1 \times p_1 \) and \( Q_2 : p_2 \times p_2 \) being orthogonal matrices satisfying
\[
\Sigma_{xx}^{-1/2}\Sigma_{xy}\Sigma_{yy}^{-1/2} = Q_1PQ_2.
\]
Hence, testing independence between \( x \) and \( y \) is equivalent to testing independence between \( \xi \) and \( \eta \). The covariance between \( \xi \) and \( \eta \) has the following simple expression
\[
\text{Var}\left(\begin{pmatrix} \xi \\
\eta \end{pmatrix}\right) =
\begin{pmatrix}
I_{p_1} & P \\
P' & I_{p_2}
\end{pmatrix}.
\]
(1.9)

In view of this, if the joint distribution of \( x \) and \( y \) is Gaussian, independence between \( x \) and \( y \) is equivalent to asserting that the population canonical correlations all vanish: \( \gamma_1 = \cdots = \gamma_{p_1} = 0 \). Details can be referred to Chapter 11 of [11]. A natural criteria for this test should be
\[
\sum_{i=1}^{p_1} \gamma_i^2.
\]
As pointed out, \( r_i \) is not a consistent estimator of the corresponding population version \( \gamma_i \) in the high dimensional case. However, fortunately, the classical sample canonical correlation coefficients \( r_1, r_2, \ldots, r_{p_1} \) or its regularized analogous still contain important information so that hypothesis testing for (1.1) is possible although the classical likelihood ratio statistic does not work well in the high dimensional case. This is due to the fact that the limits of the empirical spectral distribution (ESD) of \( r_1, \ldots, r_{p_1} \) under the null and the alternative could be different so that we may use it to distinguish dependence from independence (one may see the next section). Our approach essentially makes use of the integral of functions with respect to the ESD of canonical correlation coefficients. The proposed statistic turns out a trace of the corresponding matrices, that is, \( \sum_{i=1}^{p_1} r_i^2 \). In order to apply it to conduct tests, we further propose two modified statistics by either dividing the total samples into two groups or estimating the population covariance matrix of \( x \) in a framework of sparsity.

In addition to proposing a statistic for testing (1.1), another contribution of this paper is to establish the limit of the ESD of regularized sample canonical correlation coefficients and central limit theorems (CLT) of linear functionals of the classical and regularized sample canonical correlation coefficients \( r_1, r_2, \ldots, r_{p_1} \), respectively. This is of an independent interest in its own right in addition to providing asymptotic distributions for the proposed statistics.

To derive the CLT for linear spectral statistics of classical and regularized sample canonical correlation coefficients, the strategy is to first establish the CLT under the Gaussian case, that is, the entries of \( X \) are Gaussian distributed. In the Gaussian case, the CLT for linear spectral statistics of the matrix \( S_{xy} \) can be linked to that of an \( F \)-matrix, which has been investigated in [22]. We then extend the CLT to general distributions by bounding the difference between the characteristic func-
tions of the respective linear spectral statistics of \( S_{xy} \) under the Gaussian case and non-Gaussian case. To bound such a difference and handle the inverse of a random matrix, we use an interpolation approach and a smooth cutoff function. The approach of developing the CLT for linear spectral statistics of the matrix \( T_{xy} \) is similar to that for \( S_{xy} \), except we first have to develop CLT of perturbed sample covariance matrices in the supplement material [23] for establishing CLT of the matrix \( T_{xy} \) when the entries of \( X \) are Gaussian.

Here, we would point out some works on canonical correlation coefficients under the high dimensional scenario. In the high dimensional case, [19] investigated the limit of the ESD of the classical sample canonical correlation coefficients \( r_1, r_2, \ldots, r_p \) and [13] established the Tracy–Widom law of the maximum of sample correlation coefficients when \( A_{xx} \) and \( A_{yy} \) are Wishart matrices and \( x, y \) are independent.

The remainder of the paper is organized as follows. Section 2 proposes a new test statistic for (1.1) based on large dimensional random matrix theory and contains the main results. Two modified statistics are further provided in Section 3 and estimators for some unknown parameters in the asymptotic mean and variance for the asymptotic distribution are also proposed. Section 4 provides the powers of the test statistics. Two examples as statistical inference of independence test are explored in Section 5. Simulation results for several kinds of dependent structures are provided in Section 6. An empirical analysis of cross-sectional dependence of daily stock returns of companies from two different sections in New York Stock Exchange (NYSE) is investigated by the proposed independence test in Section 7. Some useful lemmas and proofs of all theorems and Propositions 4–5 are relegated to Appendix A while one theorem about the CLT of a sample covariance matrix plus a perturbation matrix is provided in Appendix B. All appendices are given in the supplementary material [23].

2. Methodology and theory. Throughout this paper, we make the following assumptions.

**Assumption 1.** \( p_1 = p_1(n) \) and \( p_2 = p_2(n) \) with \( \frac{p_1}{n} \to c_1 \) and \( \frac{p_2}{n} \to c_2 \), \( c_1, c_2 \in (0, 1) \), as \( n \to \infty \).

**Assumption 2.** \( p_1 = p_1(n) \) and \( p_2 = p_2(n) \) with \( \frac{p_1}{n} \to c_1' \) and \( \frac{p_2}{n} \to c_2' \), \( c_1' \in (0, +\infty) \) and \( c_2' \in (0, +\infty) \), as \( n \to \infty \).

**Assumption 3.** \( X = (X_{ij})_{i,j=1}^{p_1,n} \) and \( Y = (Y_{ij})_{i,j=1}^{p_2,n} \) satisfy \( X = \Sigma_{xx}^{1/2}W \) and \( Y = \Sigma_{yy}^{1/2}V \), where \( W = (w_1, \ldots, w_n) = (W_{ij})_{i,j=1}^{p_1,n} \) consists of i.i.d. real random variables \( \{W_{ij}\} \) with \( EW_{11} = 0 \) and \( E|W_{11}|^2 = 1 \); \( V = (v_1, \ldots, v_n) = (V_{ij})_{i,j=1}^{p_2,n} \) consists of i.i.d. real random variables with \( EV_{11} = 0 \) and \( E|V_{11}|^2 = 1 \); \( \Sigma_{xx}^{1/2} \) and
\(\Sigma_{yy}^{1/2}\) are Hermitian square roots of positive definite matrices \(\Sigma_{xx}\) and \(\Sigma_{yy}\), respectively, so that \((\Sigma_{xx}^{1/2})^2 = \Sigma_{xx}\) and \((\Sigma_{yy}^{1/2})^2 = \Sigma_{yy}\).

**Assumption 4.** \(F^{\Sigma_{xx}} \xrightarrow{D} H\), a proper cumulative distribution function.

**Remark 1.** By the definition of the matrix \(S_{xy}\), the classical canonical correlation coefficients between \(x\) and \(y\) are the same as those between \(w\) and \(v\) when \(w\) and \(\{w_i\}\) are i.i.d., and \(v\) and \(\{v_i\}\) are i.i.d.

We now introduce some results from random matrix theory. Denote the ESD of any \(n \times n\) matrix \(A\) with real eigenvalues \(\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n\) by

\[
F^A(x) = \frac{1}{n} \#\{i : \mu_i \leq x\},
\]

(2.1)

where \(#\{\cdots\}\) denotes the cardinality of the set \(\{\cdots\}\).

When the two random vectors \(x\) and \(y\) are independent and each of them consists of i.i.d. Gaussian random variables, under Assumptions 1 and 3, [19] proved that the empirical measure of the classical sample canonical correlation coefficients \(r_1, r_2, \ldots, r_{p_1}\) converges in probability to a fixed distribution whose density is given by

\[
\rho(x) = \frac{\sqrt{(x - L_1)(x + L_1)(L_2 - x)(L_2 + x)}}{\pi c_1 x(1 - x)(1 + x)}, \quad x \in [L_1, L_2],
\]

(2.2)

and atoms size of \(\max(0, (1 - c_2)/c_1)\) at zero and size \(\max(0, 1 - (1 - c_2)/c_1)\) at unity where \(L_1 = |\sqrt{c_2 - c_2c_1} - \sqrt{c_1 - c_1c_2}|\) and \(L_2 = |\sqrt{c_2 - c_2c_1} + \sqrt{c_1 - c_1c_2}|\). Here, the empirical measure of \(r_1, r_2, \ldots, r_{p_1}\) is defined as in (2.1) with \(\mu_i\) replaced by \(r_i\).

[21] proved that (2.2) also holds for classical sample canonical correlation coefficients when the entries of \(x\) and \(y\) are not necessarily Gaussian distributed. For easy reference, we state the result in the following proposition.

**Proposition 1.** In addition to Assumptions 1 and 3, suppose that \(\{X_{ij}, 1 \leq i \leq p_1, 1 \leq j \leq n\}\) and \(\{Y_{ij}, 1 \leq i \leq p_2, 1 \leq j \leq n\}\) are independent. Then the empirical measure of \(r_1, r_2, \ldots, r_{p_1}\) converges almost surely to a fixed distribution function whose density is given by (2.2).

Under Assumptions 2–4, instead of \(F^{S_{xy}}\), we analyze the ESD, \(F^{T_{xy}}\), of the regularized random matrix \(T_{xy}\) given in (1.6). To this end, define the Stieltjes transform of any distribution function \(G(x)\) by

\[
m_G = \int \frac{1}{x - z} dG(x), \quad z \in \mathbb{C}^+ = \{z \in \mathbb{C}, \Im z > 0\},
\]

where \(\Im z\) denotes the imaginary part of the complex number \(z\).
It turns out that the limit of the empirical spectral distribution (LSD) of $T_{xy}$ is connected to the LSD of $S_1S_2^{-1}$ defined below. Let

$$S_1 = \frac{1}{p_2} \sum_{k=1}^{p_2} w_k w_k^T, \quad S_2 = \frac{1}{n - p_2} \sum_{k=p_2+1}^{n} w_k w_k^T + t \frac{n}{n - p_2} \Sigma^{-1}_{xx},$$

$$y_1 = \frac{\gamma_1}{\gamma_2}, \quad y_2 = \frac{\gamma_1'}{1 - \gamma_2'}.$$

In the definition of $S_2$, we require $n > p_2$. The LSD of $S_2$ and its Stieltjes transform are denoted by $F_{y_2t}$ and $m_{y_2t}(z)$, respectively. Under Assumptions 2–4, from [17] and [16], $m_{y_2t}(z)$ is the unique solution in $\mathbb{C}^+$ to

$$m_{y_2t}(z) = m_{H_t}(z - \frac{1}{1 + y_2m_{y_2t}(z)})$$

where $m_{H_t}(z)$ denotes the Stieltjes transform of the LSD of the matrix $t \frac{n}{n - p_2} \Sigma^{-1}_{xx}$ (one may also see (1.4) in the supplement material [23]). Let $n = (n_1, n_2)$ and $y = (y_1, y_2)$ with $n_1 = p_1$ and $n_2 = n - p_2$. The Stieltjes transforms of the ESD and LSD of the matrix $S_1S_2^{-1}$ are denoted by $m_n(z)$ and $m_y(z)$, respectively, while those of the ESD and LSD of the matrix $W_1^T S_2^{-1} W_1$ are denoted by $m_n(z)$ and $m_y(z)$, respectively, where $W_1 = (w_1, w_2, \ldots, w_{p_2})$. Observe that the spectral of $S_1S_2^{-1}$ and $W_1^T S_2^{-1} W_1$ are the same except zero eigenvalues and this leads to

$$m_y(z) = -\frac{1 - y_1}{z} + y_1 m_y(z).$$

We are now in a position to state the LSD of $T_{xy}$.

**Theorem 2.1.** In addition to Assumptions 2–4, suppose that $\{X_{ij}, 1 \leq i \leq p_1, 1 \leq j \leq n\}$ and $\{Y_{ij}, 1 \leq i \leq p_2, 1 \leq j \leq n\}$ are independent.

(a) If $\gamma_2' \in (0, 1)$, then the ESD, $F_{y_2t}^T(\lambda)$, converges almost surely to a fixed distribution $\tilde{F}^T(\frac{\lambda}{q(1-\lambda)})$ with $q = \frac{\gamma_1'}{1 - \gamma_2'}$, where $\tilde{F}(\lambda)$ is a nonrandom distribution and its Stieltjes transform $m_\lambda(z)$ is the unique solution in $\mathbb{C}^+$ to

$$m_\lambda(z) = -\int \frac{dF_{y_2t}^T(1/\lambda)}{\lambda(1 - y_1 - y_1zm_\lambda(z)) - z}. $$

(b) If $\gamma_2' \in [1, \infty)$, then $F_{y_2t}^T(\lambda)$, converges almost surely to a fixed distribution $\tilde{G}(\lambda - t)$ where $\tilde{G}(\lambda)$ is a nonrandom distribution and its Stieltjes transform satisfies the equation

$$m_\tilde{G}(z) = \int \frac{dH(\lambda)}{\lambda(1 - c_1' - c_1'zm_\tilde{G}(z)) - z}.$$
REMARK 2. Indeed, taking $t = 0$ in (2.5) recovers [19]'s result (one may refer to the result of $F$ matrix in [5]).

Let us now introduce the test statistic. Under Assumptions 1 and 3, behind our test statistic is the observation that the limit of $F_{Sxy}(x)$ can be obtained from (2.2) when $x$ and $y$ are independent, while the limit of $F_{Sxy}(x)$ could be different from (2.2) when $x$ and $y$ have correlation. For example, if $y = \Sigma_1 w$ and $x = \Sigma_2 w$ with $p_1 = p_2$ and both $\Sigma_1$ and $\Sigma_2$ being invertible, then

$$ S_{xy} = I, $$

which implies that the limit of $F_{Sxy}(x)$ is a degenerate distribution. This suggests that we may make use of $F_{Sxy}(x)$ to construct a test statistic. Thus, we consider the following statistic:

$$ \int \phi(x) dF_{Sxy}(x) = \frac{1}{p_1} \sum_{i=1}^{p_1} \phi(r_i^2). $$

(2.7)

A perplexing problem is how to choose an appropriate function $\phi(x)$. For simplicity, we choose $\phi(x) = x$ in this work. That is, our statistic is

$$ S_n = \int x dF_{Sxy}(x) = \frac{1}{p_1} \sum_{i=1}^{p_1} r_i^2. $$

(2.8)

Indeed, extensive simulations based on Theorems 2.2 and 2.3 below have been conducted to help select an appropriate function $\phi(x)$. We find that other functions such as $\phi(x) = x^2$ does not have an advantage over $\phi(x) = x$.

In the classical CCA, the maximum likelihood ratio test statistic for (1.1) with fixed dimensions is

$$ \text{MLR}_n = \sum_{i=1}^{p_1} \log(1 - r_i^2) $$

(2.9)

(see [20] and [1]). That is, $\phi(x)$ in (2.7) takes $\log(1 - x)$. Note that the density $\rho(x)$ has atom size of $\max(0, 1 - (1 - c_2)/c_1)$ at unity by (2.2). Thus, the normalized statistic $\text{MLR}_n$ is not well defined when $c_1 + c_2 > 1$ [because $\int \log(1 - x^2) \rho(x) \, dx$ is not meaningful]. In addition, even when $c_1 + c_2 \leq 1$, the right end point of $\rho(x)$, $L_2$, can be equal to one so that some sample correlation coefficients $r_i$ are close to one, for example, $L_2 = 1$ when $c_1 = c_2 = 1/2$. This in turns causes a big value of the corresponding $\log(1 - r_i^2)$. Therefore, $\text{MLR}_n$ is not stable and this phenomenon is also confirmed by our simulations.

Under Assumptions 2–4, we substitute $T_{xy}$ for $S_{xy}$ and use the statistic

$$ T_n = \int x dF_{Txy}(x). $$

(2.10)
We next establish the CLTs of the statistics (2.7) and (2.10). To this end, write
\begin{align}
G^{(1)}_{p_1, p_2}(\lambda) &= p_1(F_{S_{xy}}(\lambda) - F_{c_1c_2n}(\lambda)), \\
G^{(2)}_{p_1, p_2}(\lambda) &= p_1(F_{T_{xy}}(\lambda) - F_{c'_1c'_2n}(\lambda)),
\end{align}
where $F_{c_1c_2n}(\lambda)$ and $F_{c'_1c'_2n}(\lambda)$ are obtained from $F_{c_1c_2}(\lambda)$ and $F_{c'_1c'_2}(\lambda)$ with $c_1, c_2, c'_1, c'_2$ and $H$ replaced by $c_1n = p_1 n, c_2n = p_2 n, c'_1n = p_1 n, c'_2n = p_2 n$, and $F_{S_{xy}}$ and $F_{T_{xy}}$, respectively. The density of $F_{c_1c_2}(\lambda)$ can be obtained from $\rho(x)$ in (2.2) while the density of $F_{c'_1c'_2}(\lambda)$ can be recovered from (2.5). We renormalize (2.7) and (2.10) as
\begin{align}
\int \phi(\lambda) dG^{(1)}_{p_1, p_2}(\lambda) &= p_1 \left( \int \phi(\lambda) dF_{S_{xy}}(\lambda) - \int \phi(\lambda) dF_{c_1c_2n}(\lambda) \right), \\
\int \phi(\lambda) dG^{(2)}_{p_1, p_2}(\lambda) &= p_1 \left( \int \phi(\lambda) dF_{T_{xy}}(\lambda) - \int \phi(\lambda) dF_{c'_1c'_2n}(\lambda) \right).
\end{align}
Also, let
\begin{align}
\bar{y}_1 &= \frac{c_1}{1 - c_2} \in (0, +\infty), \quad \bar{y}_2 := \frac{c_1}{c_2} \in (0, 1), \\
h &= \sqrt{\bar{y}_1 + \bar{y}_2 - \bar{y}_1\bar{y}_2}, \quad a_1 = \frac{(1 - h)^2}{(1 - \bar{y}_2)^2}, \quad a_2 = \frac{(1 + h)^2}{(1 - \bar{y}_2)^2}, \\
g_{\bar{y}_1, \bar{y}_2}(\lambda) &= \frac{1 - \bar{y}_2}{2\pi \lambda(\bar{y}_1 + \bar{y}_2\lambda)} \sqrt{(a_2 - \lambda)(\lambda - a_1)}, \quad a_1 < \lambda < a_2.
\end{align}

**Theorem 2.2.** Let $\phi_1, \ldots, \phi_s$ be functions analytic in an open region in the complex plane containing the interval $[a_1, a_2]$. In addition to Assumptions 1 and 3, suppose that
\begin{equation}
EW^{4}_{11} = 3.
\end{equation}
Then, as $n \to \infty$, the random vector
\begin{equation}
\left( \int \phi_1(\lambda) dG^{(1)}_{p_1, p_2}(\lambda), \ldots, \int \phi_s(\lambda) dG^{(1)}_{p_1, p_2}(\lambda) \right)
\end{equation}
converges weakly to a Gaussian vector $(X_{\phi_1}, \ldots, X_{\phi_s})$ with mean
\begin{equation}
EX_{\phi_i} = \lim_{r \downarrow 1} \frac{1}{4\pi i} \oint_{|\xi| = 1} f_i \left( \frac{1 + h^2 + 2h\Re(\xi)}{(1 - \bar{y}_2)^2} \right) \left[ \frac{1}{\xi - r^{-1}} + \frac{1}{\xi + r^{-1}} - \frac{2}{\xi + \bar{y}_2/h} \right] d\xi,
\end{equation}
and covariance function

\[
\text{cov}(X_{\phi_i}, X_{\phi_j}) = - \lim_{r \downarrow 1} \frac{1}{4\pi^2} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \left( f_i \left( \frac{1 + h^2 + 2h\Re(\xi_1)}{(1 - \bar{\gamma}_2)^2} \right) \times f_j \left( \frac{1 + h^2 + 2h\Re(\xi_2)}{(1 - \bar{\gamma}_2)^2} \right) \right) \frac{1}{(\xi_1 - r\xi_2)^2} d\xi_1 d\xi_2,
\]

(2.19)

where \( f_i(\lambda) = \phi_i \left( 1 + \left( \frac{1}{(1+c_2)/c_2} \right) \lambda \right) \), \( \Re \) denotes the real part of a complex number and \( r \downarrow 1 \) means that \( r \) approaches to 1 from above.

**Remark 3.** When \( \phi(x) = x \), the mean of the limit distribution in Theorem 2.2 is 0 and the variance is \( \frac{2h^2y_1^2y_2^2}{(y_1+y_2)^2} \). These are calculated in Example 4.2 of [22]. Moreover, assumption (2.16) can be replaced by \( EY_{11}^4 = 3 \) since \( X \) and \( Y \) have an equal status in the matrix \( S_{xy} \).

Before stating the CLT of the linear spectral statistics for the matrix \( T_{xy} \), we make some notation. Let \( r \) be a positive integer and introduce

\[
m_r(z) = \int \frac{dH_t(x)}{(x - z + \varpi(z))^r}, \quad \varpi(z) = \frac{1}{1 + y_2m_{y_2t}(z)},
\]

\[
g(z) = \frac{y_2(m_{y_2t}(-m_y(z)))'}{(1 + y_2m_{y_2t}(-m_y(z)))^2},
\]

\[
s(z_1, z_2) = \frac{1}{1 + y_2m_{y_2t}(z_1)} - \frac{1}{1 + y_2m_{y_2t}(z_2)},
\]

\[
h(z) = \frac{-m_y^2(z)}{1 - y_1m_y^2(z) \int (dF_{y_2t}(x))/(x + m_y(z))^2},
\]

where \( (m_{y_2t}(z))' \) stands for the derivative with respect to \( z \).

**Theorem 2.3.** Let \( \phi_1, \ldots, \phi_s \) be functions analytic in an open region in the complex plane containing the support of the LSD \( \tilde{F}(\lambda) \) whose Stieltjes transform is (2.5). In addition to Assumptions 2–4, suppose that the spectra norm of \( \Sigma_{xx} \) is bounded and

\[
(2.20) \quad EW_{11}^4 = 3.
\]
(a) If \( c'_2 \in (0, 1) \), then the random vector

\[
\left( \int \phi_1(\lambda) dG^{(2)}_{p_1, p_2}(\lambda), \ldots, \int \phi_s(\lambda) dG^{(2)}_{p_1, p_2}(\lambda) \right)
\]

converges weakly to a Gaussian vector \((X_{\phi_1}, \ldots, X_{\phi_s})\) with mean

\[
EX_{\phi_i} = -\frac{1}{2\pi i} \oint \phi_i \left( \frac{qz}{1 + qz} \right)
\]

\[
\times \left( \left( y_1 \int m_y(z)^3 x[x + m_y(z)]^{-3} dF_{y_2 t}(x) \right) \sqrt{1 - y_1 \int m_y(z)^2(x + m_y(z))^{-2} dF_{y_2 t}(x)} \right)^2
\]

\[
+ h(z)(y_2 \sigma^2(-m_y(z)) m_3(-m_y(z))
\]

\[
+ y_2^2 \sigma^4(-m_y(z)) m'_{y_2 t}(-m_y(z)) m_3(-m_y(z))
\]

\[
/ (1 - y_2 \sigma^2(-m_y(z)) m_2(-m_y(z)))
\]

\[
- h(z)(y_2^2 \sigma^3(-m_y(z)) m'_{y_2 t}(-m_y(z)) m_2(-m_y(z))
\]

\[
/ (1 - y_2 \sigma^2(-m_y(z)) m_2(-m_y(z))) \right) dz
\]

and covariance

\[
\text{cov}(X_{\phi_i}, X_{\phi_j})
\]

\[
= -\frac{1}{2\pi^2}
\]

\[
\times \oint_{C_1} \oint_{C_2} \phi_i \left( \frac{qz_1}{1 + qz_1} \right) \phi_j \left( \frac{qz_2}{1 + qz_2} \right)
\]

\[
\times \left( \frac{m'(z_1)m'(z_2)}{(m_y(z_1) - m_y(z_2))^2} - \frac{1}{(z_1 - z_2)^2}
\]

\[
- \frac{h(z_1)h(z_2)}{(-m_y(z_2) + m_y(z_1))^2}
\]

\[
+ \frac{h(z_1)h(z_2)[1 + g(z_1) + g(z_2) + g(z_1)g(z_2)]}{[-m_y(z_2) + m_y(z_1) + s(-m_y(z_1), -m_y(z_2))]^2} \right) dz_1 dz_2.
\]

Here, \( q \) is defined in Theorem 2.1. The contours in (2.22) and (2.23) [two in (2.23), which may be assumed to be nonoverlapping] are closed and are taken in the positive direction in the complex plain, each enclosing the support of \( \tilde{F}(\lambda) \).
(b) If $c_2' \in [1, +\infty)$ $(p_2 \geq n)$, (2.21) converges weakly to a Gaussian vector $(X_{\phi_1}, \ldots, X_{\phi_s})$ with mean

$$EX_{\phi_i} = -\frac{1}{2\pi i} \oint_{C_{\phi_i}} \frac{t^{-1}z}{1 + t^{-1}z} \left( \int_1^{1+s(z)3} \frac{c_1'(1 + \lambda s(z)^3 - 3s(z)3\lambda^2 dH(\lambda)}{(1 - c_1'\int s(z)2\lambda^2(1 + \lambda s(z)^3 - 2\lambda dH(\lambda))^2 d\lambda} \right) dz$$

and

$$\text{cov}(X_{\phi_i}, X_{\phi_j}) = -\frac{1}{2\pi^2} \oint_{C_1} \oint_{C_2} \phi_i \left( \frac{t^{-1}z_1}{1 + t^{-1}z_1} \right) \times \phi_j \left( \frac{t^{-1}z_2}{1 + t^{-1}z_2} \right) \frac{s'(z_1)s'(z_2)}{(s(z_1) - s(z_2))^2} dz_1 dz_2,$$

where $s(z)$ is Stieltjes transform of the LSD of the matrix $\frac{1}{n}W^T \Sigma_{xx}W$. The contours in (2.24) and (2.25) [two in (2.25), which may be assumed to be nonoverlapping] are closed and are taken in the positive direction in the complex plain, each enclosing the support of $\tilde{G}(\lambda)$.

Here, we would like to point out that the idea of testing independence between two random vectors $x$ and $y$ by CCA is based on the fact that the uncorrelatedness between $x$ and $y$ is equivalent to independence between them when the random vector of size $(p_1 + p_2)$ consisting of the components of $x$ and $y$ is a Gaussian random vector. See Wilks [20] and Anderson [1]. For non-Gaussian random vectors $x$ and $y$, uncorrelatedness is not equivalent to independence. CCA may fail in this case. Yet, since Theorems 2.2 and 2.3 hold for non-Gaussian random vectors $x$ and $y$ CCA can be still utilized to capture dependent but uncorrelated $x$ and $y$ such as ARCH type of dependence by considering the high power of their entries. See Section 6.5 for further discussion.

Following [14], condition (2.16) can be removed. However, it will significantly increase the length of this work and we will not pursue it here.

3. Test statistics. Note that the regularized statistic $\int \lambda dG^{(2)}_{p_1,p_2}(\lambda)$ in (2.14) [when $\phi(\lambda) = \lambda$] involves the unknown covariance matrix $\Sigma_{xx}$ through $F_{c_1n,c_2n}(\lambda)$. In order to apply it to conduct tests, one needs to estimate the unknown parameter. It is well known that estimating the population covariance matrix $\Sigma_{xx}$ is very challenging unless it is sparse. [8] and [3] proposed some approaches to estimate the limit of the ESD of $\Sigma_{xx}$ or its moments. However, the convergence rate is not fast enough to offset the order of $p_1$. Indeed, Theorem 1 of [3] implies that the best possible convergence rate is $O_p(\frac{1}{n})$. In view of this, we provide two methods
to deal with the problem. One is to estimate \( \int \lambda \, dF^{c_1,c_2}(\lambda) \) in a framework of sparsity while the other one is to eliminate this unknown parameter by dividing the samples into two groups.

3.1. Plug-in estimator under sparsity. When \( c_2 < 1 \), it turns out that

\[
(3.1) \quad \int \lambda \, dF^{c_1,c_2}(\lambda) = \frac{p_2}{p_1} - \frac{p_2}{p_1} \frac{1}{1 + c'_1 m_{nt}},
\]

where \( m_{nt} \) is a solution to the equation

\[
(3.2) \quad m_{nt} = a_n - \frac{a_n t}{p_1} \text{tr}(a_n^{-1} \Sigma_{xx} + t I)^{-1}
\]

with \( a_n = 1 + c'_1 m_{nt} \) (see the proof of Theorem 3.1). An estimator of \( m_{nt} \) is then proposed as \( \hat{m}_{nt} \) which is a solution to the equation

\[
(3.3) \quad \hat{m}_{nt} = \hat{a}_n - \frac{\hat{a}_n t}{p_1} \text{tr}(\hat{a}_n^{-1} \hat{\Sigma}_{xx} + t I)^{-1},
\]

with \( \hat{a}_n = 1 + c'_1 \hat{m}_{nt} \). Here, we use a thresholding estimator \( \hat{\Sigma}_{xx} \) to estimate \( \Sigma_{xx} \), slightly different from that proposed by [6]. Specifically speaking, suppose that the underlying random variables \( \{X_{ij}\} \) are mean zero and variable one. Then define \( \hat{\Sigma}_{xx} \) to be a matrix whose diagonal entries are all one and the off diagonal entries are \( \hat{\sigma}_{ij} I(|\hat{\sigma}_{ij}| \geq \ell) \) with \( \ell = M \sqrt{\frac{\log p_1}{n}} \) and \( M \) being some appropriate constant (\( M \) will be selected by cross-validation). Here \( \hat{\sigma}_{ij} \) denotes the entry at the \( (i, j) \)th position of sample covariance matrix \( \frac{1}{n} XX^T \). Therefore, the resulting test statistic is

\[
(3.4) \quad p_1 \left( \int \lambda \, dF^{T_{xy}}(\lambda) - \left( \frac{p_2}{p_1} - \frac{p_2}{p_1} \frac{1}{1 + c'_1 \hat{m}_{nt}} \right) \right).
\]

When \( p_2 \geq n \), it turns out that

\[
(3.5) \quad \int \lambda \, dF^{c_1,c_2}(\lambda) = 1 - tm_n^{(1r)},
\]

where \( m_n^{(1r)} \) satisfies the equation

\[
(3.6) \quad m_n^{(1r)} = \frac{1}{p_1} \text{tr}((1 - c'_1 + c'_1 tm_n^{(1r)}) \Sigma_{xx} + t I)^{-1}.
\]

We then propose the resulting test statistic

\[
(3.7) \quad p_1 \left( \int \lambda \, dF^{T_{xy}}(\lambda) - (1 - t \hat{m}_n^{(1r)}) \right),
\]

where \( \hat{m}_n^{(1r)} \) satisfies the equation

\[
(3.8) \quad \hat{m}_n^{(1r)} = \frac{1}{p_1} \text{tr}((1 - c'_1 + c'_1 t \hat{m}_n^{(1r)}) \hat{\Sigma}_{xx} + t I)^{-1}.
\]
THEOREM 3.1. In addition to assumptions in Theorem 2.3, suppose that
\( \text{E}X_{ij}^2 = 1, \sup_{i,j} \text{E}|X_{ij}|^{17} < \infty \) for all \( i \) and \( j \) and that
\[
(3.9) \quad s_o(p_1) \left( \frac{\log p_1}{n} \right)^{(1-q)/2} \to 0,
\]
where \( \sum_{i \neq j} |\sigma_{ij}|^q = s_o(p_1) \) with \( 0 \leq q < 1 \).

(a) If \( c'_2 < 1 \), then \( p_1 \left( \int \lambda \text{d}F^{T_{x,y}}(\lambda) - \left( \frac{p_2}{p_1} - \frac{1}{p_1 + 1 + \hat{c}_1 \hat{m}_n} \right) \right) \) converges weakly to a normal distribution with the mean and variance given in (2.22) and (2.23) with \( \phi(\lambda) = \lambda \).

(b) If \( c'_2 \geq 1 \), then \( p_1 \left( \int \lambda \text{d}F^{T_{x,y}}(\lambda) - (1 - t_{\hat{m}_n}^{(1)}) \right) \) converges weakly to a normal distribution with the mean and variance given in part (b) of Theorem 2.3 with \( \phi(\lambda) = \lambda \).

We demonstrate an example of sparse covariance matrices in the simulation parts, satisfying the sparse condition (3.9).

3.2. Strategy of dividing samples. If (3.9) is not satisfied, we then propose a strategy of dividing the total samples into two groups. Specifically speaking, we divide the \( n \) samples of \((x, y)\) into two groups, respectively, that is,
\[
\text{Group 1: } X^{(1)} = (x_1, x_2, \ldots, x_{[n/2]}), \quad Y^{(1)} = (y_1, y_2, \ldots, y_{[n/2]})
\]
and
\[
\text{Group 2: } \quad X^{(2)} = (x_{[n/2]+1}, x_{[n/2]+2}, \ldots, x_n), \quad Y^{(2)} = (y_{[n/2]+1}, y_{[n/2]+2}, \ldots, y_n),
\]
where \([n/2]\) is the largest integer not greater than \( n/2 \). When \( n \) is odd, we discard the last sample. However, if the above strategy of dividing samples into two groups is directly used, then the asymptotic means of the resulting statistic [the difference between the statistics in (2.14) obtained from two subsamples] are always zero in both null hypothesis and alternative hypothesis due to similarity of two groups so that the power of the test statistic is very low. This is also confirmed by simulations. Therefore, we further propose its modified version as follows.

For \( Y^{(2)} \) in Group 2, we extract a sub-data \( \tilde{Y}^{(2)} \), that is,
\[
\tilde{Y}^{(2)} = (\tilde{y}_{[n/2]+1}, \tilde{y}_{[n/2]+2}, \ldots, \tilde{y}_n),
\]
where \( \tilde{y}_j \) consists of the first \([p_2/2]\) components of \( y_j \), for all \( j = [n/2] + 1, [n/2] + 2, \ldots, n \). We use \( \tilde{Y}^{(2)} \) to form a new group
\[
\text{Modified Group 2: } \quad X^{(2)} = (x_{[n/2]+1}, x_{[n/2]+2}, \ldots, x_n), \quad \tilde{Y}^{(2)} = (\tilde{y}_{[n/2]+1}, \tilde{y}_{[n/2]+2}, \ldots, \tilde{y}_n).
\]
For Group 1, it follows from Theorem 2.3 that

\begin{equation}
\int \lambda \cdot d P_1(T_{xy}^{(1)}(\lambda) - F^{2c_{1n},2c_{2n}}(\lambda)) \xrightarrow{d} Z_1,
\end{equation}

where \( T_{xy}^{(1)} \) is obtained from \( T_{xy} \) with \( X \) and \( Y \) replaced by \( X^{(1)} \) and \( Y^{(1)} \), respectively, and \( Z_1 \) is a normal random variable with mean and variance given in Theorem 2.3 with \( c'_{1} \) and \( c'_{2} \) replaced by \( 2c'_{1} \) and \( 2c'_{2} \), respectively, and \( \phi(\lambda) = \lambda \).

Similarly, with Modified Group 2, by Theorem 2.3

\begin{equation}
\int \lambda \cdot d P_1(T_{xy}^{(2)}(\lambda) - F^{2c'_{1n},c'_{2n}}(\lambda)) \xrightarrow{d} Z_2,
\end{equation}

where \( T_{xy}^{(2)} \) is \( T_{xy} \) with \( X \) and \( Y \) replaced by \( X^{(2)} \) and \( Y^{(2)} \), respectively, and \( Z_2 \) is a normal random variable with the mean and variance given in Theorem 2.3 with \( \phi(\lambda) = \lambda \) and \( c'_{1} \) replaced by \( 2c'_{1} \).

We next investigate the relation between

\begin{equation}
\int \lambda \cdot d F^{2c'_{1n},2c_{2n}}(\lambda)
\end{equation}

and

\begin{equation}
\int \lambda \cdot d F^{2c_{1n},c'_{2n}}(\lambda),
\end{equation}

and then calculate some difference between the two statistics in (3.12) and (3.13) in order to eliminate the unknown parameters \( \int \lambda \cdot d F^{2c'_{1n},2c_{2n}}(\lambda) \) and \( \int \lambda \cdot d F^{2c_{1n},c'_{2n}}(\lambda) \).

When \( c'_{2} < 1/2 \), we have

\begin{equation}
\int \lambda \cdot d F^{2c'_{1n},2c_{2n}}(\lambda) = \frac{p_2}{p_1} - \frac{p_2}{p_1} \frac{1}{1 + 2c'_{1n}m_{nt}},
\end{equation}

where \( m_{nt} \) is obtained from \( m_{nt} \) satisfying (3.2) with \( c'_{1n} \) replaced by \( 2c'_{1n} \). On the other hand,

\begin{equation}
\int \lambda \cdot d F^{2c_{1n},c'_{2n}}(\lambda) = \frac{p_2/2}{p_1} - \frac{p_2/2}{p_1} \frac{1}{1 + 2c'_{1n}m_{nt}}.
\end{equation}

It follows that

\begin{equation}
\int \lambda \cdot d F^{2c'_{1n},2c_{2n}}(\lambda) = 2 \int \lambda \cdot d F^{2c_{1n},c'_{2n}}(\lambda).
\end{equation}

When \( \lfloor p_2/2 \rfloor > \lfloor n/2 \rfloor \), we have

\begin{equation}
\int \lambda \cdot d F^{2c'_{1n},2c_{2n}}(\lambda) = \int \lambda \cdot d F^{2c'_{1n},c'_{2n}}(\lambda) = 1 - t\bar{m}_{n}^{(1t)},
\end{equation}

where \( \bar{m}_{n}^{(1t)} \) is \( m_{n}^{(1t)} \) satisfying (3.6) with \( c'_{1n} \) replaced by \( 2c'_{1n} \).

The last case is \( \lfloor p_2/2 \rfloor \leq \lfloor n/2 \rfloor \) and \( c'_{2} \geq 1/2 \). For this case, if we still consider Group 1 and Modified Group 2, then

\begin{equation}
\int \lambda \cdot d F^{2c'_{1n},2c_{2n}}(\lambda) = 1 - t\bar{m}_{n}^{(1t)},
\end{equation}

\begin{equation}
\int \lambda \cdot d F^{2c_{1n},c'_{2n}}(\lambda) = \frac{[p_2/2]}{p_1} - \frac{[p_2/2]}{p_1} \frac{1}{1 + 2c'_{1n}m_{nt}}.
\end{equation}
From the above formulas, it is difficult to figure out the relation between 
\( \int \lambda \cdot dF(2\sigma_1^2 + 2\sigma_2^2) \) and \( \int \lambda \cdot dF(2\sigma_1^2 + 2\sigma_2^2) \) depending on the unknown parameter \( \Sigma \). To overcome this difficulty, we also apply a “sub-data” trick to Group 1. Specifically speaking, consider a Modified Group 1 as follows.

**Modified Group 1:** \( X^{(1)} = (x_1, x_2, \ldots, x_{[n/2]}) \), \( \tilde{Y}^{(1)} = (\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_{[n/2]}) \),

where \( \tilde{y}_k \) consists of the last \([p_2/2]\) components of \( y_k \), that is, the \( i \)th component of \( \tilde{y}_k \) is the \(([p_2/2] + i)\)th component of \( y_k \), for all \( i = 1, 2, \ldots, [p_2/2] \) and \( k = 1, 2, \ldots, [n/2] \). For Modified Group 1, by Theorem 2.3, we have

\[
\int \lambda \cdot dp_1(\tilde{T}_{xy}^{(1)}(\lambda) - F(2\sigma_1^2 + 2\sigma_2^2)) \xrightarrow{d} Z_3, \tag{3.18}
\]

where \( \tilde{T}_{xy}^{(1)} \) is \( T_{xy} \) with \( X \) and \( Y \) replaced by \( X^{(1)} \) and \( \tilde{Y}^{(1)} \), respectively; and \( Z_3 \) is a normal random variable with the mean and variance given in Theorem 2.3 with \( \phi(\lambda) = \lambda \) and \( c'_1 \) replaced by \( 2c'_1 \). Since the unknown parameters in (3.13) and (3.18) are the same the difference between (3.13) and (3.18) can be taken as the modified statistic.

The asymptotic distributions of the three resulting statistics are given in Theorem 3.2.

**Theorem 3.2.** Suppose assumptions in Theorem 2.3 hold.

(a) If \( c'_2 < 1/2 \), the statistic \( \int \lambda \cdot dF(T_{xy}^{(1)}(\lambda) - 2 \int \lambda \cdot dF(T_{xy}^{(2)}(\lambda)) \) converges weakly to a normal distribution with the mean \( (\mu_1 - 2\mu_2) \) and variance \( (\sigma_1^2 + 4\sigma_2^2) \), where \( \mu_1 \) and \( \sigma_1^2 \) are given in (2.22) and (2.23), respectively, with \( c'_1 \), \( c'_2 \) replaced by \( 2c'_1 \), \( 2c'_2 \), respectively, and \( \phi(\lambda) = \lambda \); \( \mu_2 \) and \( \sigma_2^2 \) are given in (2.22) and (2.23), respectively, with \( c'_1 \) replaced by \( 2c'_1 \) and \( \phi(\lambda) = \lambda \).

(b) If \( c'_2 \geq 1 \), the statistic \( \int \lambda \cdot dF(T_{xy}^{(1)}(\lambda) - \int \lambda \cdot dF(T_{xy}^{(2)}(\lambda)) \) converges weakly to a normal distribution with the mean zero and variance \( 2\sigma_2^2 \), where \( \sigma_3^2 \) is given in (2.25) with \( c'_1 \) replaced by \( 2c'_1 \) and \( \phi(\lambda) = \lambda \).

(c) If \( 1/2 \leq c'_2 < 1 \), the statistic \( \int \lambda \cdot dF(T_{xy}^{(1)}(\lambda) - \int \lambda \cdot dF(T_{xy}^{(2)}(\lambda)) \) converges weakly to a normal distribution with mean zero and variance \( 2\sigma_4^2 \), where \( \sigma_4^2 \) is given in (2.23) with \( c'_1 \) replaced by \( 2c'_1 \) and \( \phi(\lambda) = \lambda \).

**Remark 4.** Unlike using Group 2 of (3.11) although the asymptotic means of the statistics in the cases (b) and (c) are zero under the null hypothesis, they are not necessarily equal to zero under the alternative hypothesis so that the power of the resulting test statistic becomes much better.

Theorem 3.2 proposes test statistics which do not involve the unknown parameter \( H \), that is, the LSD of the matrix \( \Sigma_{xx} \). However, their asymptotic means and asymptotic variances contain some terms involving the unknown parameter \( H \). We below provide consistent estimators for such terms appearing in (2.22)–(2.25) in
Theorem 2.3, which, together with Slutsky’s theorem and the dominant convergence theorem, are enough for applications.

We use an estimator developed by [8] for $H$. For easy reference, we briefly state his estimator $\hat{H}_{p_1}$ for $H$ in the following proposition.

**Proposition 2** (Theorem 2 of [8]). In addition to Assumptions 2–4, suppose that the spectra norm of $\Sigma_{XX}$ is bounded. Let $J_1, J_2, \ldots$ be a sequence of integers tending to $\infty$. Let $z_0 \in \mathbb{C}^+$ and $r \in \mathbb{R}^+$ be such that $B(z_0, r) \subset \mathbb{C}^+$, where $B(z_0, r)$ denotes the closed ball of center $z_0$ and radius $r$. Let $z_1, z_2, \ldots$ be a sequence of complex variables with a limit point, all contained in $B(z_0, r)$. Let $\hat{H}_{p_1}$ be the solution of

$$
\hat{H}_{p_1} = \arg \min \max_{j \leq J_n} \left| \frac{1}{\hat{\xi}_p(z_j)} + z_j - \frac{p_1}{n} \int \frac{\lambda dG(\lambda)}{1 + \lambda \hat{\xi}_p(z_j)} \right|,
$$

where $G$ is a probability measure and $\hat{\xi}_p(z)$ is the Stieltjes transform of the ESD of the matrix $\Delta_{xx} = \frac{1}{n} X^T X$. Then we have

$$
\hat{H}_{p_1} \Rightarrow H \quad \text{a.s.}
$$

**Remark 5.** The estimator $\hat{H}_{p_1}$ given in (3.19) is proposed based on the Marčenko–Pastur equation which links the Stieltjes transform of the empirical spectral distribution of the sample covariance matrix to an integral against the population spectral distribution, that is, the LSD $H$ satisfies the Marčenko–Pastur equation

$$
-\frac{1}{\xi(z)} = z - c \int \frac{\lambda dH(\lambda)}{1 + \lambda \xi(z)} \quad \forall z \in \mathbb{C}^+,
$$

where $\xi(z)$ is the limit of $\hat{\xi}_n(z) = \frac{1}{n} \text{tr}(\Delta_{xx} - zI_n)^{-1}$.

El Karoui [8] developed an algorithm for the estimator $\hat{H}_{p_1}$ in (3.19) and we state it below.

(1) A “basis pursuit” for measure space. Instead of searching among all possible probability measures, the search space is restricted to mixtures of certain classes of probability measures in order to deal with (3.19). In other words, first select a “dictionary” of probability measures on the real line and then decompose the estimator on this dictionary, searching for the best coefficients. Hence, the problem can be formulated as

finding the best possible weights

$$
\{\hat{w}_1, \ldots, \hat{w}_K\} \quad \text{with} \quad d\hat{H}_{p_1} = \sum_{i=1}^{K} \hat{w}_i dM_i, \quad \sum_{i=1}^{K} \hat{w}_i = 1, \quad \hat{w}_i \geq 0,
$$

where the $M_i$’s are the measures in the dictionary. A “probability measures” dictionary is given as follows:
1. Point masses \( \{ \delta_{t_k}(x) \}_{k=1}^{K} \), where \( \{ t_k \}_{k=1}^{K} \) is a grid of points.

2. Probability measures that are uniform on an interval: in this case, \( dM_i^{(1)}(x) = I_{[a_i, b_i]}(x) \frac{dx}{(b_i - a_i)} \). 

3. Probability measures that have a linearly increasing (or decreasing) density on an interval \([d_j, h_j]\) and density 0 elsewhere. So, for the increasing case, \( dM_i^{(2)}(x) = I_{[d_j, h_j]}(x) \frac{2(x - d_j)}{(h_j - d_j)^2} \frac{dx}{}, \) and density 0 elsewhere.

(2) A convex optimization problem. Let

\[
e_j = \frac{1}{\hat{s}_n(z_j)} + z_j - \frac{p_1}{n} \int \frac{\lambda \cdot G(\lambda)}{1 + \lambda \hat{s}_n(z_j)}, \quad j = 1, 2, \ldots, J_n,
\]

where \( G(\cdot) \) has the form of

\[
G(x) = \sum_{i=1}^{K_1} w_i \delta_{t_i}(x) + \sum_{i=K_1+1}^{K_2} w_i dM_i^{(1)}(x) + \sum_{i=K_2+1}^{K} w_i dM_i^{(2)}(x),
\]

with all points \( t_k, k = 1, 2, \ldots, K_1 \), intervals \([a_i, b_i], i = K_1 + 1, \ldots, K_2 \) and

\[
\text{intervals } [d_j, h_j], j = K_2 + 1, \ldots, K \text{ being in the interval } [\ell_{p_1}, \ell_1]. \text{ Here, } \ell_{p_1} \text{ and } \ell_1 \text{ are, respectively, the smallest and largest eigenvalues of the sample covariance matrix } A_{xx} = \frac{1}{n}XX^T. \text{ Moreover, } 1 < K_1 < K_2 < K.
\]

The “translation” of the problem (3.19) into a convex optimization problem is

\[
\min_{w_1, \ldots, w_K, u} \quad u
\]

\[
\forall j = 1, \ldots, J_n, \quad -u \leq \Re(e_j) \leq u,
\]

\[
\forall j = 1, \ldots, J_n, \quad -u \leq \Im(e_j) \leq u
\]

subject to \( \sum_{i=1}^{K} w_i = 1 \text{ and } w_i \geq 0, \forall i = 1, 2, \ldots, K. \)

The following proposition provides consistency of the proposed algorithm above.

**Proposition 3** (Corollary 1 of [8]). Assume the same assumptions as in Proposition 2. Call \( \hat{H}_{p_1} \) the solution of (3.19), where the optimization is now over measures which are sums of atoms, the locations of which are restricted to belong to a grid (depending on \( n \)) whose step size is going to 0 as \( n \to \infty \). Then

\[
\hat{H}_{p_1} \Rightarrow H \quad \text{a.s.}
\]

Then the estimator \( \hat{H}_{p_1} \) derived from the algorithm has the form of

\[
d\hat{H}_{p_1}(x) = \sum_{i=1}^{K_1} \hat{w}_i \delta_{t_i}(x) + \sum_{i=K_1+1}^{K_2} \hat{w}_i dM_i^{(1)}(x) + \sum_{i=K_2+1}^{K} \hat{w}_i dM_i^{(2)}(x),
\]

(3.21)
where \( \{ \hat{w}_i, i = 1, 2, \ldots, K \} \) is the solution of the convex optimization problem above. In practice, we follow the implementation details in Appendix of [8] to derive the estimator \( \hat{H}_{p_1}(x) \).

Once the estimator (3.21) of the LSD \( H \) is available, we are in a position to provide estimators for \( m_H(z) \) and \( m_{y_{2t}}(z) \), the Stieltjes transforms of \( H \) and the LSD of the matrix \( S_{2t} \), respectively.

**Proposition 4.** For any \( z \in \mathbb{C}_1 \), the contour specified in Theorem 2.3, let

\[
\hat{m}_H(z) = \sum_{i=1}^{K_1} \frac{\hat{w}_i}{t_i - z} + \sum_{i=K_1+1}^{K_2} \frac{\hat{w}_i}{b_i - a_i} \log \frac{b_i - z}{a_i - z} + \sum_{i=K_2+1}^{K} \frac{2\hat{w}_i}{(h_i - d_i)^2} \left( h_i - d_i + (z - d_i) \log \frac{h_i - z}{d_i - z} \right),
\]

where \( \log \) stands for the corresponding principal branch. The estimator \( \hat{m}_{y_{2t}}(z) \) satisfies the equation

\[
\hat{m}_{y_{2t}}(z) = \hat{m}_{H_t} \left( z - \frac{1}{1 + y_{2n} \hat{m}_{y_{2t}}(z)} \right),
\]

where \( \hat{m}_{H_t}(z) \) is

\[
\hat{m}_{H_t}(z) = -\frac{1}{z} - \frac{t}{(1 - c'_{2n})z} \hat{m}_H \left( \frac{t}{(1 - c'_{2n})z} \right).
\]

Then \( \hat{m}_H(z) \) in (3.22) and \( \hat{m}_{y_{2t}}(z) \) in (3.23) are consistent estimators of \( m_H(z) \) and \( m_{y_{2t}}(z) \), respectively.

**Remark 6.** As to choices of intervals \([a_i, b_i]: i = K_1 + 1, K_1 + 2, \ldots, K_2; \) and \([h_j, d_j]: j = K_2 + 1, K_2 + 2, \ldots, K \) in (3.21), we follow the implementation details provided in Appendix of [8]. Furthermore, choice of \((z_j, \hat{s}_n(z_j))\) in the convex optimization problem, choice of interval to estimate \( H(\cdot) \), and choice of dictionary are all provided in the Appendix of [8].

With consistent estimators for \( m_H(z) \) and \( m_{y_{2t}}(z) \) in Proposition 4, we may further provide consistent estimators for all terms appearing in (2.22)–(2.25).

**Proposition 5.** 1. The estimator for \( m_y(z) \) is

\[
\hat{m}_y(z) = \frac{q_n}{(1 + q_n z)^2} \left( m_{T_{xy}} \left( \frac{q_n z}{1 + q_n z} \right) - 1 - q_n z \right),
\]

where \( y_{1n} = \frac{c_{1n}}{c_{2n}}, \ y_{2n} = \frac{c'_{1n}}{1 - c'_{2n}}, \ q_n = \frac{c'_{2n}}{1 - c'_{2n}} \) and \( m_{T_{xy}}(z) \) is the Stieltjes transform of the matrix \( T_{xy} \).
2. The estimators for $\varpi(z)$ and $s(z_1, z_2)$ are, respectively,

$$\hat{\varpi}(z) = \frac{1}{1 + y_{2n}\hat{m}_{y2}(z)},$$

$$\hat{s}(z_1, z_2) = \frac{1}{1 + y_{2n}\hat{m}_{y2}(z_1)} - \frac{1}{1 + y_{2n}\hat{m}_{y2}(z_2)}.$$

3. The estimators for $m_r(z)$ with $r = 2, 3$ are

$$\hat{m}_2(z) = \hat{m}_2^{(1)}(z - \hat{\varpi}(z)), \quad \hat{m}_3(z) = \frac{1}{2}\hat{m}_2^{(2)}(z - \hat{\varpi}(z)),$$

respectively, where $\hat{m}_2^{(j)}(z)$ is the $j$th derivative of $\hat{m}_2(z)$ with respect to $z$ with $j = 1, 2.$

4. The estimators for $g(z)$ and $h(z)$ are

$$\hat{g}(z) = \frac{y_{2n}(\hat{m}_y^{(1)}(-\hat{m}_y(z)))}{(1 + y_{2n}\hat{m}_y(-\hat{m}_y(z)))^2}, \quad \hat{h}(z) = \frac{-\hat{m}_y^2(z)}{1 - y_{1n}\hat{m}_y^2(z)\hat{m}_y^{(1)}(-\hat{m}_y(z))},$$

respectively, where $\hat{m}_y(z) = -1 - y_{1n} + y_{1n}\hat{m}_y(z)$.

5. The estimators for $\varpi_1 = \int m_y^3(z)x[x + m_y(z)]^{-3}dF_{y2}(x)$ and $\varpi_2 = \int m_y^3(z)[x + m_y(z)]^{-2}dF_{y2}(x)$ are

$$\hat{\varpi}_1 = \hat{m}_y^3(z)\hat{m}_y^{(1)}(-\hat{m}_y(z)) - \frac{1}{2}\hat{m}_y^4(z)\hat{m}_y^{(2)}(-\hat{m}_y(z)),$$

$$\hat{\varpi}_2 = \hat{m}_y^2(z)\hat{m}_y^{(1)}(-\hat{m}_y(z)),$$

respectively.

6. The estimators for $\varpi_3 = \int [1 + \lambda \hat{s}_n(z)]^{-3}\hat{s}_n(z)^3\lambda^2 dH(\lambda)$ and $\varpi_4 = \int \hat{s}_n^2(z) \times \lambda^2[1 + \lambda \hat{s}_n(z)]^{-2}dH(\lambda)$ are

$$\hat{\varpi}_3 = \frac{1}{\hat{s}_n(z)}\hat{m}_H \left( -\frac{1}{\hat{s}_n(z)} \right) - \frac{2}{\hat{s}_n(z)^2}\hat{m}_H^{(1)} \left( -\frac{1}{\hat{s}_n(z)} \right) + \frac{1}{2\hat{s}_n(z)}\hat{m}_H^{(2)} \left( -\frac{1}{\hat{s}_n(z)} \right),$$

$$\hat{\varpi}_4 = 1 - \frac{2}{\hat{s}_n(z)}\hat{m}_H \left( -\frac{1}{\hat{s}_n(z)} \right) + \frac{1}{\hat{s}_n(z)}\hat{m}_H^{(1)} \left( -\frac{1}{\hat{s}_n(z)} \right),$$

respectively, where $\hat{s}_n(z)$ is defined as $\hat{s}_n(z) = \frac{1}{n}\text{tr}(A_{xx} - zI_n)^{-1}$ with $A_{xx} = \frac{1}{n}X^TX$.

All estimators listed above are consistent for the corresponding unknown parameters.

The proofs of Propositions 4 and 5 are provided in Appendix A of [23].
4. The power under local alternatives. This section is to evaluate the power of $S_n$ or $T_n$ under a kind of local alternatives. Consider the alternative hypothesis

$$H_1 : \mathbf{x} \text{ and } \mathbf{y} \text{ are dependent},$$

satisfying condition (4.1) below. Draw $n$ samples from such alternatives $\mathbf{x}$ and $\mathbf{y}$ to form the respective analogues of (1.5) and (1.6) and denote them by $S$ and $T$, respectively. Suppose that the underlying random variables involved in $S_{xy}$, $T_{xy}$ and $S$, $T$ are in the same probability space $(\Omega, P)$.

Recall the definitions of $G^{(i)}_{p_1, p_2}, i = 1, 2$ in (2.11) and (2.12), and let $R^{(i)}_n = \int \lambda dG^{(i)}_{p_1, p_2}$.

**Theorem 4.1.** In addition to assumptions in Theorems 2.2 or 2.3 suppose that for any $M > 0$

$$P(|\text{tr}(S - S_{xy})| \geq M) \rightarrow 1, \quad P(|\text{tr}(T - T_{xy})| \geq M) \rightarrow 1.$$  

Then

$$\lim_{n \rightarrow \infty} P(R^{(i)}_n > z^{(i)}_{1-\alpha} \text{ or } R^{(i)}_n < z^{(i)}_{\alpha} | H_1) = 1,$$

where $z^{(i)}_{1-\alpha}$ and $z^{(i)}_{\alpha}$ are, respectively, $(1 - \alpha)$ and $\alpha$ quantiles of the asymptotic distribution of the statistic $R^{(i)}_n$ under the null hypothesis.

**Remark 7.** For example, one may take $S = (XLX^T)^{-1}XLY^T(YLY^T)^{-1} \times YLY^T$ and $S_{xy} = (XX^T)^{-1}XP_xX^T$ with $L$ being a random matrix and $P_y = Y^T(YY^T)^{-1}Y$. Particularly, if $L = I + ee^T$ with $e = x^2(1, 1, \ldots, 1)$ and $x^2$ having finite moment, then under assumptions in Theorems 2.2 or 2.3 it can be proved that

$$\text{tr}(S - S_{xy}) = O_p(n)$$

satisfying (4.1).

Next, we evaluate the powers of the modified statistics with the dividing-sample method. Draw $n$ samples from alternatives $\mathbf{x}$ and $\mathbf{y}$ to form the respective analogues of $T_{xy}^{(i)}, i = 1, 2, \tilde{T}_{xy}^{(1)}$ and denote them by $T^{(i)}, i = 1, 2, \tilde{T}^{(1)}$, respectively. Let

$$J_n^{(1)} = \int \lambda dF^{(1)}(\lambda) - 2 \int \lambda dF^{(2)}(\lambda),$$

$$J_n^{(2)} = \int \lambda dF^{(1)}(\lambda) - \int \lambda dF^{(2)}(\lambda),$$

$$J_n^{(3)} = \int \lambda d\tilde{F}^{(1)}(\lambda) - \int \lambda dF^{(2)}(\lambda).$$
THEOREM 4.2. In addition to assumptions in Theorem 2.3, suppose that for any $M > 0$,

$$P\left(\left|\operatorname{tr}(T^{(1)}) - 2\operatorname{tr}(T^{(2)}) - (\operatorname{tr}(T_{xy}^{(1)}) - 2\operatorname{tr}(T_{xy}^{(2)})\right) \geq M\right) \rightarrow 1,$$

if $c'_2 < 1/2$;\hspace{1cm}(4.3)

$$P\left(\left|\operatorname{tr}(T^{(1)}) - \operatorname{tr}(T^{(2)}) - (\operatorname{tr}(T_{xy}^{(1)}) - \operatorname{tr}(T_{xy}^{(2)})\right) \geq M\right) \rightarrow 1,$$

if $c'_2 \geq 1$;\hspace{1cm}(4.4)

$$P\left(\left|\operatorname{tr}(\tilde{T}^{(1)}) - \operatorname{tr}(T^{(2)}) - (\operatorname{tr}(T_{xy}^{(1)}) - \operatorname{tr}(T_{xy}^{(2)})\right) \geq M\right) \rightarrow 1,$$

if $1/2 \leq c'_2 < 1$.\hspace{1cm}(4.5)

Then

$$\lim_{n \to \infty} P\left(J_n^{(i)} > z^{(i)}_{1-\alpha} \text{ or } J_n^{(i)} < z^{(i)}_\alpha \mid \mathbb{H}_1\right) = 1, \quad i = 1, 2, 3,$$

where $z^{(i)}_{1-\alpha}$ and $z^{(i)}_\alpha$ are, respectively, $(1 - \alpha)$ and $\alpha$ quantiles of the asymptotic distribution of the statistic $J_n^{(i)}$ under the null hypothesis, $i = 1, 2, 3$.

5. Applications of CCA. This section explores some applications of the proposed test. We consider two examples from multivariate analysis and time series analysis, respectively.

5.1. Multivariate regression test with CCA. Consider the multivariate regression (MR) model as follows:

$$Y = XB + E,$$

(5.1)

where

$$Y = [y_1, y_2, \ldots, y_{p_1}]_{n \times p_1}, \quad X = [1_n, x_1, x_2, \ldots, x_{p_2}]_{n \times p_2},$$

$$B = [\beta_1, \beta_2, \ldots, \beta_{p_1}]_{p_2 \times p_1}, \quad E = [e_1, e_2, \ldots, e_{p_1}]_{n \times p_1},$$

and each of the vectors $y_j$, $x_j$, $e_j$, for $j = 1, 2, \ldots, p_1$ is $n \times 1$ vectors and $\{\beta_i, i = 1, 2, \ldots, p_1\}$ are $p_2 \times 1$ vectors.

Let $A_{xy} = \frac{1}{n}X^TY$ and $A_{xx} = \frac{1}{n}X^TX$. We have the least square estimate of $B$

$$\hat{B} = A_{xx}^{-1}A_{xy}.$$ \hspace{1cm}(5.2)

The most common hypothesis testing is to test whether there exists linear relationship between the two sets of variables (response variables and predictor variables) or the overall regression test

$$\mathbb{H}_0 : B = 0.$$ \hspace{1cm}(5.3)
To test \( H_0 : \mathbf{B} = 0 \), Wilks’ \( \Lambda \) criterion is
\[
\Lambda = \frac{\det(\mathbf{E})}{\det(\mathbf{E} + \mathbf{H})} = \prod_{i=1}^{s} (1 + \lambda_i)^{-1},
\]
where
\[
\mathbf{E} = \mathbf{Y}^T (\mathbf{I} - \mathbf{X}(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T) \mathbf{Y}
\]
and
\[
\mathbf{H} = \mathbf{B}^T (\mathbf{X}^T \mathbf{X}) \mathbf{B};
\]
and \( \{\lambda_i : i = 1, \ldots, s\} \) are the roots of \( \det(\mathbf{H} - \lambda \mathbf{E}) = 0 \), \( s = \min(k, p) \). An alternative form for \( \Lambda \) is to employ sample covariance matrices. That is, \( \mathbf{H} = \mathbf{A}_{yx} \mathbf{A}_{xx}^{-1} \mathbf{A}_{xy} \) and \( \mathbf{E} = \mathbf{A}_{yy} - \mathbf{A}_{yx} \mathbf{A}_{xx}^{-1} \mathbf{A}_{xy} \), so that \( \det(\mathbf{H} - \lambda \mathbf{E}) = 0 \) becomes \( \det(\mathbf{A}_{yx} \mathbf{A}_{xx}^{-1} \mathbf{A}_{xy} - \lambda (\mathbf{A}_{yy} - \mathbf{A}_{yx} \mathbf{A}_{xx}^{-1} \mathbf{A}_{xy})) = 0 \). From Theorem 2.6.8 of [18], we have \( \det(\mathbf{H} - \theta(\mathbf{H} + \mathbf{E})) = \det(\mathbf{A}_{yx} \mathbf{A}_{xx}^{-1} \mathbf{A}_{xy} - \theta \mathbf{A}_{yy}) = 0 \) so that
\[
\Lambda = \prod_{i=1}^{s} (1 + \lambda_i)^{-1} = \prod_{i=1}^{s} (1 - \theta_i) = \frac{\det(\mathbf{A}_{yy} - \mathbf{A}_{yx} \mathbf{A}_{xx}^{-1} \mathbf{A}_{xy})}{\det(\mathbf{A}_{yy})}. 
\]
Evidently, the quantities \( r_i^2 = \theta_i, i = 1, \ldots, s \) are sample canonical correlation coefficients. Therefore, the test statistic (5.4) can be rewritten as
\[
\log \Lambda = \sum_{i=1}^{s} \log(1 - r_i^2).
\]
From this point of view, the multiple regression test is equivalent to the independence test based on canonical correlation coefficients. As stated in the last section, our test statistic \( S_n \) or \( T_n \) can be applied in the MR test.

5.2. Testing for cointegration with CCA. Consider an \( n \)-dimensional vector process \( \{\mathbf{y}_t\} \) that has a first-order error correction representation
\[
\Delta \mathbf{y}_t = -\mathbf{\alpha} \mathbf{\beta}' \mathbf{y}_{t-1} + \mathbf{\epsilon}_t, \quad t = 1, \ldots, T,
\]
where \( \mathbf{\alpha} \) and \( \mathbf{\beta} \) are full rank \( n \times r \) matrices \( (r < n) \) and the \( n \)-dimensional innovation \( \{\mathbf{\epsilon}_t\} \) is i.i.d. with zero mean and positive covariance matrix \( \mathbf{\Omega} \). Select \( \mathbf{\alpha} \) and \( \mathbf{\beta} \) so that the fact that \( |\mathbf{I}_n - (\mathbf{I}_n - \mathbf{\alpha} \mathbf{\beta}')| z = 0 \) implies that either \( |z| > 1 \) or \( z = 1 \) and that \( \mathbf{\alpha}' \mathbf{\beta}' \) is of full rank, where \( \mathbf{\alpha}' \) and \( \mathbf{\beta}' \) are full rank \( n \times (n - r) \) matrices orthogonal to \( \mathbf{\alpha} \) and \( \mathbf{\beta} \). Under these assumptions, \( \{\mathbf{y}_t\} \) is \( I(1) \) with \( r \) cointegration relations among its elements; that is, \( \{\mathbf{\beta}' \mathbf{y}_t\} \) is \( I(0) \). Here, \( I(d) \) denotes integrated of order \( d \).

The goal is to test
\[
H_0 : r = 0 \quad (\mathbf{\alpha} = \mathbf{\beta} = 0); \quad \text{against} \quad H_1 : r > 0;
\]
that is, whether there exists cointegration relationships among the elements of the time series \( \{\mathbf{y}_t\} \).
This cointegration test is equivalent to testing:

\[ H_0 : \Delta y_t \text{ is independent with } \Delta y_{t-1}; \]

\[ H_1 : \Delta y_t \text{ is dependent with } \Delta y_{t-1}. \]

In order to apply canonical correlation coefficients to cointegration test (5.10), we construct random matrices

\[ X = (\Delta y_2, \Delta y_4, \ldots, \Delta y_{2r-2}, \Delta y_{2r}, \ldots, \Delta y_T), \]

\[ Y = (\Delta y_1, \Delta y_3, \ldots, \Delta y_{2r-1}, \Delta y_{2r+1}, \ldots, \Delta y_{T-1}). \]  

6. Simulation results. This section reports some simulated examples to show the finite sample performance of the proposed test.

6.1. Empirical sizes and empirical powers. First, we introduce the method of calculating empirical sizes and empirical powers. Let \( z_{1-\alpha} \) be the 100(1 - \( \alpha \))% quantile of the asymptotic null distribution of the test statistic \( S_n \). With \( K \) replications of the data set simulated under the null hypothesis, we calculate the empirical size as

\[ \hat{\alpha} = \frac{\# \text{ of } S_n^H \geq z_{1-\alpha}}{K}, \]  

where \( S_n^H \) represents the values of the test statistic \( S_n \) based on the data simulated under the null hypothesis.

The empirical power is calculated as

\[ \hat{\beta} = \frac{\# \text{ of } S_n^A \geq \hat{z}_{1-\alpha}}{K}, \]

where \( S_n^A \) represents the values of the test statistic \( S_n \) based on the data simulated under the alternative hypothesis.

In our simulations, we choose \( K = 1000 \) as the number of repeated simulations. The significance level is \( \alpha = 0.05 \).

6.2. Testing independence. Consider the data generating process

\[ x = \Sigma_{xx}^{1/2}w, \quad y = \Sigma_{yy}^{1/2}v, \]

with

(a) \( \Sigma_{xx} = I_{p_1}, \quad \Sigma_{yy} = I_{p_2}; \)

(b) \( \Sigma_{xx} = (\sigma_{SP})_{k,h=1}^{p_1}, \quad \Sigma_{yy} = I_{p_2}; \)

(c) \( \Sigma_{xx} = (\sigma_{AR})_{k,h=1}^{p_1}, \quad \Sigma_{yy} = I_{p_2}; \)

(d) \( \Sigma_{xx} = B' \text{cov}(f_i)B + \Sigma_u, \quad \Sigma_{yy} = I_{p_2}, \)
where

\[
\sigma_{kh}^{SP} = \begin{cases} 
1, & k = h; \\
\theta, & k = 1; h = 2, 3, \ldots, \left[p_1^{1/3}\right]; \text{or } h = 1; k = 2, 3, \ldots, \left[p_1^{1/3}\right]; \\
0, & \text{others}
\end{cases}
\]

with \( \theta = 0.2 \) and

\[
\sigma_{kh}^{AR} = \frac{\phi^{|k-h|}}{1 - \phi^2}, \quad k, h = 1, 2, \ldots, p_1, \phi = 0.8.
\]

Here, \( B = \frac{1}{\sqrt{p_1}} (b_1, b_2, \ldots, b_{p_1}) \) is a deterministic matrix. In the simulation, each \( b_i : r \times 1 \) is generated independently from a normal distribution with covariance matrix being an \( r \times r \) identity matrix and mean \( \mu_B \) consisting of all 1. \( \text{cov}(f_t) \) is also an \( r \times r \) identity matrix and \( \Sigma_u \) is a \( p_1 \times p_1 \) identity matrix.

The empirical sizes of the proposed statistics \( S_n \) for data generating processes (DGPs) (a) and (b) are listed in Table 1. Moreover, the empirical sizes for the renor-

<table>
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<tr>
<th>((p_1, p_2, n))</th>
<th>(S_n) DGP (a)</th>
<th>(S_n) DGP (b)</th>
<th>(\text{MLR}_n) DGP (a)</th>
<th>(\text{MLR}_n) DGP (b)</th>
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<td>0.0461</td>
<td>0.0481</td>
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</tr>
<tr>
<td>(120, 240, 480)</td>
<td>0.0513</td>
<td>0.0511</td>
<td>0.0400</td>
<td>0.0432</td>
</tr>
<tr>
<td>(130, 260, 520)</td>
<td>0.0511</td>
<td>0.0511</td>
<td>0.0520</td>
<td>0.0560</td>
</tr>
<tr>
<td>(140, 280, 560)</td>
<td>0.0469</td>
<td>0.0474</td>
<td>0.0582</td>
<td>0.0580</td>
</tr>
<tr>
<td>(150, 300, 600)</td>
<td>0.0495</td>
<td>0.0500</td>
<td>0.0590</td>
<td>0.0593</td>
</tr>
<tr>
<td>(160, 320, 640)</td>
<td>0.0514</td>
<td>0.0517</td>
<td>0.0437</td>
<td>0.0559</td>
</tr>
<tr>
<td>(170, 340, 680)</td>
<td>0.0498</td>
<td>0.0500</td>
<td>0.0428</td>
<td>0.0430</td>
</tr>
<tr>
<td>(180, 360, 720)</td>
<td>0.0509</td>
<td>0.0510</td>
<td>0.0580</td>
<td>0.0577</td>
</tr>
<tr>
<td>(190, 380, 760)</td>
<td>0.0488</td>
<td>0.0485</td>
<td>0.0388</td>
<td>0.0499</td>
</tr>
<tr>
<td>(200, 400, 800)</td>
<td>0.0491</td>
<td>0.0491</td>
<td>0.0462</td>
<td>0.0499</td>
</tr>
<tr>
<td>(210, 420, 840)</td>
<td>0.0491</td>
<td>0.0500</td>
<td>0.0450</td>
<td>0.0555</td>
</tr>
<tr>
<td>(220, 440, 880)</td>
<td>0.0515</td>
<td>0.0510</td>
<td>0.0572</td>
<td>0.0588</td>
</tr>
<tr>
<td>(230, 460, 920)</td>
<td>0.0493</td>
<td>0.0498</td>
<td>0.0470</td>
<td>0.0488</td>
</tr>
<tr>
<td>(240, 480, 960)</td>
<td>0.0482</td>
<td>0.0479</td>
<td>0.0521</td>
<td>0.0561</td>
</tr>
<tr>
<td>(250, 500, 1000)</td>
<td>0.0452</td>
<td>0.0450</td>
<td>0.0527</td>
<td>0.0545</td>
</tr>
</tbody>
</table>
The empirical sizes of the proposed test $T_n$ at 0.05 significance level for DGP (a)–DGP (d) are listed in Table 2. For DGP (a), we use the original statistic $T_n$ in Theorem 2.3; for DGP (b), the statistic in Theorem 3.1 is used; for DGPs (c) and (d), the dividing-sample statistic in Theorem 3.2 is utilized. For Theorem 3.2, we follow the implementation details in Appendix of [8] to estimate the LSD $H$ for the matrix $\Sigma_{xx}$.

From the results in Tables 1 and 2, the proposed statistics $S_n$ and $T_n$ work well under Assumptions 1 and 2, respectively.

6.3. Factor model dependence. We consider the factor model as follows:

$\mathbf{x}_t = \Lambda_1 \mathbf{f}_t + \mathbf{u}_t, \quad \mathbf{y}_t = \Lambda_2 \mathbf{f}_t + \mathbf{v}_t, \quad t = 1, 2, \ldots, n,$

where $\Lambda_1 = \frac{1}{\sqrt{p_1}}(\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_r^{(1)})$ and $\Lambda_2 = \frac{1}{\sqrt{p_2}}(\lambda_1^{(2)}, \lambda_2^{(2)}, \ldots, \lambda_r^{(2)})$ are $p_1 \times r$ and $p_2 \times r$ deterministic matrices, respectively. In the simulation, all the components of $\lambda_k^{(j)}: k = 1, 2, \ldots, r; j = 1, 2$ are generated from a normal distribution with mean being 0.8 and variance being 1. $\mathbf{f}_t, t = 1, 2, \ldots, n$ are $r \times 1$ random vectors with i.i.d. standard Gaussian distributed elements and $\mathbf{u}_t$ and $\mathbf{v}_t, t = 1, 2, \ldots, n$ are independent random vectors whose elements are all standard Gaussian distributed.

For this model, $\mathbf{x}_t$ and $\mathbf{y}_t$ are not independent if $r \neq 0$. The proposed test statistic $S_n$ and $T_n$ can be used to detect this dependent structure. Tables 3 and 4 illustrate...
### Table 3

Empirical powers of the proposed test $S_n$ at 0.05 significance level for factor models

<table>
<thead>
<tr>
<th>$(p_1, p_2, n)$</th>
<th>$r = 3$</th>
<th>$r = 5$</th>
<th>$r = 7$</th>
<th>$r = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 20, 40)</td>
<td>0.3750</td>
<td>0.5200</td>
<td>0.5910</td>
<td>0.9320</td>
</tr>
<tr>
<td>(30, 60, 120)</td>
<td>0.3070</td>
<td>0.6240</td>
<td>0.8450</td>
<td>0.9330</td>
</tr>
<tr>
<td>(50, 100, 200)</td>
<td>0.3090</td>
<td>0.6700</td>
<td>0.7980</td>
<td>0.9700</td>
</tr>
<tr>
<td>(70, 140, 280)</td>
<td>0.3520</td>
<td>0.6470</td>
<td>0.8330</td>
<td>0.9850</td>
</tr>
<tr>
<td>(90, 180, 360)</td>
<td>0.3670</td>
<td>0.6720</td>
<td>0.8230</td>
<td>0.9880</td>
</tr>
<tr>
<td>(110, 220, 440)</td>
<td>0.3570</td>
<td>0.6690</td>
<td>0.8490</td>
<td>0.9850</td>
</tr>
<tr>
<td>(130, 260, 520)</td>
<td>0.3440</td>
<td>0.6390</td>
<td>0.8510</td>
<td>0.9960</td>
</tr>
<tr>
<td>(150, 300, 600)</td>
<td>0.3780</td>
<td>0.6440</td>
<td>0.8370</td>
<td>0.9990</td>
</tr>
<tr>
<td>(170, 340, 680)</td>
<td>0.3580</td>
<td>0.6580</td>
<td>0.8590</td>
<td>1.0000</td>
</tr>
<tr>
<td>(190, 380, 760)</td>
<td>0.3490</td>
<td>0.6620</td>
<td>0.8720</td>
<td>1.0000</td>
</tr>
<tr>
<td>(210, 420, 840)</td>
<td>0.3460</td>
<td>0.6790</td>
<td>0.8890</td>
<td>1.0000</td>
</tr>
<tr>
<td>(230, 460, 920)</td>
<td>0.3800</td>
<td>0.6930</td>
<td>0.8770</td>
<td>1.0000</td>
</tr>
<tr>
<td>(250, 500, 1000)</td>
<td>0.3470</td>
<td>0.6890</td>
<td>0.8940</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The powers are under the alternative hypothesis that $x$ and $y$ satisfy the factor model (6.4). $r$ is the number of factors.

The powers of the proposed statistics $S_n$ and $T_n$, respectively, as $r$ increases from 3 to 10. For $T_n$, we use its modified version in Theorem 3.2. Results in these tables indicate that for one triple $(p_1, p_2, n)$, the power increases as the number of factors $r$ increases. This phenomenon makes sense since the dependence between $x_t$ and $y_t$ is described by the $r$ common factors contained in the factor vector $f_t$. Stronger

### Table 4

Empirical powers of the proposed test $T_n$ at 0.05 significance level for factor models

<table>
<thead>
<tr>
<th>$(p_1, p_2, n)$</th>
<th>$r = 3$</th>
<th>$r = 5$</th>
<th>$r = 7$</th>
<th>$r = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(100, 50, 80)</td>
<td>0.3680</td>
<td>0.6380</td>
<td>0.7330</td>
<td>0.9470</td>
</tr>
<tr>
<td>(140, 70, 120)</td>
<td>0.3380</td>
<td>0.6440</td>
<td>0.8690</td>
<td>0.9520</td>
</tr>
<tr>
<td>(180, 90, 150)</td>
<td>0.3290</td>
<td>0.6190</td>
<td>0.8890</td>
<td>0.9740</td>
</tr>
<tr>
<td>(200, 100, 170)</td>
<td>0.3410</td>
<td>0.6270</td>
<td>0.8920</td>
<td>0.9820</td>
</tr>
<tr>
<td>(240, 120, 180)</td>
<td>0.3340</td>
<td>0.6290</td>
<td>0.8840</td>
<td>0.9790</td>
</tr>
<tr>
<td>(280, 140, 250)</td>
<td>0.3570</td>
<td>0.6480</td>
<td>0.8730</td>
<td>0.9870</td>
</tr>
<tr>
<td>(320, 160, 270)</td>
<td>0.3490</td>
<td>0.7120</td>
<td>0.8890</td>
<td>0.9940</td>
</tr>
<tr>
<td>(360, 180, 290)</td>
<td>0.3690</td>
<td>0.6890</td>
<td>0.8930</td>
<td>0.9920</td>
</tr>
<tr>
<td>(400, 200, 310)</td>
<td>0.3830</td>
<td>0.7080</td>
<td>0.9030</td>
<td>0.9980</td>
</tr>
<tr>
<td>(440, 220, 330)</td>
<td>0.3920</td>
<td>0.7040</td>
<td>0.8930</td>
<td>1.0000</td>
</tr>
<tr>
<td>(480, 240, 350)</td>
<td>0.3970</td>
<td>0.6990</td>
<td>0.9110</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The powers are under the alternative hypothesis that $x$ and $y$ satisfy the factor model (6.4). $r$ is the number of factors. The parameter $t$ in the statistic $T_n$ takes value of 40. For $T_n$, we use its modified dividing-sample version in Theorem 3.2.
dependence between $x_t$ and $y_t$ exists while more common factors are included in the model.

Here, we would like to point out that using CCA based on the sample covariance matrices with sample mean will incorrectly conclude that $x_t$ and $y_t$ can be independent even if $r > 0$ but $f_t = f$ independent of $t$ because CCA of $x_t$ and $y_t$ is the same as that of $u_t$ and $v_t$. This is why (1.4) and (1.6) are used.

6.4. Uncorrelated but dependent. The construction of (2.8) is based on the idea that the limit of $F^{S_{xy}}(x)$ could not be determined from (2.2) when $x$ and $y$ have correlation. Thus, a natural question is whether our statistic works in the uncorrelated but dependent case. Below is such an example to demonstrate the power of the test statistic in detecting uncorrelatedness.

Let $x_t = (X_{1t}, X_{2t}, \ldots, X_{p1t})^T, t = 1, 2, \ldots, n$ be i.i.d. normally distributed random vectors with zero means and unit variances. Define $y_t = (Y_{1t}, Y_{2t}, \ldots, Y_{p2t})^T, t = 1, 2, \ldots, n$ by

$$Y_{it} = Z_{it} \sqrt{\alpha_0 + \alpha_1 X_{i1t}^2}, \quad i = 1, 2, \ldots, \min(p_1, p_2);$$

if $p_1 < p_2$, let $Y_{jt} = Z_{jt}, j = p_1 + 1, \ldots, p_2; t = 1, \ldots, n$, where $Z_{jt}, j = p_1 + 1, \ldots, p_2; t = 1, \ldots, n$ are i.i.d. normal distributed random variables and independent with $x_t$ and $k$ is a positive integer.

**Remark 8.** For standard normal random variable $X_{it}$, the $2k$th moment is $E X_{it}^{2k} = 2^{-k} (2k)! / k!$.

For this model, $x_t$ and $y_t$ are uncorrelated since $\text{cov}(X_{it}, Y_{it}) = E X_{it}^{2k+1} - E X_{it} E X_{it}^{2k} = 0$. Simulation results in Tables 5 and 6 provide the empirical powers of $S_n$ and $T_n$ by taking $k = 2$ and $k = 5$, respectively. They show that $S_n$ and $T_n$ can distinguish this kind of dependent relationship well when $k = 5$. For the statistic $T_n$, since the covariance matrix of $x$ is an identity matrix, we use the original statistic $T_n$ in Theorem 2.3.

6.5. ARCH type dependence. The statistic works in the above example because the limit of $F^{S_{xy}}(x)$ cannot be determined from (2.2) if $x$ and $y$ are uncorrelated. However, the limit of $F^{S_{xy}}(x)$ might be the same as (2.2) when $x$ and $y$ are uncorrelated. We consider such an example as follows.

Consider two random vectors $x_t = (X_{1t}, X_{2t}, \ldots, X_{p1t})$ and $y_t = (Y_{1t}, Y_{2t}, \ldots, Y_{p2t})$ as follows:

$$Y_{it} = Z_{it} \sqrt{\alpha_0 + \alpha_1 X_{i1t}^2}, \quad i = 1, 2, \ldots, \min(p_1, p_2);$$

$$Y_{jt} = Z_{jt}, \quad j = p_1 + 1, \ldots, p_2;$$

where $Z_t = (Z_{1t}, Z_{2t}, \ldots, Z_{p2t})$ is a random vector consisting of i.i.d. elements generated from Normal $(0, 1)$ and $\{Z_t : t = 1, \ldots, n\}$ are independent across $t$; $x_t = (X_{1t}, X_{2t}, \ldots, X_{p1t})$ is also a random vector with i.i.d. elements generated from
### Table 5

Empirical powers of the proposed test $S_n$ at 0.05 significance level for uncorrelated but dependent case

<table>
<thead>
<tr>
<th>$(p_1, p_2, n)$</th>
<th>$\omega = 4$</th>
<th>$\omega = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10, 20, 40)</td>
<td>0.8140</td>
<td>0.9690</td>
</tr>
<tr>
<td>(30, 60, 120)</td>
<td>0.8200</td>
<td>0.9510</td>
</tr>
<tr>
<td>(50, 100, 200)</td>
<td>0.8220</td>
<td>0.9600</td>
</tr>
<tr>
<td>(70, 140, 280)</td>
<td>0.8100</td>
<td>0.9610</td>
</tr>
<tr>
<td>(90, 180, 360)</td>
<td>0.8210</td>
<td>0.9640</td>
</tr>
<tr>
<td>(110, 220, 440)</td>
<td>0.8110</td>
<td>0.9670</td>
</tr>
<tr>
<td>(130, 260, 520)</td>
<td>0.8320</td>
<td>0.9740</td>
</tr>
<tr>
<td>(150, 300, 600)</td>
<td>0.8420</td>
<td>0.9740</td>
</tr>
<tr>
<td>(170, 340, 680)</td>
<td>0.8450</td>
<td>0.9760</td>
</tr>
<tr>
<td>(190, 380, 760)</td>
<td>0.8580</td>
<td>0.9680</td>
</tr>
<tr>
<td>(210, 420, 840)</td>
<td>0.8420</td>
<td>0.9670</td>
</tr>
<tr>
<td>(230, 460, 920)</td>
<td>0.8440</td>
<td>0.9810</td>
</tr>
<tr>
<td>(250, 500, 1000)</td>
<td>0.8620</td>
<td>0.9810</td>
</tr>
</tbody>
</table>

The powers are under the alternative hypothesis that $Y_{it} = X_{it}^{\omega} - EX_{it}^{\omega}, i = 1, 2, \ldots, p_1$ and $Y_{jt} = \varepsilon_{jt}, j = p_1 + 1, \ldots, p_2; t = 1, \ldots, n$, where $\varepsilon_{jt}, j = p_1 + 1, \ldots, p_2; t = 1, \ldots, n$ are standard normal distributed and independent with $X_{it}$ and $\omega = 4, 10$. The parameter $t$ in the statistic $T_n$ takes value of 40. The original statistic $T_n$ in Theorem 2.3 is used.

### Table 6

Empirical powers of the proposed test $T_n$ at 0.05 significance level for uncorrelated but dependent case

<table>
<thead>
<tr>
<th>$(p_1, p_2, n)$</th>
<th>$\omega = 4$</th>
<th>$\omega = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(100, 50, 80)</td>
<td>0.7240</td>
<td>0.8690</td>
</tr>
<tr>
<td>(140, 70, 120)</td>
<td>0.7940</td>
<td>0.8890</td>
</tr>
<tr>
<td>(180, 90, 150)</td>
<td>0.7830</td>
<td>0.8940</td>
</tr>
<tr>
<td>(200, 100, 170)</td>
<td>0.7910</td>
<td>0.9340</td>
</tr>
<tr>
<td>(240, 120, 180)</td>
<td>0.8420</td>
<td>0.9290</td>
</tr>
<tr>
<td>(280, 140, 250)</td>
<td>0.8680</td>
<td>0.9580</td>
</tr>
<tr>
<td>(320, 160, 270)</td>
<td>0.9010</td>
<td>0.9820</td>
</tr>
<tr>
<td>(360, 180, 290)</td>
<td>0.9190</td>
<td>0.9940</td>
</tr>
<tr>
<td>(400, 200, 310)</td>
<td>0.9530</td>
<td>0.9990</td>
</tr>
<tr>
<td>(440, 220, 330)</td>
<td>0.9820</td>
<td>1.0000</td>
</tr>
<tr>
<td>(480, 240, 350)</td>
<td>0.9940</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The powers are under the alternative hypothesis that $Y_{it} = X_{it}^{\omega} - EX_{it}^{\omega}, i = 1, 2, \ldots, p_1$ and $Y_{jt} = \varepsilon_{jt}, j = p_1 + 1, \ldots, p_2; t = 1, \ldots, n$, where $\varepsilon_{jt}, j = p_1 + 1, \ldots, p_2; t = 1, \ldots, n$ are standard normal distributed and independent with $X_{it}$ and $\omega = 4, 10$. The parameter $t$ in the statistic $T_n$ takes value of 40. The original statistic $T_n$ in Theorem 2.3 is used.
Table 7
Empirical powers of the proposed test $S_n$ at 0.05 significance level for $x$ and $y$ with ARCH(1) dependent type

<table>
<thead>
<tr>
<th>$(p_1, p_2, n)$</th>
<th>$(0.9, 0.1)$</th>
<th>$(0.8, 0.2)$</th>
<th>$(0.7, 0.3)$</th>
<th>$(0.6, 0.4)$</th>
<th>$(0.5, 0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(10, 20, 40)$</td>
<td>0.3480</td>
<td>0.4670</td>
<td>0.6380</td>
<td>0.7650</td>
<td>0.8500</td>
</tr>
<tr>
<td>$(30, 60, 120)$</td>
<td>0.4840</td>
<td>0.8090</td>
<td>0.9820</td>
<td>0.9990</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(50, 100, 200)$</td>
<td>0.6190</td>
<td>0.9730</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(70, 140, 280)$</td>
<td>0.7020</td>
<td>0.9980</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(90, 180, 360)$</td>
<td>0.7900</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(110, 220, 440)$</td>
<td>0.8620</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(130, 260, 520)$</td>
<td>0.8970</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(150, 300, 600)$</td>
<td>0.9440</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(170, 340, 680)$</td>
<td>0.9520</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(190, 380, 760)$</td>
<td>0.9810</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(210, 420, 840)$</td>
<td>0.9880</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(230, 460, 920)$</td>
<td>0.9950</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$(250, 500, 1000)$</td>
<td>0.9980</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The powers are under the alternative hypothesis that $Y_{it} = Z_{it} \sqrt{\alpha_0 + \alpha_1 X_{it}^2} , i = 1, 2, \ldots, p_1; Y_{jt} = Z_{jt}, j = p_1 + 1, \ldots, p_2$. The pair of two numbers in this table is the value of $(\alpha_0, \alpha_1)$.

Normal $(0, 1)$ and $\{x_t : t = 1, \ldots, n\}$ are independent across $t$. Moreover, $\{z_t : t = 1, \ldots, n\}$ are independent of $\{x_t : t = 1, \ldots, n\}$.

For this model, $x_t$ and $y_t$ are dependent but uncorrelated. Simulation results indicate that the proposed test statistic $S_n$ cannot detect the dependence between them. Nevertheless, if we substitute the elements $X_{it}^2$ and $Y_{it}^2$ for $X_{it}$ and $Y_{jt}$, respectively, in the matrix $S_{xy}$, then the new resulting statistic $S_n$ can capture the dependence of this type. This efficiency is due to the correlation between the high powers of $X_{it}$ and $Y_{it}$.

Tables 7 and 8 list the powers of the proposed statistics $S_n$ and $T_n$ for testing model (6.5) in several cases, that is, $\alpha_0$ and $\alpha_1$ take different values. For the statistic $T_n$, since the covariance matrix of $x$ is an identity matrix, we use the original statistic $T_n$ in Theorem 2.3. From the table, we can find the phenomenon that as $\alpha_1$ increases, the powers also increase. This is consistent with our intuition because larger $\alpha_1$ brings about larger correlation between $Y_{it}$ and $X_{it}$.

7. Empirical applications. As an application of the proposed independence test, we test the cross-sectional dependence of daily stock returns of companies between two different sections from New York Stock Exchange (NYSE) during the period 2000.1.1–2002.1.1, including consumer service section, consumer duration section, consumer nonduration section, energy section, finance section, transport section, healthcare section, capital goods section, basic industry section and public utility section. The data set is obtained from Wharton Research Data Services (WRDS) database.
We randomly choose $p_1$ and $p_2$ companies from two different sections, respectively, such as the transport and finance section. At each time $t$, denote the closed stock prices of these companies from the two different sections by $x_t = (x_{1t}, x_{2t}, \ldots, x_{p_1t})^T$ and $y_t = (y_{1t}, y_{2t}, \ldots, y_{p_2t})^T$, respectively. We consider daily stock returns $r^x_t = (r^x_{1t}, r^x_{2t}, \ldots, r^x_{p_1t})$ and $r^y_t = (r^y_{1t}, r^y_{2t}, \ldots, r^y_{p_2t})$ with $r^x_{it} = \log \frac{x_{it}}{x_{i,t-1}}$, $i = 1, 2, \ldots, p_1$ and $r^y_{jt} = \log \frac{y_{jt}}{y_{j,t-1}}$, $j = 1, 2, \ldots, p_2$. The goal is to test the dependence between $r^x_t$ and $r^y_t$.

The proposed test $S_n$ is applied to testing dependence of $r^x_t$ and $r^y_t$. For each $(p_1, p_2, n)$, we randomly choose $p_1$ and $p_2$ companies from two different sections, construct the corresponding sample matrices $X = (r^x_1, r^x_2, \ldots, r^x_{p_1})$ and $Y = (r^y_1, r^y_2, \ldots, r^y_{p_2})$, and then calculate the $P$-value by applying the proposed test. Repeat this procedure 100 times and derive 100 $P$-values to see whether the cross-sectional “dependence” feature is popular between the tested two sections.

We test independence of daily stock returns of companies from three pairs of sections, that is, basic industry section and capital goods section, public utility section and capital goods section, finance section and healthcare section. From Tables 9, 10 and 11, we can see that, as the pair of numbers of companies $(p_1, p_2)$ increases, more experiments are rejected in terms of the $P$-values below 0.05. It shows that cross-sectional dependence exists and is popular for different sections in NYSE. This suggests that the assumption that cross-sectional independence in such empirical studies may not be appropriate.

### Table 8

<table>
<thead>
<tr>
<th>$(p_1, p_2, n)$</th>
<th>(0.9, 0.1)</th>
<th>(0.8, 0.2)</th>
<th>(0.7, 0.3)</th>
<th>(0.6, 0.4)</th>
<th>(0.5, 0.5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(100, 50, 80)</td>
<td>0.6020</td>
<td>0.6180</td>
<td>0.7270</td>
<td>0.8930</td>
<td>0.9660</td>
</tr>
<tr>
<td>(140, 70, 120)</td>
<td>0.6370</td>
<td>0.7890</td>
<td>0.8020</td>
<td>0.8990</td>
<td>0.9820</td>
</tr>
<tr>
<td>(180, 90, 150)</td>
<td>0.7490</td>
<td>0.8280</td>
<td>0.9090</td>
<td>0.9920</td>
<td>1.0000</td>
</tr>
<tr>
<td>(200, 100, 170)</td>
<td>0.8130</td>
<td>0.8730</td>
<td>0.9930</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(240, 120, 180)</td>
<td>0.8920</td>
<td>0.9720</td>
<td>0.9950</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(280, 140, 250)</td>
<td>0.9470</td>
<td>0.9870</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(320, 160, 270)</td>
<td>0.9900</td>
<td>0.9980</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(360, 180, 290)</td>
<td>0.9910</td>
<td>0.9940</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(400, 200, 310)</td>
<td>0.9980</td>
<td>0.9950</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(440, 220, 330)</td>
<td>0.9990</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>(480, 240, 350)</td>
<td>0.9980</td>
<td>0.9970</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

The powers are under the alternative hypothesis that $Y_{jt} = \sqrt{\alpha_0 + \alpha_1 X_{jt}^T}, i = 1, 2, \ldots, p_1; Y_{jt} = Z_{jt}, j = p_1 + 1, \ldots, p_2$. The pair of two numbers in this table is the value of $(\alpha_0, \alpha_1)$. The parameter $t$ in the statistic $T_n$ takes value of 40. The original statistic $T_n$ in Theorem 2.3 is used.
TABLE 9
P-values for \((p_1, p_2)\) companies from basic industry section and capital goods section of NYSE

<table>
<thead>
<tr>
<th>(P)-value interval</th>
<th>No. of exp. ((p_1, p_2, n))</th>
<th>((p_1, p_2, n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0.05])</td>
<td>56</td>
<td>60</td>
</tr>
<tr>
<td>([0.05, 0.1])</td>
<td>22</td>
<td>20</td>
</tr>
<tr>
<td>([0.1, 0.2])</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>([0.2, 0.3])</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>([0.3, 0.4])</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>([0.4, 0.5])</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>([0.6, 0.7])</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([0.8, 0.9])</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([0.9, 1])</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

These are \(P\)-values for \((p_1, p_2)\) companies from different two sections of NYSE: basic industry section and capital goods section, each of which has \(n\) daily stock returns during the period 2000.1.1–2002.1.1. The number of repeated experiments is 100. All the closed stock prices are from WRDS database. No. of Exp. is the number of experiments whose \(P\)-values are in the corresponding interval.

TABLE 10
P-values for \((p_1, p_2)\) companies from public utility section and capital goods section of NYSE

<table>
<thead>
<tr>
<th>(P)-value interval</th>
<th>No. of exp. ((p_1, p_2, n))</th>
<th>((p_1, p_2, n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>([0, 0.05])</td>
<td>76</td>
<td>84</td>
</tr>
<tr>
<td>([0.05, 0.1])</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>([0.1, 0.2])</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>([0.2, 0.3])</td>
<td>7</td>
<td>1</td>
</tr>
<tr>
<td>([0.3, 0.4])</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>([0.4, 0.5])</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>([0.6, 0.7])</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>([0.8, 0.9])</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>([0.9, 1])</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

These are \(P\)-values for \((p_1, p_2)\) companies from different two sections of NYSE: public utility section and capital goods section, each of which has \(n\) daily stock returns during the period 2000.1.1–2002.1.1. The number of repeated experiments is 100. All the closed stock prices are from WRDS database. No. of Exp. is the number of experiments whose \(P\)-values are in the corresponding interval.
Table 11

<table>
<thead>
<tr>
<th>P-values:</th>
<th>No. of exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>P-value interval</td>
<td>(p₁, p₂, n)</td>
</tr>
<tr>
<td>(10, 15, 20)</td>
<td>(15, 20, 25)</td>
</tr>
<tr>
<td>[0, 0.05]</td>
<td>90</td>
</tr>
<tr>
<td>[0.05, 0.1]</td>
<td>4</td>
</tr>
<tr>
<td>[0.1, 0.2]</td>
<td>5</td>
</tr>
<tr>
<td>[0.2, 0.3]</td>
<td>1</td>
</tr>
<tr>
<td>[0.3, 0.4]</td>
<td>0</td>
</tr>
<tr>
<td>[0.4, 0.5]</td>
<td>0</td>
</tr>
<tr>
<td>[0.6, 0.7]</td>
<td>0</td>
</tr>
<tr>
<td>[0.8, 0.9]</td>
<td>0</td>
</tr>
<tr>
<td>[0.9, 1]</td>
<td>0</td>
</tr>
</tbody>
</table>

These are P-values for (p₁, p₂) companies from different two sections of NYSE: finance section and healthcare section, each of which has n daily stock returns during the period 2000.1.1–2002.1.1. The number of repeated experiments is 100. All the closed stock prices are from WRDS database. No. of Exp. is the number of experiments whose P-values are in the corresponding interval.

8. Acknowledgement. The authors would like to thank the Editor, an Associate Editor and the referees for their constructive comments and suggestions which significantly improved this paper.

SUPPLEMENTARY MATERIAL

Supplement to “Independence test for high dimensional data based on regularized canonical correlation coefficients” (DOI: 10.1214/14-AOS1284SUPP; .pdf). The supplementary material is divided into Appendices A and B. Some useful lemmas, and proofs of all theorems and Proposition 4–5 are given in Appendix A while one theorem related to CLT of a sample covariance matrix plus a perturbation matrix is provided in Appendix B.

REFERENCES


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