Gaussian estimates for solutions of some one-dimensional stochastic equations

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Abstract

Using covariance identities based on the Clark-Ocone representation formula we derive Gaussian density bounds and tail estimates for the probability law of the solutions of several types of stochastic differential equations, including Stratonovich equations with boundary condition and irregular drifts, and equations driven by fractional Brownian motion. Our arguments are generally simpler than the existing ones in the literature as our approach avoids the use of the inverse of the Ornstein-Uhlenbeck operator.

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1 Introduction

Gaussian density estimates for classes of stochastic equations have been extensively studied in recent years, see e.g. [1], [10], [11]. On the other hand, density estimates for random variables on the Wiener space have been obtained in [8] based on covariance representations using the number (or Ornstein-Uhlenbeck) operator $-L$ and its inverse $(-L)^{-1}$. Recently, those tools have been combined in [2], [5] and [6] with the Malliavin calculus in order to derive bounds for the density of solutions of stochastic differential equations driven by fractional Brownian motion and for the density of additive functionals of stochastic equations

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with irregular drifts.

Although the Ornstein-Uhlenbeck operator $-L$ has nice contractivity properties as well as an integral representation, it can be quite technical to compute in practice. For example, its application in [2] requires the use of quadratic programming. In this paper we propose to use covariance representations based on the Clark-Ocone formula instead of the Ornstein-Uhlenbeck operator. Such covariance representations have been recently applied in [14] to the Malliavin calculus approach to the Stein method [7]. In contrast with covariance identities based on the Ornstein-Uhlenbeck operator $-L$, which relies on the divergence-gradient composition, the Clark-Ocone formula only requires the computation of a gradient and a conditional expectation.

This paper is organized as follows. In Section 2, for a one-dimensional random variable on the Wiener space, we present general Gaussian estimates for the density and the tail probabilities, see Theorem 2.4.

Our main results are then proved in Sections 3, 4 and 5. In Section 3 we apply the formulas of Section 2 to derive Gaussian estimates for the density and the tails of the solution X_t of a one-dimensional stochastic differential equation (SDE) of the form

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\circ \mathrm{d}B_t, \qquad t \in [0,T], \tag{1.1}$$

under anticipating boundary condition $X_0 = f(X_T)$, see Theorem 3.2. Here the symbol $\circ$ denotes the Stratonovich stochastic differential with respect to the standard Brownian motion $(B_t)_{t\in[0,T]}$ and f is a differentiable function with bounded derivative. To the best of our knowledge Gaussian bounds for solutions of these kind of SDEs are not available in the existing literature. The existence and uniqueness of X_t when f is linear and such that $f' < 0$ has been proved in [3], cf. also Theorem 3.3.5 of [9]. This result is generalized in Proposition 3.1 below.

In Section 4 we apply the formulas of Section 2 to derive Gaussian estimates for the tails of additive functionals of the solution to (1.1), see Theorem 4.1. Particularly, we allow bounded and measurable drift coefficients and positive, bounded from below and $\mathcal{C}_b^\infty(\mathbb{R})$ diffusion coefficients. Our proof combines the arguments in [5, 6] and the above mentioned

Clark-Ocone type covariance representation.

Finally, in Theorem 5.2 of Section 5 we provide Gaussian type bounds for the density and the tails of the stochastic process

$$X_t = x_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s^H, \quad t \in [0, T] \tag{1.2}$$

where $(B_t^H)_{t\in[0,T]}$ is a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and the coefficients b and σ satisfy suitable regularity conditions. Even if our result has some similarities with the corresponding result in [2], we allow b and σ depend explicitly on the time and our approach is somewhat simpler due to the use of the Clark-Ocone type representation. We remark that the existence and uniqueness of the solution to (1.2) is provided in Lemma 5.1.

2 Density formula and tail probabilities by integration by parts

Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, P)$, where $\mathcal{F}_t = \sigma(\mathcal{F}_t^B, \mathcal{N})$, $(\mathcal{F}_t^B)_{t\in\mathbb{R}_+}$ is the natural filtration of $(B_t)_{t\in\mathbb{R}_+}$ and $\mathcal{N} = \{A \in \mathcal{F} : \ P(A) = 0\}$. We denote by $\mathcal{S}_b$ the class of smooth random variables F of the form

$$F = f(I_1(u_1), \ldots, I_1(u_n)), \quad f \in \mathcal{C}_b^1(\mathbb{R}^n) \tag{2.1}$$

where $I_1(u_i) = \int_0^\infty u_i(t)\,\mathrm{d}B_t$ and $u_1, \ldots, u_n \in L^2(\mathbb{R}_+)$. The Malliavin gradient of a smooth random variable F of the form (2.1) is given by

$$D_t F = \sum_{i=1}^n \partial_i f(I_1(u_1), \ldots, I_1(u_n)) u_i(t), \quad t \geq 0.$$

Let ℓ be the Lebesgue measure on $\mathbb{R}_+$. It turns out that the operator

$$D : \mathcal{S}_b \subset L^2(\Omega, \mathcal{F}, P) \longrightarrow L^2(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+), P \otimes \ell)$$

defined by $DF = (D_t F)_{t\in\mathbb{R}_+}$ is closable and we shall denote by $\mathbb{D}^{1,2}$ the domain of the minimal closed extension of D, still denoted by D, meaning that $\mathbb{D}^{1,2}$ is the closure of the class of smooth random variables $\mathcal{S}_b$ with respect to the norm

$$\|F\|_{1,2} = \left(\mathrm{E}[|F|^2] + \mathrm{E}\left[\int_0^\infty |D_t F|^2\,\mathrm{d}t \right] \right)^{1/2}.$$

We define similarly the space $\mathbb{D}^{1,p}$, $p \geq 2$. Recall that for any $\phi \in \mathcal{C}_b^1(\mathbb{R})$ and $F \in \mathbb{D}^{1,2}$ we have $\phi(F) \in \mathbb{D}^{1,2}$ and D satisfies the chain rule of derivation

$$D_t \phi(F) = \phi'(F) D_t F, \tag{2.2}$$

cf. Proposition 1.2.3 in [9]. The operator D also satisfies the Clark-Ocone representation formula

$$F = \mathrm{E}[F] + \int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t] \, \mathrm{d}B_t, \qquad F \in \mathbb{D}^{1,2}, \tag{2.3}$$

see e.g. Corollary 5.2.2 in [13] and the following covariance identity

$$\mathrm{Cov}(F, G) = \mathrm{E}\left[\int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t] D_t G \, \mathrm{d}t \right], \quad F, G \in \mathbb{D}^{1,2}, \tag{2.4}$$

cf. Proposition 3.4.1 in [13], p. 121.

Let now $(-L)^{-1}$ denote the inverse of the Ornstein-Uhlenbeck operator $-L = \delta D$ where δ is the divergence operator i.e. the dual of the gradient operator D on the Wiener space. We recall the following result, cf. Theorem 3.1 of [8]. We denote by $\mathrm{Supp}(f)$ the support of any given function f.

Proposition 2.1 *Let $F \in \mathbb{D}^{1,2}$ be such that $\mathrm{E}[F] = 0$. The law of F has a density p_F with respect to the Lebesgue measure if and only if the function*

$$g_F(x) := -\mathrm{E}\left[\int_0^\infty D_t F D_t L^{-1} F \, \mathrm{d}t \,\middle|\, F = x \right], \qquad x \in \mathbb{R},$$

satisfies $g_F(F) > 0$ a.s. In this case $\mathrm{Supp}(p_F)$ is a closed interval of $\mathbb{R}$ containing 0 and we have

$$p_F(z) = \frac{\mathrm{E}[|F|]}{2 g_F(z)} \exp\left(-\int_0^z \frac{u}{g_F(u)} \, \mathrm{d}u \right), \quad a.e.\ z \in \mathrm{Supp}(p_F). \tag{2.5}$$

Similarly, by Theorem 4.1 of [8] we have the following result.

Proposition 2.2 *Let $F \in \mathbb{D}^{1,2}$ be such that $\mathrm{E}[F] = 0$. If in addition*

$$0 < g_F(F) \leq \alpha F + \beta, \qquad a.s.$$

for some $\alpha \geq 0$ and $\beta > 0$, then

$$P(F \geq x) \leq \exp\left(-\frac{x^2}{2\alpha x + 2\beta} \right) \quad \text{and} \quad P(F \leq -x) \leq \exp\left(-\frac{x^2}{2\beta} \right), \qquad x > 0. \tag{2.6}$$

The results of this paper rely on the following proposition.

Proposition 2.3 *Let $F \in \mathbb{D}^{1,2}$ be centered. Then we have $g_F(F) = \varphi_F(F)$ a.s., where the function φ_F is defined by*

$$\varphi_F(x) := \mathrm{E}\left[\int_0^\infty D_t F\, \mathrm{E}[D_t F \,|\, \mathcal{F}_t]\, \mathrm{d}t \,\Big|\, F = x\right], \qquad a.e.\ x \in \mathbb{R}. \tag{2.7}$$

Proof We start by checking that the random variable $\varphi_F(F)$ defined by (2.7) is integrable for any $F \in \mathbb{D}^{1,2}$. Indeed

$$\mathrm{E}[|\varphi_F(F)|] \leq \mathrm{E}\left[\int_0^\infty |D_t F|\, \mathrm{E}[|D_t F| \,|\, \mathcal{F}_t]\, \mathrm{d}t\right] \leq \int_0^\infty \mathrm{E}[|D_t F|^2]\, \mathrm{d}t < \infty.$$

By (2.4) and the properties of the gradient operator, for any $\phi \in \mathcal{C}_b^1(\mathbb{R})$ we have

$$\begin{aligned}
\mathrm{E}[\phi'(F)\varphi_F(F)] &= \mathrm{E}\left[\mathrm{E}\left[\phi'(F)\int_0^\infty D_t F \mathrm{E}[D_t F \,|\, \mathcal{F}_t]\, \mathrm{d}t \mid F\right]\right] \\
&= \mathrm{E}\left[\phi'(F)\int_0^\infty D_t F \mathrm{E}[D_t F \,|\, \mathcal{F}_t]\, \mathrm{d}t\right] \\
&= \mathrm{E}\left[\int_0^\infty \mathrm{E}[D_t F \,|\, \mathcal{F}_t] D_t \phi(F)\, \mathrm{d}t\right] \\
&= \mathrm{Cov}(F, \phi(F)) = \mathrm{E}[\phi'(F) g_F(F)],
\end{aligned}$$

where this latter equality follows by formula (3.15) of Theorem 3.1 in [8]. Combining the above relation with an approximation argument we have

$$\mathrm{E}[\mathbb{1}_B(F)\varphi_F(F)] = \mathrm{E}[\mathbb{1}_B(F) g_F(F)], \quad \text{for any Borel set } B \subseteq \mathbb{R}. \tag{2.8}$$

Since $g_F(F) \geq 0$ a.s. (see Proposition 3.9 in [7]), this relation and the integrability of $\varphi_F(F)$ yield the integrability of $g_F(F)$. Using again (2.8) we finally have $\varphi_F(F) = g_F(F)$ a.s., and the proof is completed. □

The new representation of the conditional expectation given in the above proposition avoids the use of $(-L)^{-1}$ and will provide us with simpler arguments for the derivation of Gaussian estimates for the density and the tail probabilities of a centered $F \in \mathbb{D}^{1,2}$. In particular it leads to the following result which will play a key role in the next sections.

Theorem 2.4 *Let $F \in \mathbb{D}^{1,2}$ be a centered random variable such that*

$$0 < g \leq \int_0^\infty D_s F \mathrm{E}[D_s F \,|\, \mathcal{F}_s]\, \mathrm{d}s \leq G \quad a.s., \tag{2.9}$$

where $g, G > 0$ are positive constants. Then:

(i) the density p_F satisfies

$$\frac{\mathrm{E}[|F|]}{2G}\exp\left(-\frac{z^2}{2g}\right)\leq p_F(z)\leq\frac{\mathrm{E}[|F|]}{2g}\exp\left(-\frac{z^2}{2G}\right),\quad a.e.\ z\in\mathbb{R}. \tag{2.10}$$

(ii) the tail probabilities satisfy

$$P(F\geq x)\leq\exp\left(-\frac{x^2}{2G}\right)\quad and\quad P(F\leq -x)\leq\exp\left(-\frac{x^2}{2G}\right),\qquad x>0. \tag{2.11}$$

Proof. The inequalities (2.10) follow by Proposition 2.1 and Proposition 2.3. In particular, relation $\mathrm{Supp}(p_F)=\mathbb{R}$ may be found in the proof of Corollary 3.3 in [8]. The bounds (2.11) follow by Proposition 2.2 and again Proposition 2.3 taking $\alpha=0$ and $\beta=G$. □

3 Gaussian estimates of one-dimensional SDEs with boundary conditions

Let $b:\mathbb{R}\to\mathbb{R}$ be a continuous function and $\sigma:\mathbb{R}\to\mathbb{R}$ be a continuously differentiable function with bounded derivative of the first order. We assume that

$$b_1:=b+\frac{1}{2}\sigma\sigma'$$

is a Lipschitz continuous function. The aim of this section is to obtain Gaussian estimates for the density and the tail probabilities of the solution of the one-dimensional Stratonovich stochastic differential equation

$$\mathrm{d}X_t=b(X_t)\,\mathrm{d}t+\sigma(X_t)\circ\mathrm{d}B_t,\qquad t\in[0,T], \tag{3.1}$$

with anticipating boundary condition

$$X_0=f(X_T), \tag{3.2}$$

see Theorem 3.2 below. In the case where f is linear with $f'<0$ and the functions σ and b_1 are of class $\mathcal{C}^4(\mathbb{R})$ with bounded derivatives, the existence and uniqueness (in a suitable class of processes) of the solution to (3.1)-(3.2) was proved in [3], see also Theorem 3.3.5 in [9].

Let $\varphi_t(x)$, $x\in\mathbb{R}$, be the stochastic flow associated with the coefficients of the one-dimensional Stratonovich stochastic differential equation (3.1), i.e. the solution of the following stochastic differential equation with initial value $x\in\mathbb{R}$:

$$\varphi_t(x)=x+\int_0^t b(\varphi_s(x))\,\mathrm{d}s+\int_0^t\sigma(\varphi_s(x))\circ\mathrm{d}B_s$$

$$= x + \int_0^t b_1(\varphi_s(x))\,\mathrm{d}s + \int_0^t \sigma(\varphi_s(x))\,\mathrm{d}B_s, \quad t \in [0,T]. \tag{3.3}$$

In the following, for any function $h : \mathbb{R} \to \mathbb{R}$ we set

$$\underline{h} := \inf_{x\in\mathbb{R}} |h(x)| \quad \text{and} \quad \overline{h} := \sup_{x\in\mathbb{R}} |h(x)|.$$

In Proposition 3.1 below we start by proving the existence and uniqueness (in a suitable class of processes) of the solution to (3.1)-(3.2) for a more general class of boundary data functions f, allowing e.g. f to be nonlinear or $f' > 0$.

Before stating Proposition 3.1, we recall some useful sets of random variables (see e.g. [9] for more details). If $\mathbb{L}$ is a family of random variables we denote by $\mathbb{L}_{loc}$ the set of random variables X such that there exists a sequence $\{(\Omega_n, X_n)\}_{n\geq 1} \subset \mathcal{F} \times \mathbb{L}$ such that $\Omega_n \uparrow \Omega$ and $X = X_n$ almost surely on Ω_n. We denote by $\mathbb{L}^{1,p}(\mathbb{R})$, $p \geq 2$, the family of real-valued processes $\{X_t\}_{t\in[0,T]} \in L^p([0,T]\times\Omega)$, $T > 0$, such that $X_t \in \mathbb{D}^{1,p}$ for Lebesgue almost all $t \in [0,T]$ and there exists a measurable version of the two parameter process $D_s X_t$ verifying

$$\mathrm{E}\left[\int_{[0,T]^2} |D_s X_t|^p \,\mathrm{d}s\mathrm{d}t\right] < \infty.$$

Finally, we denote by $\mathbb{L}_2^{1,4}(\mathbb{R})$ the class of processes $\{X_t\}_{t\in[0,T]} \in \mathbb{L}^{1,4}(\mathbb{R})$ such that

$$\lim_{n\to\infty} \int_0^1 \sup_{s\leq t\leq \min\{s+n^{-1},1\}} \mathrm{E}[|D_s X_t - U_s|^2]\,\mathrm{d}s = 0$$

and

$$\lim_{n\to\infty} \int_0^1 \sup_{\max\{s-n^{-1},0\}\leq t\leq s} \mathrm{E}[|D_s X_t - V_s|^2]\,\mathrm{d}s = 0$$

for some processes $\{U_t\}_{t\in[0,1]}, \{V_t\}_{t\in[0,1]} \in L^2([0,1]\times\Omega)$.

Proposition 3.1 *Assume that σ and $b_1 := b + \sigma\sigma'/2$ are of class $\mathcal{C}^4(\mathbb{R})$ with bounded derivatives (up to the fourth order). In addition, suppose that the boundary data function f is differentiable with bounded derivative. If one of the following two sets of conditions holds:*
(C1) b' is bounded and

$$\sup_{x\in\mathbb{R}} f'(x) < 0 \tag{3.4}$$

(C2)

$$0 < \underline{\sigma}, \overline{\sigma} < \infty, \; K := \sup_{x\in\mathbb{R}} \left| \frac{b'(x)\sigma(x) - b(x)\sigma'(x)}{\sigma(x)} \right| < \infty \tag{3.5}$$

and

$$f \text{ is bounded and } \sup_{x\in\mathbb{R}} f'(x) < (\underline{\sigma}/\overline{\sigma})\mathrm{e}^{-KT}, \tag{3.6}$$

then (3.1)-(3.2) *has a solution which is given by* $X_t = \varphi_t(X_0)$, *where* X_0 *is the unique random variable satisfying* $X_0 = f(\varphi_T(X_0))$. *In addition, this solution is the unique continuous solution in* $\mathbb{L}^{1,4}_{2,loc}(\mathbb{R})$.

The following is the main result of this section.

Theorem 3.2 *Assume that* σ *and* $b_1 := b+\sigma\sigma'/2$ *are of class* $\mathcal{C}^4(\mathbb{R})$ *with bounded derivatives (up to the fourth order) and* (3.5). *In addition, suppose that the boundary data function* f *is differentiable with bounded derivative and let* $(X_t)_{t\in[0,T]}$ *be the unique solution to* (3.1)-(3.2). *If one of the following two sets of conditions holds:*
(C3)

$$-(\underline{\sigma}/\overline{\sigma})^2 e^{-3KT} < \inf_{x\in\mathbb{R}} f'(x) \le \sup_{x\in\mathbb{R}} f'(x) < 0$$

(C4)

$$f \text{ is bounded and } \frac{(\underline{\sigma}/\overline{\sigma})e^{-KT} - \sup_{x\in\mathbb{R}} f'(x)}{\overline{f'}} > (\overline{\sigma}/\underline{\sigma})^3 e^{4KT},$$

where the constant K *is defined in* (3.5), *then, for each* $t \in (0,T]$, *the density* p_{X_t} *satisfies the bounds*

$$\frac{E[|X_t - EX_t|]}{2G(t)} \exp\left(-\frac{(x-E[X_t])^2}{2g(t)}\right) \le p_{X_t}(x) \le \frac{E[|X_t - E[X_t]|]}{2g(t)} \exp\left(-\frac{(x-E[X_t])^2}{2G(t)}\right), \tag{3.7}$$

a.e. $x \in \mathbb{R}$, *and*

$$P(X_t - E[X_t] \ge x) \le \exp\left(-\frac{x^2}{2G(t)}\right), \qquad x > 0$$

$$P(X_t - E[X_t] \le -x) \le \exp\left(-\frac{x^2}{2G(t)}\right), \qquad x > 0$$

where the functions $g, G : (0,T] \to (0,\infty)$ *are defined as follows:*

(a) Under (C3), $g(t) := (c + c_*)^2 t$ *and* $G(t) := C^2 t + c_*^2 (T-t)$.

(b) Under (C4), $g(t) := (c - C_*)^2 t$ *and* $G(t) := (C + C_*)^2 t + C_*^2 (T-t)$.
The constants c, C, c_*, C_* *are given by*

$$c := \underline{\sigma} \exp(-KT), \qquad C := \overline{\sigma} \exp(KT), \tag{3.8}$$

$$c_* := \frac{C^2}{\underline{\sigma}} \inf_{x\in\mathbb{R}} f'(x), \qquad C_* := \frac{C\overline{\sigma}}{c\underline{\sigma}} \frac{C\overline{f'}}{\underline{\sigma}/C - \sup_{x\in\mathbb{R}} f'(x)}.$$

Note that the assumptions of Theorem 3.2 hold if $b \in \mathcal{C}_b^4(\mathbb{R})$, σ is bounded away from zero, $\sigma \in \mathcal{C}_b^5(\mathbb{R})$ and the boundary data function f is bounded with bounded negative derivative. SDEs with boundary conditions have applications in several fields, among others we cite [16] for applications in quantum mechanics.

For the proofs of Proposition 3.1 and Theorem 3.2 we will need the following Lemmas 3.3 and 3.4 which will be proved at the end of this section.

Lemma 3.3 *Assume that σ and b_1 are of class $\mathcal{C}^3(\mathbb{R})$, with bounded derivatives (up to the third order). Then $(\varphi_t(x))_{t\in[0,T]} \in \mathbb{D}^{1,2}$ and $\varphi_t'(x) := \frac{\partial \varphi_t(x)}{\partial x}$ exists. If in addition, σ is strictly positive and bounded from below, then we have the following expressions: for $0 \le t \le T$ and $x \in \mathbb{R}$,*

$$D_\theta \varphi_t(x) = \sigma(\varphi_t(x)) \exp\left(\int_\theta^t \frac{b'(\varphi_s(x))\sigma(\varphi_s(x)) - \sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma(\varphi_s(x))} \, \mathrm{d}s \right) \mathbb{1}_{[0,t]}(\theta), \quad (3.9)$$

$$\varphi_t'(x) = \frac{\sigma(\varphi_t(x))}{\sigma(x)} \exp\left(\int_0^t \frac{b'(\varphi_s(x))\sigma(\varphi_s(x)) - \sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma(\varphi_s(x))} \, \mathrm{d}s \right), \quad t \in [0,T]. \quad (3.10)$$

Lemma 3.4 *Assume that σ and $b_1 := b + \sigma\sigma'/2$ are of class $\mathcal{C}^4(\mathbb{R})$ with bounded derivatives (up to the fourth order) and f is differentiable with bounded derivative. Then:*
(i) If in addition we suppose Condition $(C1)$ of Proposition 3.1, then $X_0 \in \mathbb{D}^{1,p}_{loc}$, for all $p \ge 2$, and

$$D_s X_0 = \frac{f'(\varphi_T(X_0))(D_s\varphi_T)(X_0)}{1 - f'(\varphi_T(X_0))\varphi_T'(X_0)}, \quad a.s., \text{ for } 0 \le s \le T \quad (3.11)$$

where $(D_s\varphi_T)(X_0) = D_s\varphi_T(x)|_{x=X_0}$.
(ii) If in addition we suppose Condition $(C2)$ of Proposition 3.1, then $X_0 \in \mathbb{D}^{1,2}$ and relation (3.11) *holds.*

Proof of Proposition 3.1.
Existence. By Theorem 3.3.1 in [9], for any random variable X_0 the stochastic process $(\varphi_t(X_0))_{t\in[0,T]}$ is a solution of (3.1). Hence, in order to show the existence of a solution to (3.1)-(3.2) we only need to prove that there is a unique random variable X_0 such that

$$X_0 = f(\varphi_T(X_0)). \quad (3.12)$$

Consider the function $h(x) := x - f(\varphi_T(x))$, $x \in \mathbb{R}$. By Lemma 3.3 we have that h is differentiable with derivative

$$h'(x) = 1 - f'(\varphi_T(x))\varphi_T'(x).$$

Part 1.

We first prove the existence under Condition $(C1)$. It is proved in [3] (see p. 171) that

$$\varphi_t'(x) = \exp\left(\int_0^t \left(b_1'(\varphi_s(x)) - \frac{\sigma'(\varphi_s(x))^2}{2}\right) \mathrm{d}s + \int_0^t \sigma'(\varphi_s(x))\,\mathrm{d}B_s\right), \quad t \in [0,T].$$

Therefore $\varphi_T' > 0$ and so $h' > 0$. Since f decreases and φ_T increases we have

$$h(x) > x - f(\varphi_T(0)), \quad \text{for any } x > 0$$

and

$$h(x) < x - f(\varphi_T(0)), \quad \text{for any } x < 0.$$

Letting x goes to $+\infty$ in the first relation and x goes to $-\infty$ in the second relation we easily have

$$\lim_{x\to+\infty} h(x) = +\infty \quad \text{and} \quad \lim_{x\to-\infty} h(x) = -\infty. \tag{3.13}$$

Therefore there exists a unique zero of h, and this implies the existence of a unique random variable X_0 satisfying (3.12). In other words, $(\varphi_t(X_0))_{t\in[0,T]}$ is solution to (3.1)-(3.2).

Part 2.

We now prove the existence under Condition $(C2)$. By (3.10) it easily follows that

$$\frac{c}{\overline{\sigma}} \le \varphi_t'(x) \le \frac{C}{\underline{\sigma}}, \quad (t,x) \in [0,T]\times\mathbb{R} \tag{3.14}$$

where the constants c and C are defined in (3.8). For any $x, y \in \mathbb{R}$, by the non-negativity of φ_T', we have

$$\begin{aligned} 1 - f'(x)\varphi_T'(y) &\ge 1 - \varphi_T'(y)\sup_{x\in\mathbb{R}} f'(x) \\ &\ge 1 - \frac{C}{\underline{\sigma}}\sup_{x\in\mathbb{R}} f'(x) =: M > 0 \end{aligned} \tag{3.15}$$

where the inequality in (3.15) follows by (3.6). Consequently, we have $h' > 0$. Since f is bounded we have (3.13), and we conclude as in the previous step.

Uniqueness. According to Theorem 3.3.2 in [9], to show that $(\varphi_t(X_0))_{t\in[0,T]}$ is the unique solution in $\mathbb{L}^{1,4}_{2,loc}(\mathbb{R})$ which is continuous it suffices to check that X_0 belongs to $\mathbb{D}^{1,p}_{loc}$, for some $p > 4$. Since, under Condition $(C1)$ this is guaranteed by Lemma 3.4(i), we only need to show the claim under Condition $(C2)$. By the mean value Theorem, we have

$$f(\varphi_T(X_0)) = f(0) + f'(\varepsilon_1)\varphi_T(X_0) \quad \text{and} \quad \varphi_T(X_0) = \varphi_T(0) + \varphi_T'(\varepsilon_2)X_0,$$

where $\varepsilon_1, \varepsilon_2$ are two random variables lying between 0 and $\varphi_T(X_0)$ and 0 and X_0, respectively. Since $X_0 = f(\varphi_T(X_0))$, we have

$$X_0 = f(0) + f'(\varepsilon_1)\varphi_T(X_0) = f(0) + f'(\varepsilon_1)[\varphi_T(0) + \varphi'_T(\varepsilon_2)X_0],$$

and therefore

$$X_0 = \frac{f(0) + f'(\varepsilon_1)\varphi_T(0)}{1 - f'(\varepsilon_1)\varphi'_T(\varepsilon_2)} \tag{3.16}$$

if the denominator is different from zero. We preliminary note that the denominator of (3.16) is different from zero due to (3.15). By (3.15), (3.16) and the boundedness of f', we deduce that there exist two finite constants $M_1, M_2 > 0$ such that

$$|X_0| \leq M_1 + M_2|\varphi_T(0)|, \quad a.s..$$

By the boundedness of σ' and b' we have that σ and b are Lipschitz continuous and have at most linear growth. Therefore by Lemma 2.2.1 in [9] we deduce $\varphi_T(0) \in \bigcap_{p\geq 1} L^p(\Omega)$ and so $X_0 \in \bigcap_{p\geq 1} L^p(\Omega)$. By Lemma 3.4($ii$) we have $X_0 \in \mathbb{D}^{1,2}$ and (3.11). By Lemma 3.3 we have

$$c \leq (D_s\varphi_t)(x) \leq C, \qquad s, t \in [0, T], \quad x \in \mathbb{R}. \tag{3.17}$$

Using this relation, the boundedness of f' and (3.15), by (3.11) it follows that there exists a constant $M_3 > 0$ so that

$$|D_s X_0| \leq M_3, \quad \text{a.s. for any } s \in [0, T]$$

which implies $\mathrm{E}\left[\int_0^T |D_s X_0|^p \, \mathrm{d}s\right] < \infty$ for any $p \geq 1$. By Proposition 1.5.5 in [9] we then have $X_0 \in \mathbb{D}^{1,p}$ for any $p \geq 2$. □

Proof of Theorem 3.2.

We preliminary note that under Condition ($C3$) is satisfied Condition ($C1$) of Proposition 3.1 and under Condition ($C4$) is satisfied Condition ($C2$) of Proposition 3.1. By Lemmas 3.3 and 3.4 and Exercise 1.3.6 p. 52 in [9] we have $X_t \in \mathbb{D}^{1,2}_{loc}$, $t \in [0, T]$, and, for any $s, t \in [0, T]$,

$$D_s X_t = D_s\varphi_t(X_0) = \begin{cases} (D_s\varphi_t)(X_0) + \varphi'_t(X_0)D_s X_0, & s \leq t, \\ \\ \varphi'_t(X_0)D_s X_0, & s > t. \end{cases} \tag{3.18}$$

In fact, $X_t \in \mathbb{D}^{1,2}$ because $(\varphi_t(x))_{t\in[0,T]} \in \mathbb{D}^{1,2}$ for any $x \in \mathbb{R}$ and $X_0 \in \mathbb{D}^{1,2}$ (this can be proved as in the proof of Lemma 3.4(ii) below, using that σ is bounded). By (3.11) we deduce

$$\varphi'_t(X_0)D_s X_0 = \frac{\varphi'_t(X_0)(D_s\varphi_T)(X_0)}{1 - \varphi'_T(X_0)f'(\varphi_T(X_0))} f'(\varphi_T(X_0)). \tag{3.19}$$

Part 1.

We first prove the claim under Condition ($C3$). We have

$$\inf_{x\in\mathbb{R}} f'(x) \leq f'(\varphi_T(X_0)) \leq \sup_{x\in\mathbb{R}} f'(x) < 0. \tag{3.20}$$

Combining this with (3.14) and (3.17), we deduce

$$0 \leq \frac{\varphi_t'(X_0)(D_s\varphi_T)(X_0)}{1-\varphi_T'(X_0)f'(\varphi_T(X_0))} \leq \varphi_t'(X_0)(D_s\varphi_T)(X_0) \leq \frac{C^2}{\underline{\sigma}}. \tag{3.21}$$

By (3.19), (3.20) and (3.21), we have

$$c_* := \frac{C^2}{\underline{\sigma}} \inf_{x\in\mathbb{R}} f'(x) \leq \varphi_t'(X_0)D_sX_0 < 0.$$

Combining the above with (3.17), by the expression of D_sX_t computed in (3.18) we find

$$c + c_* \leq D_sX_t \leq C, \qquad 0 \leq s \leq t,$$

and

$$c_* \leq D_sX_t < 0, \qquad t < s \leq T.$$

We also note that $c + c_* > 0$, indeed

$$c_* = \frac{C^2}{\underline{\sigma}} \inf_{x\in\mathbb{R}} f'(x) > -\frac{C^2}{\underline{\sigma}}\frac{c\underline{\sigma}}{C^2} = -c.$$

These inequalities yield

$$(c+c_*)^2 t \leq \int_0^T D_sX_t\mathrm{E}[D_sX_t \,|\, \mathcal{F}_s]\,\mathrm{d}s \leq C^2t + c_*^2(T-t), \quad t \in [0,T].$$

The claim follows applying Theorem 2.4 to $F := X_t - \mathrm{E}[X_t]$.

Part 2.

We now prove the claim under Condition ($C4$). By Condition ($C4$) and (3.14) we have

$$\frac{1}{\varphi_T'(X_0)} - f'(\varphi_T(X_0)) \geq \frac{\underline{\sigma}}{C} - \sup_{x\in\mathbb{R}} f'(x) > 0.$$

Combining this inequality with (3.14) and (3.17) yields

$$0 \leq \frac{\varphi_t'(X_0)(D_s\varphi_T)(X_0)}{\varphi_T'(X_0)\left(\frac{1}{\varphi_T'(X_0)} - f'(\varphi_T(X_0))\right)} \leq \frac{C\overline{\sigma}}{c\underline{\sigma}}\frac{C}{\frac{\underline{\sigma}}{C} - \sup_{x\in\mathbb{R}} f'(x)}.$$

Recalling (3.11), we have

$$|\varphi_t'(X_0)D_sX_0| \leq \frac{C\overline{\sigma}}{c\underline{\sigma}}\frac{C\overline{f'}}{\underline{\sigma}/C - \sup_{x\in\mathbb{R}} f'(x)} = C_*.$$

Therefore, by (3.17) and the expression of D_sX_t computed at the beginning of the proof, we have

$$c - C_* \le D_sX_t \le C + C_*, \quad 0 \le s \le t \tag{3.22}$$

and

$$-C_* \le D_sX_t \le C_*, \quad t < s \le T. \tag{3.23}$$

Using the foregoing assumption ($C4$), one may easily see that $c - C_* > 0$. Thus, by (3.22) we deduce

$$(c - C_*)^2 t \le \int_0^t D_sX_t \mathrm{E}[D_sX_t \,|\, \mathcal{F}_s] \,\mathrm{d}s \le (C + C_*)^2 t. \tag{3.24}$$

Since $D_sX_t\mathrm{E}[D_sX_t \,|\, \mathcal{F}_s] \ge 0$, by (3.23) we have

$$0 \le \int_t^T D_sX_t \mathrm{E}[D_sX_t \,|\, \mathcal{F}_s] \,\mathrm{d}s \le C_*^2(T - t). \tag{3.25}$$

Finally, combining (3.24) and (3.25) we deduce

$$(c - C_*)^2 t \le \int_0^T D_sX_t \mathrm{E}[D_sX_t \,|\, \mathcal{F}_s] \,\mathrm{d}s \le (C + C_*)^2 t + C_*^2(T - t), \quad t \in [0, T].$$

The claim follows applying Theorem 2.4 to $F := X_t - \mathrm{E}[X_t]$. □

Proof of Lemma 3.3. The Malliavin differentiability of $(\varphi_t(x))_{t\in[0,T]}$ and the existence of $\varphi_t'(x)$ were proved in [12] (see also Theorem 2.2.1 in [9] for the Malliavin differentiability). Let us check the expressions (3.9) and (3.10). Set

$$F(x) := \int_0^x \frac{1}{\sigma(z)} \,\mathrm{d}z, \qquad x \ge 0,$$

and $\Phi_t(x) := F(\varphi_t(x))$. By the Itô formula and (3.3), we have

$$\begin{aligned}
\mathrm{d}\Phi_t(x) &= \left(F'(\varphi_t(x)) b_1(\varphi_t(x)) + \frac{1}{2} F''(\varphi_t(x)) (\sigma(\varphi_t(x)))^2 \right) \mathrm{d}t + F'(\varphi_t(x)) \sigma(\varphi_t(x)) \,\mathrm{d}B_t \\
&= \left(\frac{b_1(\varphi_t(x))}{\sigma(\varphi_t(x))} - \frac{1}{2} \sigma'(\varphi_t(x)) \right) \mathrm{d}t + \mathrm{d}B_t.
\end{aligned}$$

i.e.

$$\Phi_t(x) = F(x) + \int_0^t \frac{b(\varphi_s(x))}{\sigma(\varphi_s(x))} \,\mathrm{d}s + B_t \tag{3.26}$$

Since $(\varphi_t(x))_{t\in[0,T]}$ is Malliavin differentiable and F is continuously differentiable with bounded first derivative (because σ is positive and bounded from below), then by Proposition

1.2.3 in [9] we have $(\Phi_t(x))_{t\in[0,T]}\in\mathbb{D}^{1,2}$. By Theorem 2.2.1 in [9] (note that the function b/σ is Lipschitz since by the assumptions on b and σ it follows $(b/\sigma)'$ bounded) we have

$$D_\theta\Phi_t(x)=D_\theta\left(F(x)+\int_0^t\frac{b(\varphi_s(x))}{\sigma(\varphi_s(x))}\,\mathrm{d}s+B_t\right)=0,\quad\text{for }\theta>t$$

and, for $0\le\theta\le t$,

$$D_\theta\Phi_t(x)=1+\int_\theta^t\frac{b'(\varphi_s(x))\sigma(\varphi_s(x))-\sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma^2(\varphi_s(x))}D_\theta\varphi_s(x)\,\mathrm{d}s.\tag{3.27}$$

By Proposition 1.2.3 in [9], for $0\le\theta\le t$, we also have

$$D_\theta\Phi_t(x)=F'(\varphi_t(x))D_\theta\varphi_t(x)=\frac{1}{\sigma(\varphi_t(x))}D_\theta\varphi_t(x).\tag{3.28}$$

Combining this with (3.27), for $0\le\theta\le t$, we deduce

$$D_\theta\Phi_t(x)=1+\int_\theta^t\frac{b'(\varphi_s(x))\sigma(\varphi_s(x))-\sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma(\varphi_s(x))}D_\theta\Phi_s(x)\,\mathrm{d}s.$$

This is a linear ordinary differential equation with initial condition $D_\theta\Phi_\theta(x)=1$, and therefore

$$D_\theta\Phi_t(x)=\exp\left(\int_\theta^t\frac{b'(\varphi_s(x))\sigma(\varphi_s(x))-\sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma(\varphi_s(x))}\,\mathrm{d}s\right)\mathbb{1}_{[0,t]}(\theta).$$

Relation (3.9) follows by this latter equality noticing that by Theorem 2.2.1 in [9], for $\theta>t$, one has $D_\theta\varphi_t(x)=0$ and by (3.28), for $0\le\theta\le t$, it holds $D_\theta\varphi_t(x)=\sigma(\varphi_t(x))D_\theta\Phi_t(x)$. It remains to verify (3.10). Since

$$\Phi_t'(x)=\frac{\varphi_t'(x)}{\sigma(\varphi_t(x))},\tag{3.29}$$

differentiating (3.26) with respect to x we deduce

$$\begin{aligned}\Phi_t'(x)&=\frac{1}{\sigma(x)}+\int_0^t\frac{b'(\varphi_s(x))\sigma(\varphi_s(x))-\sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma^2(\varphi_s(x))}\varphi_s'(x)\,\mathrm{d}s\\&=\frac{1}{\sigma(x)}+\int_0^t\frac{b'(\varphi_s(x))\sigma(\varphi_s(x))-\sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma(\varphi_s(x))}\Phi_s'(x)\,\mathrm{d}s,\quad t\in[0,T].\end{aligned}\tag{3.30}$$

Solving (3.30) we have

$$\Phi_t'(x)=\frac{1}{\sigma(x)}\exp\left(\int_0^t\frac{b'(\varphi_s(x))\sigma(\varphi_s(x))-\sigma'(\varphi_s(x))b(\varphi_s(x))}{\sigma(\varphi_s(x))}\,\mathrm{d}s\right),\quad t\in[0,T].$$

The claim follows combining this equality with (3.29). □

Proof of Lemma 3.4. As shown in the part on the *Existence* of the proof of Proposition 3.1, equation (3.1)-(3.2) has a solution which is given by $X_t = \varphi_t(X_0)$, where X_0 is the unique random variable satisfying $X_0 = f(\varphi_T(X_0))$.

Proof of (i).

By e.g. Exercise 2.2.2 in [9] one can represent the flow $\varphi_t(x)$ as a Frechet differentiable function of the Brownian motion B. Using this and the implicit function Theorem (note that the first order derivative of f^{-1} is always different from zero) one deduces that $X_0 \in \mathbb{D}^{1,p}_{loc}$ for any $p \geq 2$. By Lemma 3.3 and Exercise 1.3.6 p. 52 in [9], for any $s \in [0, T]$,

$$D_s X_0 = (f \circ \varphi_T)'(X_0) D_s X_0 + D_s (f \circ \varphi_T)(X_0)$$

where

$$D_s (f \circ \varphi_T)(X_0) := D_s (f \circ \varphi_T)(x)|_{x=X_0}.$$

By the chain rule in Proposition 1.2.3 of [9], we have

$$D_s (f \circ \varphi_T)(X_0) := D_s (f \circ \varphi_T)(x)|_{x=X_0} = f'(\varphi_T(x)) D_s \varphi_T(x)|_{x=X_0} = f'(\varphi_T(X_0))(D_s \varphi_T)(X_0),$$

and so

$$D_s X_0 = \frac{f'(\varphi_T(X_0))(D_s \varphi_T)(X_0)}{1 - f'(\varphi_T(X_0))\varphi_T'(X_0)},$$

note that our assumptions guarantee that the denominator is different from zero.

Proof of (ii).

Letting $(\Omega, \mathcal{F}, P)$ denote the canonical space of the standard Brownian motion indexed on $[0, T]$, for $q \in L^2([0, T], \mathrm{d}x)$ and $\omega \in \Omega$, we set $\omega_q(\cdot) = \omega(\cdot) + \int_0^{\cdot} q(s)\,\mathrm{d}s$. In the following we write $\varphi_t(\omega, x)$ in place of $\varphi_t(x)$ to explicit the dependence on ω of $\varphi_t(x)$, i.e.

$$\varphi_t(\omega, x) = x + \int_0^t b_1(\varphi_s(\omega, x))\,\mathrm{d}s + \int_0^t \sigma(\varphi_s(\omega, x))\,\mathrm{d}B_s(\omega).$$

By the mean value Theorem, we have

$$\begin{aligned} X_0(\omega_q) - X_0(\omega) &= f(\varphi_T(\omega_q, X_0(\omega_q))) - f(\varphi_T(\omega, X_0(\omega))) \\ &= f(\varphi_T(\omega_q, X_0(\omega_q))) - f(\varphi_T(\omega_q, X_0(\omega))) + f(\varphi_T(\omega_q, X_0(\omega))) - f(\varphi_T(\omega, X_0(\omega))) \\ &= f'(\varphi_T(\omega_q, \xi(\omega, \omega_q)))\varphi_T'(\omega_q, \xi(\omega, \omega_q))[X_0(\omega_q) - X_0(\omega)] \\ &\quad + f(\varphi_T(\omega_q, X_0(\omega))) - f(\varphi_T(\omega, X_0(\omega))), \end{aligned}$$

where $\xi(\omega, \omega_q)$ is a random variable between $X_0(\omega)$ and $X_0(\omega_q)$. Therefore,

$$X_0(\omega_q) - X_0(\omega) = \frac{f(\varphi_T(\omega_q, X_0(\omega))) - f(\varphi_T(\omega, X_0(\omega)))}{1 - f'(\varphi_T(\omega_q, \xi(\omega, \omega_q)))\varphi_T'(\omega_q, \xi(\omega, \omega_q))},$$

note that our assumptions guarantee that the denominator is different from zero. By the boundedness of f' and σ, for some constant $C > 0$ (which may vary from line to line) we have

$$
\begin{aligned}
|X_0(\omega_q) - X_0(\omega)| &\leq C|\varphi_T(\omega_q, X_0(\omega)) - \varphi_T(\omega, X_0(\omega))| \\
&= C\left| \int_0^T \sigma(\varphi_s(X_0(\omega)))\mathrm{d}B_s(\omega_q) - \int_0^T \sigma(\varphi_s(X_0(\omega)))\mathrm{d}B_s(\omega) \right| \\
&\leq C \int_0^T |q(s)| \, \mathrm{d}s \qquad (3.31) \\
&\leq C \left(\int_0^T |q(s)|^2 \, \mathrm{d}s \right)^{1/2}, \qquad (3.32)
\end{aligned}
$$

where (3.31) follows by noticing that on the canonical space $B_t(\omega) = \omega(t)$ and (3.32) is a consequence of the Cauchy-Schwarz inequality. So $X_0 \in \mathbb{D}^{1,2}$ by Exercise 1.2.9 in [9]. Relation (3.11) follows repeating the arguments of the previous step. □

4 Gaussian estimates of one-dimensional additive functionals of SDEs with irregular drifts

Here we consider the one-dimensional diffusion equation on the probability space $(\Omega, \mathcal{F}, P)$

$$
X_t = x_0 + \int_0^t b(X_s)\,\mathrm{d}s + \int_0^t \sigma(X_s) \circ \mathrm{d}B_s, \quad x_0 \in \mathbb{R}, \quad t \in [0,T], \tag{4.1}
$$

where $b : \mathbb{R} \to \mathbb{R}$ is measurable and bounded, σ is smooth, i.e. $\sigma \in \mathcal{C}_b^\infty(\mathbb{R})$, and such that $\underline{\sigma} := \inf_{x\in\mathbb{R}} \sigma(x) > 0$ and the stochastic integral is the Stratonovich integral. Under the above assumptions, Kohatsu-Higa and Tanaka [6] proved the existence and smoothness of the density of the random variable

$$
Y_t = \int_0^t \psi(X_s)\,\mathrm{d}s, \quad t \in [0,T], \tag{4.2}
$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a smooth function.

In this section we provide Gaussian estimates for the tails of Y_t. Letting $(W_t)_{t\in[0,T]}$ denote a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, Q)$, we consider the system (X_t, Y_t) defined by

$$
X_t = x_0 + \int_0^t \sigma(X_s) \circ \mathrm{d}W_s, \tag{4.3}
$$

and

$$Y_t = \int_0^t \psi(X_s)\,\mathrm{d}s, \quad t \in [0,T]. \tag{4.4}$$

We define a new probability measure $\tilde{Q}$ on $(\Omega, \mathcal{F}_T)$ by

$$\mathrm{d}\tilde{Q} = Z_T \mathrm{d}Q,$$

where

$$Z_t := \exp\left(\int_0^t b_0(X_s)\,\mathrm{d}W_s - \frac{1}{2}\int_0^t b_0^2(X_s)\,\mathrm{d}s\right), \quad t \in [0,T],$$

and b_0 is the bounded and measurable function $b_0 := b/\sigma$. By Girsanov's theorem, the stochastic process $\tilde{B}_t = W_t - \int_0^t b_0(X_s)\,\mathrm{d}s$ is a Brownian motion under the probability measure $\tilde{Q}$, and so the solution of the system (4.3)-(4.4) under $\tilde{Q}$ is equal in law to the solution of the system (4.1)-(4.2).

Theorem 4.1 *Under the foregoing assumptions, if moreover* $\underline{\psi'} := \inf_{x\in\mathbb{R}} \psi'(x) > 0$*, then, for each* $t \in (0,T]$*, the tails of* Y_t *satisfy the bounds*

$$P(Y_t \geq x) \leq \exp\left(-\frac{(x - \mathrm{E}_Q[Y_t])^2}{4G(t)}\right)\exp\left(\frac{1}{2}\|b_0\|_\infty^2 t\right), \quad x > \mathrm{E}_Q[Y_t],$$

$$P(Y_t \leq -x) \leq \exp\left(-\frac{(x + \mathrm{E}_Q[Y_t])^2}{4G(t)}\right)\exp\left(\frac{1}{2}\|b_0\|_\infty^2 t\right), \quad x < -\mathrm{E}_Q[Y_t].$$

where $\|b_0\|_\infty := \sup_{x\in\mathbb{R}} |b_0(x)|$, $G(t) := (\overline{\psi'}\overline{\sigma})^2 t^3/3$, $\overline{\sigma} := \sup_{x\in\mathbb{R}} \sigma(x)$, $\overline{\psi'} := \sup_{x\in\mathbb{R}} \psi'(x)$ *and* E_Q *denotes the expectation under the probability measure* Q*.*

Proof. For any $x \in \mathbb{R}$ we have

$$P(Y_t \geq x) = \mathrm{E}_{\tilde{Q}}[\mathbb{1}_{[x,\infty)}(Y_t)] = \mathrm{E}_Q[\mathbb{1}_{[x,\infty)}(Y_t)Z_t], \qquad t \in [0,T].$$

By this relation and the Cauchy-Schwarz inequality, we have

$$P(Y_t \geq x) \leq \mathrm{E}_Q[\mathbb{1}_{[x,\infty)}(Y_t)^2]^{1/2}\mathrm{E}_Q[Z_t^2]^{1/2} = \mathrm{E}_Q[\mathbb{1}_{[x,\infty)}(Y_t)]^{1/2}\mathrm{E}_Q[Z_t^2]^{1/2}, \qquad t \in [0,T]. \tag{4.5}$$

Since b_0 is bounded, standard computations (see e.g. Lemma 1 in [5]) give

$$\mathrm{E}_Q[Z_t^p] \leq \exp\left(\frac{1}{2}|p^2 - p|\|b_0\|_\infty^2 t\right), \quad \text{for any } p \in \mathbb{R}. \tag{4.6}$$

Combining (4.5) with (4.6) we deduce

$$P(Y_t \geq x) \leq Q(Y_t \geq x)^{1/2} \exp\left(\frac{1}{2}\|b_0\|_\infty^2 t\right). \tag{4.7}$$

Now we bound the probability $Q(Y_t \geq x)$. By Lemma 3.3 we have that the Malliavin derivative of X_s with respect to W is

$$D_\theta X_s = \sigma(X_s)\mathbf{1}_{[0,s]}(\theta), \qquad s \in \mathbb{R}_+,$$

therefore by Proposition 1.2.3 in [9] we have

$$D_\theta Y_t = \int_\theta^t D_\theta \psi(X_s)\,\mathrm{d}s = \mathbb{1}_{[0,t]}(\theta)\int_\theta^t \psi'(X_s)\sigma(X_s)\,\mathrm{d}s,$$

where the exchange between D_θ and the integral can be justified by standard closability arguments. Consequently we have

$$\underline{\psi'}\underline{\sigma}(t-\theta)\mathbb{1}_{[0,t]}(\theta) \leq D_\theta Y_t \leq \overline{\psi'}\overline{\sigma}(t-\theta)\mathbb{1}_{[0,t]}(\theta),$$

hence

$$g(t) \leq \int_0^T D_s Y_t \mathrm{E}_Q[D_s Y_t \,|\, \mathcal{F}_s]\,\mathrm{d}s = \int_0^t D_s Y_t \mathrm{E}_Q[D_s Y_t \,|\, \mathcal{F}_s]\,\mathrm{d}s \leq G(t),$$

where

$$g(t) := (\underline{\psi'}\underline{\sigma})^2 t^3/3.$$

By Theorem 2.4, for each $t \in (0,T]$ and $x > \mathrm{E}_Q[Y_t]$, we have

$$Q(Y_t \geq x) = Q(Y_t - \mathrm{E}_Q[Y_t] \geq x - \mathrm{E}_Q[Y_t]) \leq \exp\left(-\frac{(x-\mathrm{E}_Q[Y_t])^2}{2G(t)}\right). \tag{4.8}$$

The Gaussian bound for the tail $P(Y_t \geq x)$ follows by (4.7) and (4.8). The Gaussian bound for the tail $P(Y_t \leq -x)$ can be proved similarly. □

5 Gaussian estimates of one-dimensional SDEs driven by fractional Brownian motion

In this section, we provide Gaussian estimates for the density and the tail probabilities of solutions to stochastic differential equations driven by the fractional Brownian motion. Some results in this direction can be found in [2].

Recall that a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = (B_t^H)_{t\in\mathbb{R}_+}$ with covariance function

$$R_H(t,s) := \mathrm{E}[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H > 1/2$, B_t^H admits the so-called Volterra representation (see e.g. [9] pp. 277-279)

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}B_s, \tag{5.1}$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion,

$$K_H(t,s) := c_H\, s^{1/2-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-1/2}\,\mathrm{d}u, \quad s \le t$$

and

$$c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}}, \quad \text{where } \beta \text{ is the Beta function.}$$

We suppose that $(B_t)_{t\in[0,T]}$ is defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, where $\mathcal{F} = \sigma(\mathcal{F}_t^B, \mathcal{N})$, being $\mathcal{N}$ the family of sets with probability zero. Consequently, by (5.1) the fBm $(B_t^H)_{t\in[0,T]}$ is $\mathcal{F}_t$-adapted.

Hereafter, we denote by $\mathcal{C}_b^{1,1}([0,T]\times\mathbb{R})$ the space of bounded functions $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ with bounded partial derivatives of the first order and we define

$$f_1'(t,x) := \frac{\partial f}{\partial t}(t,x), \quad f_2'(t,x) := \frac{\partial f}{\partial x}(t,x).$$

Throughout this section we will work under the following condition.

Assumption 5.1 $H \in (1/2, 1)$, $b, \sigma \in \mathcal{C}_b^{1,1}([0,T]\times\mathbb{R})$, *there exists a constant* $c > 0$ *so that* $|\sigma(t,x)| \ge c$ *for all* $(t,x) \in [0,T]\times\mathbb{R}$.

Under this assumption the stochastic integral $\int_0^t \sigma(s, X_s)\,\mathrm{d}B_s^H$, $t \in [0,T]$, exists as a pathwise Riemann-Stieltjes integral if the stochastic process $(X_t)_{t\in[0,T]}$ is Hölder continuous of order $H - \varepsilon$ for all $\varepsilon \in (0, H)$. (see [15]; see also [9] p. 312).

Before stating the main result of this section, we consider the following lemma, whose proof will be given later on.

Lemma 5.1 *Under Assumption 5.1, there exists a unique strong solution of*

$$X_t = x_0 + \int_0^t b(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}B_s^H, \quad t \in [0,T], \tag{5.2}$$

where by definition a strong solution to (5.2) *is an* $\mathcal{F}_t$*-adapted process* $(X_t)_{t\in[0,T]}$ *which satisfies* (5.2) *pathwise and whose trajectories are Hölder continuous of order* $H - \varepsilon$ *for all* $\varepsilon \in (0, H)$.

The following estimates hold.

Theorem 5.2 *Suppose Assumption 5.1 and let $(X_t)_{t\in[0,T]}$ be the unique strong solution to (5.2). Then, for each $t \in (0,T]$, the density p_{X_t} and the tails of X_t satisfy the bounds*

$$\frac{\mathrm{E}[|X_t - \mathrm{E}[X_t]|]}{2G(t)} \exp\left(-\frac{(x-\mathrm{E}[X_t])^2}{2g(t)}\right) \le p_{X_t}(x)$$
$$\le \frac{\mathrm{E}[|X_t - \mathrm{E}[X_t]|]}{2g(t)} \exp\left(-\frac{(x-\mathrm{E}[X_t])^2}{2G(t)}\right), \quad a.e.\ x \in \mathbb{R}, \quad (5.3)$$

$$P(X_t - \mathrm{E}[X_t] \ge x) \le \exp\left(-\frac{x^2}{2G(t)}\right), \qquad x > 0,$$

and

$$P(X_t - \mathrm{E}[X_t] \le -x) \le \exp\left(-\frac{x^2}{2G(t)}\right), \qquad x > 0,$$

where the functions $g, G : (0,T] \to (0,\infty)$ are defined by $G(t) := C^2 \mathrm{e}^{2MT} t^{2H}$ and $g(t) := c^2 \mathrm{e}^{-2MT} t^{2H}$, with $C = \sup_{(t,x)} |\sigma(t,x)|$ and

$$M = \sup_{(t,x)} \left| b_2'(t,x) - \frac{b(t,x)\sigma_2'(t,x)}{\sigma(t,x)} - \frac{\sigma_1'(t,x)}{\sigma(t,x)} \right| < \infty.$$

The proof of this theorem uses the next lemma which provides the Malliavin derivative of the unique strong solution to (5.2).

Lemma 5.3 *Under Assumption 5.1 the unique strong solution $(X_t)_{t\in[0,T]}$ to (5.2) is such that $(X_t)_{t\in[0,T]} \in \mathbb{D}^{1,2}$ with*

$$D_s X_t = \sigma(t,X_t) \mathbb{1}_{[0,t]}(s) \left(\int_s^t (K_H)_1'(v,s) \exp\left(\int_v^t \left(b_2' - \frac{b\sigma_2'}{\sigma} - \frac{\sigma_1'}{\sigma} \right)(u, X_u)\, \mathrm{d}u \right) \mathrm{d}v \right).$$

Proof of Theorem 5.2. We assume $\sigma(t,X_t) > 0$, i.e. $c \le \sigma(t,X_t) \le C$. The case $\sigma(t,X_t) < 0$, i.e. $-C \le \sigma(t,X_t) \le -c$ can be treated similarly. By Lemma 5.3, since $(K_H)_1'$ is non-negative we have

$$c\mathrm{e}^{-MT} \int_s^t (K_H)_1'(v,s)\, \mathrm{d}v \le D_s X_t \le C\mathrm{e}^{MT} \int_s^t (K_H)_1'(v,s)\, \mathrm{d}v, \quad s \le t,$$

i.e.

$$c\mathrm{e}^{-MT} K_H(t,s) \le D_s X_t \le C\mathrm{e}^{MT} K_H(t,s), \quad s \le t.$$

This yields

$$c^2 \mathrm{e}^{-2MT} \int_0^t K_H^2(t,s)\, \mathrm{d}s \le \int_0^T D_s X_t \mathrm{E}[D_s X_t \,|\, \mathcal{F}_s]\, \mathrm{d}s$$
$$= \int_0^t D_s X_t \mathrm{E}[D_s X_t \,|\, \mathcal{F}_s]\, \mathrm{d}s \le C^2 \mathrm{e}^{2MT} \int_0^t K_H^2(t,s)\, \mathrm{d}s.$$

Since $\int_0^t K_H^2(t,s)\,\mathrm{d}s = \mathrm{E}[|B_t^H|^2] = t^{2H}$, we have

$$0 < c^2 \mathrm{e}^{-2MT} t^{2H} \le \int_0^T D_s X_t \mathrm{E}[D_s X_t \,|\, \mathcal{F}_s]\,\mathrm{d}s \le C^2 e^{2MT} t^{2H}.$$

The claim then follows by Theorem 2.4. □

Proof of Lemma 5.1. Define the function

$$F(t,x) := \int_0^x \frac{1}{\sigma(t,z)}\,\mathrm{d}z, \quad (t,x) \in [0,T] \times \mathbb{R}.$$

For $(t,u) \in [0,T] \times \mathbb{R}$, consider the function $\Phi(t,u) := F(t,u) - x$, where $x \in \mathbb{R}$ is fixed. Since $\Phi_2'(t,u) = \sigma(t,u)^{-1} \neq 0$, by the Implicit Function Theorem we have that there exists a function $G(t,x)$ so that $\Phi(t, G(t,x)) = 0$, i.e. $F(t, G(t,x)) = x$. Note that

$$F_2'(t,x) = \sigma(t,x)^{-1} \quad \text{and} \quad F_1'(t,x) = -\int_0^x \frac{\sigma_1'(t,z)}{\sigma(t,z)^2}\,\mathrm{d}z,$$

$$G_2'(t,x) = (F_2'(t, G(t,x)))^{-1} = \sigma(t, G(t,x)), \tag{5.4}$$

$$G_1'(t,x) = -\frac{F_1'(t,G(t,x))}{F_2'(t,G(t,x))} = -F_1'(t, G(t,x))\sigma(t, G(t,x)). \tag{5.5}$$

Existence. We start by showing that there exists a strong solution of

$$Y_t = y_0 + \int_0^t A(s, Y_s)\,\mathrm{d}s + B_t^H, \tag{5.6}$$

where $y_0 = F(0, x_0)$ and

$$A(t,y) = F_1'(t, G(t,y)) + \frac{b(t, G(t,y))}{\sigma(t, G(t,y))}. \tag{5.7}$$

A straightforward computation shows that $A_2'(t,y)$ is uniformly bounded, and so for some positive constant $C > 0$ (which may vary from line to line)

$$|A(t,x) - A(t,y)| \le C|x-y| \quad \text{and} \quad |A(t,x)|^2 \le C(1+|x|^2) \quad \text{for all } (t,x,y) \in [0,T] \times \mathbb{R}^2. \tag{5.8}$$

For $n \ge 0$, consider the processes $(Y_t^{(n)})_{t\in[0,T]}$ defined recursively by

$$\begin{cases} Y_t^{(0)} = y_0 + B_t^H, \\ Y_t^{(n+1)} = y_0 + \int_0^t A(s, Y_s^{(n)})\,\mathrm{d}s + B_t^H, \quad n \ge 0. \end{cases}$$

For some constant $C > 0$, by the Cauchy-Schwarz inequality and (5.8), we have

$$\mathrm{E}\left[\sup_{t\in[0,T]} |Y_t^{(1)} - Y_t^{(0)}|^2\right] \le \mathrm{E}\left[\left(\int_0^T |A(s, Y_s^{(0)})|\,\mathrm{d}s\right)^2\right]$$

$$
\begin{aligned}
&\le C\mathrm{E}\left[\int_0^T |A(s,Y_s^{(0)})|^2\,\mathrm{d}s\right]\\
&\le C\mathrm{E}\left[\int_0^T (1+|Y_s^{(0)}|^2)\,\mathrm{d}s\right]\\
&\le CT + 2C\mathrm{E}\left[\int_0^T (y_0^2+|B_s^H|^2)\,\mathrm{d}s\right]\\
&= CT(1+2y_0^2) + 2C\int_0^T s^{2H}\,\mathrm{d}s =: C_1. \qquad (5.9)
\end{aligned}
$$

For any $n \ge 1$ and some constant $C > 0$, by the Cauchy-Schwarz inequality and (5.8) we have

$$
\begin{aligned}
\mathrm{E}\left[\sup_{u\in[0,t]} |Y_u^{(n+1)} - Y_u^{(n)}|^2\right] &\le \mathrm{E}\left[\left(\int_0^t |A(s,Y_s^{(n)}) - A(s,Y_s^{(n-1)})|\,\mathrm{d}s\right)^2\right]\\
&\le \mathrm{E}\left[\left(\int_0^t \sup_{u\in[0,s]} |A(u,Y_u^{(n)}) - A(u,Y_u^{(n-1)})|\,\mathrm{d}s\right)^2\right]\\
&\le C\mathrm{E}\left[\left(\int_0^t \sup_{u\in[0,s]} |Y_u^{(n)} - Y_u^{(n-1)}|\,\mathrm{d}s\right)^2\right]\\
&\le Ct\int_0^t \mathrm{E}\left[\sup_{u\in[0,s]} |Y_u^{(n)} - Y_u^{(n-1)}|^2\right]\,\mathrm{d}s\\
&\le CT\int_0^t \mathrm{E}\left[\sup_{u\in[0,s]} |Y_u^{(n)} - Y_u^{(n-1)}|^2\right]\,\mathrm{d}s.
\end{aligned}
$$

Iterating this inequality and using (5.9), we deduce

$$
\begin{aligned}
\mathrm{E}\left[\sup_{u\in[0,t]} |Y_u^{(n+1)} - Y_u^{(n)}|^2\right] &\le (CT)^n \int_0^t\int_0^{s_1}\cdots\int_0^{s_{n-1}} \mathrm{E}\left[\sup_{u\in[0,s_n]} |Y_u^{(1)} - Y_u^{(0)}|^2\right]\,\mathrm{d}s_n\ldots\mathrm{d}s_2\mathrm{d}s_1\\
&\le C_1\frac{(CT^2)^n}{n!}.
\end{aligned}
$$

By this inequality we easily have that there exists a process $(Y_t^*)_{t\in[0,T]}$ such that $Y_t^{(n')} \to Y_t^*$ almost surely for any $t \in [0,T]$ and some subsequence $(Y_t^{(n')})_{n'}$ of $(Y_t^{(n)})_{n\ge0}$. Then one easily sees that $(Y_t^*)_{t\in[0,T]}$ satisfies (5.6) pathwise and Y_t^* is $\mathcal{F}_t$-measurable as a.s. limit of $\mathcal{F}_t$-measurable random variables. It remains to check that the paths of $(Y_t^*)_{t\in[0,T]}$ are Hölder continuous of order $(H-\varepsilon)$. By (5.6), the boundedness of A and the fact that the paths of $(B_t^H)_{t\in[0,T]}$ are Hölder continuous of order $H-\varepsilon$, for any $\varepsilon > 0$ (see e.g. [9] p. 274), for any $t_1, t_2 \in [0,T]$, some positive constants C, C' and random variable G, we have

$$
|Y_{t_2}^* - Y_{t_1}^*| = \left|\int_{t_1}^{t_2} A(s,Y_s^*)\,\mathrm{d}s\right| + |B_{t_2}^H - B_{t_1}^H|
$$

$$
\begin{aligned}
&\le C|t_2-t_1| + G|t_1-t_2|^{H-\varepsilon} \\
&= C|t_2-t_1|^{[1-(H-\varepsilon)]+(H-\varepsilon)} + G|t_1-t_2|^{H-\varepsilon} && (5.10)\\
&\le C'|t_2-t_1|^{H-\varepsilon} + G|t_1-t_2|^{H-\varepsilon} = (C'+G)|t_1-t_2|^{H-\varepsilon}, && (5.11)
\end{aligned}
$$

where the latter inequality follows by $1 > H-\varepsilon$. Now, we show that the process $(X_t^*)_{t\in[0,T]}$ defined by $X_t^* := G(t, Y_t^*)$ is solution of (5.2). Clearly, X_t^* is $\mathcal{F}_t$-measurable. We also have that X_t^* is $(H-\varepsilon)$-Hölder continuous. Indeed, for any $t, s \in [0,T]$, there exist θ_1 between s and t and θ_2 between Y_s^* and Y_t^*, and some positive constant $C > 0$, so that

$$
\begin{aligned}
|X_t^* - X_s^*| &\le |G(t,Y_t^*) - G(s,Y_t^*)| + |G(s,Y_t^*) - G(s,Y_s^*)| \\
&= |G_1'(\theta_1, Y_t^*)(t-s)| + |G_2'(s,\theta_2)(Y_t^* - Y_s^*)| \\
&\le C(|t-s| + |Y_t^* - Y_s^*|).
\end{aligned}
$$

So the $(H-\varepsilon)$-Hölderianity of the paths of $(X_t^*)_{t\in[0,T]}$ follows by the $(H-\varepsilon)$-Hölderianity of the paths of $(Y_t^*)_{t\in[0,T]}$ and by treating the term $|t-s|$ exactly as in (5.10)-(5.11). It remains to check that $(X_t^*)_{t\in[0,T]}$ satisfies (5.2) pathwise. To this aim we are going to apply formula (31) in [15]. By the mean value theorem we have

$$
|G_2'(t,Y_t^*) - G_2'(s,Y_s^*)| = |\sigma(t,Y_t^*) - \sigma(s,Y_s^*)| \le C(|t-s| + |Y_t^* - Y_s^*|),
$$

where C is some positive constant. Therefore, the paths $t \mapsto G_2'(t, Y_t^*)$ are Hölder continuous of order $H-\varepsilon$. Since $2H - \bar{\varepsilon} > 1$ for some $\bar{\varepsilon} > 0$, by formula (31) in [15] and the fact that $(Y_t^*)_{t\in[0,T]}$ is solution of (5.6), we deduce

$$
\begin{aligned}
G(t,Y_t^*) - G(0,Y_0^*) &= \int_0^t G_1'(s,Y_s^*)\,\mathrm{d}s + \int_0^t G_2'(s,Y_s^*)\,\mathrm{d}Y_s^* \\
&= \int_0^t G_1'(s,Y_s^*)\,\mathrm{d}s + \int_0^t G_2'(s,Y_s^*)(A(s,Y_s)\,\mathrm{d}s + \mathrm{d}B_t^H) \\
&= \int_0^t G_1'(s,Y_s^*)\,\mathrm{d}s \\
&\quad + \int_0^t G_2'(s,Y_s^*)\left(F_1'(s,G(s,Y_s^*))\,\mathrm{d}s + \frac{b(s,G(s,Y_s^*))}{\sigma(s,G(s,Y_s^*))}\,\mathrm{d}s + \mathrm{d}B_t^H\right) && (5.12)\\
&= -\int_0^t F_1'(s,G(s,Y_s^*))\sigma(s,G(s,Y_s^*))\,\mathrm{d}s \\
&\quad + \int_0^t \sigma(s,G(s,Y_s^*))\left(F_1'(s,G(s,Y_s^*))\,\mathrm{d}s + \frac{b(s,G(s,Y_s^*))}{\sigma(s,G(s,Y_s^*))}\,\mathrm{d}s + \mathrm{d}B_t^H\right) && (5.13)\\
&= \int_0^t b(s,G(s,Y_s^*))\,\mathrm{d}s + \int_0^t \sigma(s,G(s,Y_s^*))\mathrm{d}B_t^H,
\end{aligned}
$$

where in (5.12) we used (5.7) and in (5.13) we used (5.4) and (5.5). The claim follows noticing that by the definition of G and y_0 we have $G(0, y_0) = G(0, F(0, x_0)) = x_0$.

Uniqueness Let $(X_t^{**})_{t\in[0,T]}$ be another solution of (5.2). Since X_t^{**} is $(H-\varepsilon)$-Hölder continuous, we have that $\sigma_2'(t, X_t^{**})$ is $(H-\varepsilon)$-Hölder continuous and so $F_2'(t, X_t^{**})$ is Hölder continuous of order $H-\varepsilon$. Applying again formula (31) in [15], we have

$$
\begin{aligned}
F(t, X_t^{**}) &= y_0 + \int_0^t F_1'(s, X_s^{**})\,\mathrm{d}s + \int_0^t F_2'(s, X_s^{**})\,\mathrm{d}X_s^{**} \\
&= y_0 + \int_0^t F_1'(s, X_s^{**})\,\mathrm{d}s + \int_0^t F_2'(s, X_s^{**})(b(s, X_s^{**})\,\mathrm{d}s + \sigma(s, X_s^{**})\,\mathrm{d}B_s^H) \\
&= y_0 + \int_0^t \left(F_1'(s, X_s^{**}) + \frac{b(s, X_s^{**})}{\sigma(s, X_s^{**})} \right) \mathrm{d}s + B_t^H.
\end{aligned}
$$

By applying Gronwall's Lemma one may easily check that the solution to (5.6) is indeed unique. Therefore, for any $t \in [0,T]$, $F(t, X^*) = F(t, X_t^{**})$ a.s., i.e. for any $t \in [0,T]$, $X^* = X_t^{**}$ a.s.. □

Proof of Lemma 5.3. Throughout this proof we use the notation introduced in the proof of the previous lemma. We first show $(Y_t)_{t\in[0,T]} \in \mathbb{D}^{1,2}$ and

$$
D_s Y_t = \mathbb{1}_{[0,t]}(s) \left(\int_s^t A_2'(u, Y_u) D_s Y_u \,\mathrm{d}u + K_H(t,s) \right),
$$

where $(Y_t)_{t\in[0,T]}$ is the unique strong solution to (5.6). Consider the sequence of stochastic processes $((Y_t^{(n)})_{t\in[0,T]})_{n\geq 0}$ defined recursively in the proof of Lemma 5.1. We start showing by induction on $n \geq 0$ that $((Y_t^{(n)})_{t\in[0,T]})_{n\geq 0} \in \mathbb{D}^{1,2}$ and

$$
D_s Y_t^{(n)} = \mathbb{1}_{[0,t]}(s) \left(\int_s^t A_2'(u, Y_u^{(n-1)}) D_s Y_u^{(n-1)} \,\mathrm{d}u + K_H(t,s) \right), \quad n \geq 0 \tag{5.14}
$$

where $D_s Y_t^{(-1)} = 0$. On the space of step functions on $[0,T]$ define the linear operator

$$
K_H^* \mathbb{1}_{[0,t]}(s) = K_H(t,s) \mathbb{1}_{[0,t]}(s).
$$

Letting D^{B_H} denote the Malliavin derivative operator with respect to the fBm, we have

$$
D_s^{B_H} Y_t^{(0)} = \mathbb{1}_{[0,t]}(s)
$$

and so by Proposition 5.2.1 in [9] it follows $(Y_t^{(0)})_{t\in[0,T]} \in \mathbb{D}^{1,2}$ and

$$
D_s Y_t^{(0)} = K_H^* \mathbb{1}_{[0,t]}(s) = K_H(t,s) \mathbb{1}_{[0,t]}(s).
$$

This proves the basis of the induction. Now assume the claim for $n-1$, $n \geq 2$. Since $(Y_t^{(n-1)})_{t\in[0,T]} \in \mathbb{D}^{1,2}$, there exists a sequence of smooth random variables $(Y_{k,t}^{(n-1)})_{k\geq 1}$ such that $\|Y_{k,t}^{(n-1)} - Y_t^{(n-1)}\|_{1,2} \to 0$, as $k\to\infty$. Consider the sequences

$$\begin{cases} Y_{k,t}^{(n)} = y_0 + \int_0^t A(u, Y_{k,u}^{(n-1)})\,\mathrm{d}u + B_t^H, \\ \\ D_s Y_{k,t}^{(n)} = \mathbb{1}_{[0,t]}(s)\left(\int_s^t A_2'(u, Y_{k,u}^{(n-1)}) D_s Y_{k,u}^{(n-1)}\,\mathrm{d}u + K_H(t,s)\right) \end{cases}$$

(where the term $D_s Y_{k,t}^{(n)}$ is computed by using Proposition 1.2.3 in [9]). Since $A(u,\cdot) \in \mathcal{C}_b^1(\mathbb{R})$, we have that $(Y_{k,t}^{(n)})_{k\geq 1} \in \mathbb{D}^{1,2}$. Using the boundedness of $A_2'(u,\cdot)$ and that $\|Y_{k,t}^{(n-1)} - Y_t^{(n-1)}\|_{1,2} \to 0$, as $k\to\infty$, one may easily check that $\|Y_{k,t}^{(n)} - Y_t^{(n)}\|_{1,2} \to 0$, as $k\to\infty$, where

$$\begin{cases} Y_t^{(n)} = y_0 + \displaystyle\int_0^t A(u, Y_u^{(n-1)})\,\mathrm{d}u + B_t^H, \\ \\ D_s Y_t^{(n)} = \mathbb{1}_{[0,t]}(s)\left(\displaystyle\int_s^t A_2'(u, Y_u^{(n-1)}) D_s Y_u^{(n-1)}\,\mathrm{d}u + K_H(t,s)\right). \end{cases}$$

Therefore, $((Y_t^{(n)})_{t\in[0,T]})_{n\geq 0} \in \mathbb{D}^{1,2}$ for all $n \geq 0$. We already showed (see the proof of the previous lemma) that Y_t is the L^2-limit of $Y_t^{(n)}$. Moreover, since A_2' is bounded, for some constant $C > 0$,

$$\begin{aligned} \mathrm{E}\left[\int_0^t |D_s Y_t^{(n+1)}|^2\,\mathrm{d}s\right] &\leq 2\mathrm{E}\left[\int_0^t \left(\left(\int_s^t A_2'(u, Y_u^{(n)}) D_s Y_u^{(n)}\,\mathrm{d}u\right)^2 + K_H^2(t,s)\right)\mathrm{d}s\right] \\ &\leq 2C\int_0^t\int_s^t \mathrm{E}[|D_s Y_u^{(n)}|^2]\,\mathrm{d}u\mathrm{d}s + 2t^{2H} \\ &\leq 2C\int_0^t\int_0^u \mathrm{E}[|D_s Y_u^{(n)}|^2]\,\mathrm{d}s\mathrm{d}u + 2t^{2H}. \end{aligned}$$

Setting $\psi_n(u) = \int_0^u \mathrm{E}[|D_s Y_u^{(n)}|^2]\,\mathrm{d}s$, we have

$$\psi_{n+1}(t) \leq 2C\int_0^t \psi_n(u)\,\mathrm{d}u + 2T^{2H}. \tag{5.15}$$

We note that $\psi_0(u) = \int_0^u \mathrm{E}[|D_s Y_u^{(0)}|^2]\,\mathrm{d}s = u^{2H} \leq 2T^{2H}$. Then, we have

$$\psi_1(t) \leq 4CT^{2H}t + 2T^{2H},\ \psi_2(t) \leq 2T^{2H}\frac{(2Ct)^2}{2!} + 2T^{2H}\frac{2Ct}{1!} + 2T^{2H},$$

and

$$\psi_n(t) \leq 2T^{2H}\sum_{k=0}^n \frac{(2Ct)^k}{k!}.$$

Consequently,

$$\sup_n \mathrm{E}\left[\int_0^t |D_s Y_t^{(n)}|^2 \,\mathrm{d}s\right] = \sup_n \psi_n(t) \le 2T^{2H} e^{2CT} < \infty.$$

By Lemma 1.2.3 in [9], we deduce that $Y_t \in \mathbb{D}^{1,2}$ (and $DY_t^{(n)} \to DY_t$ in the weak topology of $L^2(\Omega, L^2[0,T])$). Applying the operator D to the equation (5.6) we deduce

$$D_s Y_t = \mathbb{1}_{[0,t]}(s) \left(\int_s^t A_2'(u, Y_u) D_s Y_u \,\mathrm{d}u + K_H(t,s) \right). \tag{5.16}$$

Solving the equation (5.16) with the initial condition $D_s Y_s = K(s,s) = 0$, we have

$$D_s Y_t = \int_s^t (K_H)_1'(v,s) \exp\left(\int_v^t A_2'(u, Y_u) \,\mathrm{d}u \right) \mathrm{d}v, \quad s \le t.$$

On the other hand, by simple computations we deduce

$$A_2'(u, Y_u) = \left(b_2' - \frac{\sigma_2' b}{\sigma} - \frac{\sigma_1'}{\sigma} \right)(u, G(u, Y_u)) = \left(b_2' - \frac{\sigma_2' b}{\sigma} - \frac{\sigma_1'}{\sigma} \right)(u, X_u),$$

hence

$$D_s Y_t = \int_s^t (K_H)_1'(v,s) \exp\left(\int_v^t \left(b_2' - \frac{b\sigma_2'}{\sigma} - \frac{\sigma_1'}{\sigma} \right)(u, X_u) \,\mathrm{d}u \right) \mathrm{d}v, \quad 0 \le s \le t.$$

The claim follows noticing that from the relation $X_t = G(t, Y_t)$ and Proposition 1.2.3 in [9] we have $(X_t)_{t\in[0,T]} \in \mathbb{D}^{1,2}$ and

$$D_s X_t = G_2'(t, Y_t) D_s Y_t = \sigma(t, X_t) D_s Y_t.$$

□

For the sake of completeness we recall the related bounds proved in [2]. They considered a stochastic differential equation of the form

$$X_t = x_0 + \int_0^t V_0(X_s) \,\mathrm{d}s + \int_0^t V_1(X_s) \,\mathrm{d}B_s^H, \quad t \in [0,1], \tag{5.17}$$

and proved the following two Gaussian estimates for the density of X_t: (i) If $V_1 \equiv \sigma > 0$ is a positive constant, V_0 is such that $M := \sup_{x\in\mathbb{R}} |V_0'(x)| \in (0,\infty)$ and $H \in (0,1)$ then, for any $t \in (0,1]$, the density p_{X_t} of X_t satisfies

$$\frac{\mathrm{E}[|X_t - \mathrm{E}[X_t]|]}{c_1 \sigma^2 t^{2H}} \exp\left(-\frac{(x - \mathrm{E}[X_t])^2}{c_2 \sigma^2 t^{2H}} \right) \le p_{X_t}(x) \le \frac{\mathrm{E}[|X_t - \mathrm{E}[X_t]|]}{c_2 \sigma^2 t^{2H}} \exp\left(-\frac{(x - \mathrm{E}[X_t])^2}{c_1 \sigma^2 t^{2H}} \right), \tag{5.18}$$

for any $x \in \mathbb{R}$ and some constants $c_1, c_2 > 0$ depending only on M and H, see Theorem 3.2 in [2]; (ii) If $H \in (1/2, 1)$, $V_0, V_1 \in \mathcal{C}_b^1(\mathbb{R})$ and $c \leq \inf_{x\in\mathbb{R}} |V_1(x)| \leq \sup_{x\in\mathbb{R}} |V_1(x)| \leq C$, for some constants $c, C > 0$, then, for any $t \in [0,1]$, the density p_{X_t} of X_t satisfies

$$\frac{1}{C_1\sqrt{2\pi t^{2H}}} \exp\left(-C_1 \frac{(x-x_0)^2}{2t^{2H}}\right) \leq p_{X_t}(x) \leq \frac{1}{C_2\sqrt{2\pi t^{2H}}} \exp\left(-C_2 \frac{(x-x_0)^2}{2t^{2H}}\right), \qquad (5.19)$$

for any $x \in \mathbb{R}$ and some constants $C_1, C_2 > 0$, see Theorem 4.2 in [2].

Note that the stochastic differential equation (5.2) that we considered is more general than (5.17) since we allow the drift and diffusion coefficients to depend on the time. In this generality we are able to prove existence and uniqueness of a strong solution to (5.2) under Assumption 5.1 which in turn requires $H \in (1/2, 1)$. For this reason our Gaussian estimates hold only if the Hurst parameter lies in $(1/2, 1)$. Note that the estimates (5.3) are similar to the estimates (5.18), indeed the derivation of both these inequalities is based on Theorem 3.1 in [8] (see Proposition 2.1). In particular, note that if $T = 1$, $b(t,x) = b(x)$, $M := \sup_{x\in\mathbb{R}} |b'(x)|$ and $\sigma(t,x) \equiv \sigma$ then the estimates (5.3) coincide with the estimates (5.18) with $c_1 := 2\mathrm{e}^{-2M}$ and $c_2 := 2\mathrm{e}^{2M}$. Finally, we note that the estimates (5.3) are different from the estimates (5.19), which are indeed obtained by using Malliavin type techniques which do not rely on Theorem 3.1 in [8].

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