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Citation	Sun, C., Hu, G., & Xie, L. (2014). Fast finite-time consensus tracking for first-order multi-agent systems with unmodelled dynamics and disturbances. Proceedings of 2014 11th IEEE International Conference on Control and Automation (ICCA), 249-254.
Date	2014
URL	<a href="http://hdl.handle.net/10220/25743">http://hdl.handle.net/10220/25743</a>
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# Fast Finite-Time Consensus Tracking for First-Order Multi-agent Systems with Unmodelled Dynamics and Disturbances

Chao Sun, Guoqiang Hu, Lihua Xie

**Abstract**—In this paper, we study fast finite-time consensus tracking problem for a class of first-order multi-agent systems with unmodelled dynamics and unknown disturbances. A continuous control law is proposed to guarantee fast finite-time consensus tracking of a desired signal despite of uncertainties and disturbances in the system dynamics. Sufficient conditions for finite-time consensus tracking are established by using Lyapunov analysis methods. A numerical example is provided to show the effectiveness of the proposed algorithm.

## I. INTRODUCTION

Consensus problem of multi-agent systems has attracted considerable attention in recent years (see [1]–[9]). This is partly due to its wide range of applications in many fields such as robotics, sensor networks, formation control and unmanned air vehicles (UAVs). Consensus problem has been studied from different perspectives. For example, [10] discussed the distributed tracking control problem for second-order multi-agent systems. [11] and [12] proposed continuous distributed consensus protocols to achieve asymptotic consensus tracking of first-order and second-order multi-agent systems with disturbances and unmodelled agent dynamics. [13] developed an adaptive control algorithm to enable a class of second-order nonlinear multi-agent systems to synchronize with a virtual leader. [14] studied the distributed containment control problem for a group of autonomous vehicles modeled by double-integrator dynamics with multiple dynamic leaders using only position information.

In the above literature, most of the algorithms are available for asymptotic consensus. However, this doesn't disclose how long it will take to achieve consensus. In many real applications, one prefers to make sure that consensus is reached in finite-time. In addition, when considering the control accuracy and the robustness to external disturbances, finite-time consensus algorithms usually have better performance [15]. There are many results about finite-time stability and some attempts on finite-time consensus for multi-agent systems. [16] presented nonsmooth tools to analyze finite-time stability properties of continuous time systems where the differential equations have discontinuous right-hand side, and then extended the results to networked finite-time consensus.

[17] proposed a distributed finite-time consensus algorithm for second-order nonlinear multi-agent systems based on the principle of terminal sliding mode control (TSM). [18] discussed the finite-time consensus problem for first-order multi-agent systems under unidirectional and bidirectional intersection topologies. [19] addressed the finite-time consensus problem for leaderless and leader-follower multi-agent systems with external disturbances. [20] studied finite-time consensus for multi-agent systems with Lipschitz-like nonlinearity. In [21], the authors considered the finite-time consensus control problem for multiple manipulators with unmodelled dynamics based on homogeneous function theory. Although finite-time consensus and robust consensus of multi-agent systems with unknown dynamics and disturbances have been studied respectively, few work have been reported that consider these two factors simultaneously. A closely related work can be found in [22], the proposed non-smooth control law can drive the high-order uncertain nonlinear systems to reach fast finite-time consensus. However, it is well-known that discontinuous control may generate some unexpected characteristics, such as chattering, which may excite unmodelled dynamics with high frequency and then lead to degraded performance of the control strategy.

The objective of this paper is to explore continuous control techniques for faster finite-time consensus tracking of multi-agent systems with unknown disturbances and unmodelled dynamics. The differences of this paper from previous results are summarized as below. Compared with [20] and [21], we consider non-Lipschitz uncertainties in the agent dynamics, which exist in many practical scenarios (e.g., in systems with disturbances generated by sensor fault). Different from the approaches in [17] and [22], the proposed control law in this paper is continuous.

The rest of this paper is organized as follows. In Section II, background of graph theory and fast finite-time convergence are presented. In Section III, we formulate our problem and a continuous distributed consensus tracking algorithm is presented. Lyapunov stability methods are used to prove the closed-loop finite-time stability. In Section IV, simulation results show the effectiveness of the proposed algorithm and finally Section V concludes the paper.

## II. NOTATION AND PRELIMINARIES

### A. Graph Theory

Graph theory is widely used for investigating multi-agent systems. Let  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$  represent a directed graph, and  $\mathcal{V} = \{1, \dots, N\}$  denote the set of vertices. Every agent is represented by a vertex. The set of edges is denoted as  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ . An edge is an ordered pair  $(i, j) \in \mathcal{E}$  if agent  $j$  can be directly supplied with information from agent  $i$ . In this paper, we assume that there is no self loop in the graph, that is,  $(i, i) \notin \mathcal{E}$ .  $\mathcal{N}_i = \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$  denotes the neighborhood set of vertex  $i$ . Graph  $\mathcal{G}$  is said to be undirected if for any edge  $(i, j) \in \mathcal{E}$ , edge  $(j, i) \in \mathcal{E}$ . Hence, an undirected graph is a special case of a directed graph. A path is referred by the sequence of its vertices. Path  $\mathcal{P}$  between two vertices  $v_0$  and  $v_k$  is the sequence  $\{v_0, \dots, v_k\}$  where  $(v_{i-1}, v_i) \in \mathcal{E}$  for  $i = 1, \dots, k$  and the vertices are distinct. The number  $k$  is defined as the length of path  $\mathcal{P}$ . Graph  $\mathcal{G}$  is strongly connected if any two vertices are linked with a path in  $\mathcal{G}$ . Graph  $\mathcal{G}$  contains a directed spanning tree if there is a vertex which can reach all the other vertices through a directed path.  $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$  denotes the adjacency matrix of  $\mathcal{G}$ , where  $a_{ij} > 0$  if and only if  $(j, i) \in \mathcal{E}$  else  $a_{ij} = 0$ .  $L = D - \mathcal{A}$  is called Laplacian matrix of  $\mathcal{G}$ , where  $D = [d_{ii}] \in \mathbb{R}^{N \times N}$  is a diagonal matrix with  $d_{ii} = \sum_{j=1}^N a_{ij}$ .

Next, some properties of the Laplacian matrix are presented as follows.

**Lemma 1:** Zero is an eigenvalue of  $L$  for both directed and undirected graphs. Zero is a simple eigenvalue of  $L$  and the associated eigenvector is  $\mathbf{1}$  where  $\mathbf{1} \in \mathbb{R}^N$  is a unitary column vector, if and only if the undirected graph is connected or if the directed graph has a directed spanning tree. All of the nonzero eigenvalues of  $L$  are positive for an undirected graph or have positive real parts for a directed graph.

### B. Fast Finite-time stability

The following lemma provides a sufficient condition for fast finite-time stability.

**Lemma 2:** [22] For a nonlinear system  $\dot{x} = f(x)$ , with  $f(0) = 0$ , suppose there exists a Lyapunov function  $V$  and real numbers  $\alpha > 0$ ,  $\beta > 0$ , and  $0 < \gamma < 1$ , such that (1)  $V(x)$  is positive for any nonzero  $x$ , (2)  $\dot{V}(x) + \alpha V(x) + \beta V^\gamma(x) \leq 0$ . Then the origin of this system is globally fast finite-time stable, and the setting time  $T$  satisfies

$$T \leq \frac{1}{\alpha(1-\gamma)} \ln\left(\frac{\alpha V^{1-\gamma}(x(0)) + \beta}{\beta}\right).$$

**Remark 1:** It is well known that the Lyapunov function satisfying  $\dot{V}(x) + \beta V^\gamma(x) \leq 0$  can guarantee global finite-time stability of the closed-loop system. However, as mentioned in [22], it slows down the convergence rate when the state  $x(t)$  is far away from the origin compared with the Lyapunov function  $V(x)$  satisfying  $\dot{V}(x) + \beta V(x) \leq 0$  which can ensure global asymptotic stability of the closed

system. This is because  $V^\gamma(x) \leq V(x)$  for  $0 < \gamma < 1$ , and  $V(x) \geq 1$ . The modified Lyapunov function which satisfies  $\dot{V}(x) + \alpha V(x) + \beta V^\gamma(x) \leq 0$  combines the properties of both and can guarantee finite-time convergence and a faster convergence rate.

## III. PROTOCOL DESIGN AND STABILITY ANALYSIS

### A. Problem Formulation

Consider a multi-agent system with  $N$  agents. The dynamics of the  $i$ -th agent is described by

$$\dot{x}_i = u_i + f_i(x_i) + d_i(t), \quad (1)$$

where  $x_i(t) \in \mathbb{R}$  is the state of the  $i$ -th agent,  $u_i \in \mathbb{R}$  represents its control input,  $f_i(x_i) \in \mathbb{R}$  represents the unmodelled dynamics, and  $d_i(t) \in \mathbb{R}$  denotes the disturbance.

The control objective is to ensure all the states of this multi-agent system reach consensus and track a desired signal in finite time. Suppose the desired tracking signal is governed by the following dynamics

$$\dot{x}_d = f_d(t, x_d), \quad (2)$$

where  $x_d(t) \in \mathbb{R}$  represents the state variable of this signal, and  $f_d(t, x_d) \in \mathbb{R}$  is some uncertain term.

To facilitate the subsequent stability analysis, we make the following assumptions.

**Assumption 1:** The disturbance term  $d_i(t)$  and its first-order time derivative are bounded (i.e.,  $d_i(t), \dot{d}_i(t) \in \mathcal{L}_\infty, i \in \{1, \dots, N\}$ ).

**Assumption 2:** If  $x_i(t)$  is bounded (i.e.,  $x_i(t) \in \mathcal{L}_\infty$ ), then the unmodelled dynamics  $f_i(x_i)$  and its first-order derivatives with respect to  $x_i$  is bounded (i.e.,  $f_i(x_i), \frac{\partial f_i(x_i)}{\partial x_i} \in \mathcal{L}_\infty, i \in \{1, \dots, N\}$ ).

**Assumption 3:** The state variable  $x_d$  and its first-order and second-order derivative are bounded (i.e.,  $x_d(t), \dot{x}_d(t), \ddot{x}_d(t) \in \mathcal{L}_\infty, i \in \{1, \dots, N\}$ ).

### B. Fast Finite-time Consensus Protocol Design

Since not all the agents have access to the desired trajectory information, we define a state tracking error  $e_i$  as

$$e_i = \sum_{j=1}^N a_{ij}(x_i - x_j) + b_i(x_i - x_d), \quad (3)$$

where  $b_i, i \in \{1, \dots, N\}$  represents the access of agent  $i$  to the desired trajectory signal. If  $b_i$  is equal to 1, then the  $i$ -th agent has access to the desired trajectory; otherwise, if  $b_i = 0$ , then the  $i$ -th agent doesn't have access to the desired trajectory. It can be noted that  $e_i$  only uses local information exchanged from neighboring agents.

To facilitate the subsequent development and stability analysis, we define the concatenated vectors  $E(t), U(t)$ ,

$F(t), X(t), B, D(t)$  as

$$\begin{aligned} E &= [e_1, e_2, \dots, e_N]^T, \\ U &= [u_1, u_2, \dots, u_N]^T, \\ F &= [f_1, f_2, \dots, f_N]^T, \\ X &= [x_1, x_2, \dots, x_N]^T, \\ B &= \text{diag}\{b_1, b_2, \dots, b_N\}, \\ D &= [d_1, d_2, \dots, d_N]^T. \end{aligned}$$

Based on (1) and (3), we have

$$\dot{X} = U + F + D, \quad (4)$$

$$E = HX - Bx_d \mathbf{1}, \quad (5)$$

where  $H$  is defined as  $H = L + B = D - A + B$  and  $D$  and  $A$  are defined in Section II.

Based on (4) and (5), the derivative of  $E$  satisfies

$$\begin{aligned} \dot{E} &= H\dot{X} - B\dot{x}_d \mathbf{1} \\ &= H(U + F + D) - B\dot{x}_d \mathbf{1} \\ &= HU + \Psi, \end{aligned}$$

where  $\Psi = [\varphi_1, \varphi_2, \dots, \varphi_N]^T$  is defined as

$$\Psi = HF + HD - B\dot{x}_d \mathbf{1}.$$

Motivated by [23], the following distributed control law is proposed

$$u_i = \frac{1}{\kappa_i} [-k_1 \text{sig}(e_i) - k_2 e_i + \sum_{j=1, j \neq i}^N a_{ij} u_j + r_i] \quad (6)$$

$$\dot{r}_i = -k_3 \text{sgn}(e_i) - k_4 e_i, \quad (7)$$

where

$$\begin{aligned} \kappa_i &= \sum_{j=1, j \neq i}^N a_{ij} + b_i, \\ \text{sig}(e_i) &= \text{sgn}(e_i) |e_i|^{\frac{1}{2}}, \end{aligned}$$

and  $k_1, k_2, k_3, k_4$  are control gains that will be determined later.

**Remark 2:** The control input of agent  $i$  in (6) utilizes the control information  $u_j$  exchanged from neighboring agents. Meanwhile, the information of  $u_i$  is required when calculating  $u_j$ . This brings up an implementation loop issue. In case there exists mutual information exchange between agents  $i$  and  $j$ , we use  $u_j$  obtained during the previous sampling period to avoid this implementation issue. Here, we assume that the sample frequency is large enough and the communication delay is sufficiently small, so that this implementation strategy doesn't affect the stability of the closed-loop system. In the future work, we will consider a more realistic sampled-data system with communication delay and study how the sample frequency and time delay affect the closed-loop stability. For special information-exchange topologies where there is no loop in the directed graph, the

controller can be implemented without worrying about the loop issue.

From (6) and (7), we have  $(D + B)U = -k_1 \text{sig}(E) - k_2 E + AU + R$  and  $\dot{R} = -k_3 \text{sgn}(E) - k_4 E$ , where  $\text{sig}(E) = [\text{sig}(e_1), \dots, \text{sig}(e_N)]^T$  and  $\text{sgn}(E) = [\text{sgn}(e_1), \dots, \text{sgn}(e_N)]^T$ . Noticing that  $D + B - A = L + B = H$ , we have

$$U = H^{-1}[-k_1 \text{sig}(E) - k_2 E + R], \quad (8)$$

$$\dot{R} = -k_3 \text{sgn}(E) - k_4 E. \quad (9)$$

Then, the closed-loop system can be rewritten as

$$\dot{E} = -k_1 \text{sig}(E) - k_2 E + Y$$

$$\dot{Y} = -k_3 \text{sgn}(E) - k_4 E + \Phi, \quad (10)$$

where  $Y = [y_1, y_2, \dots, y_N]^T$  is an unmeasurable variable, and  $\Phi = \dot{\Psi} = [\psi_1, \psi_2, \dots, \psi_N]^T$ .

Based on (10), for agent  $i$ , we have

$$\dot{e}_i = -k_1 \text{sig}(e_i) - k_2 e_i + y_i$$

$$\dot{y}_i = -k_3 \text{sgn}(e_i) - k_4 e_i + \psi_i. \quad (11)$$

With Assumptions 1, 2 and 3, we suppose that

$$|\psi_i| \leq \sigma. \quad (12)$$

**Remark 3:** Since (10) and (11) have discontinuous right-hand sides, the solutions of above differential inclusion are in the sense of Filippov.

### C. Stability Analysis

The major result of this paper is given as follows.

**Theorem 1:** Under Assumptions 1, 2 and 3, the proposed robust tracking protocols given in (6) and (7) guarantee fast finite-time consensus tracking of the desired trajectory, provided that the control gains  $k_1, k_2, k_3, k_4$  are selected according to the following sufficient conditions:

$$\begin{aligned} k_4 &> 2k_2^2, \\ k_3 &> \frac{(k_1^2 + 2\sigma)^2(5k_2^2 + 2k_4)}{4k_1^2(k_4 - 2k_2^2)} + \sigma - \frac{1}{2}k_1^2, \end{aligned} \quad (13)$$

where  $\sigma$  is described in (12).

*Proof:* Let  $\text{sig}^T(E) = (\text{sig}(E))^T$  and define a vector  $Z(t) = [\text{sig}^T(E), E^T, Y^T]^T \in \mathbb{R}^{3N}$ . Choose a Lyapunov function candidate  $V(t, Z) \in \mathbb{R}$  as

$$V(t, Z) = \frac{1}{2} Z^T(t) (P \otimes I_N) Z(t), \quad (14)$$

where

$$P = \begin{bmatrix} (4k_3 + k_1^2) & k_1 k_2 & -k_1 \\ k_1 k_2 & 2k_4 + k_2^2 & -k_2 \\ -k_1 & -k_2 & 2 \end{bmatrix}.$$

It is easy to verify that  $P$  is positive definite and symmetric if we choose  $k_1, k_2, k_3$ , and  $k_4 > 0$ . Thus, the Rayleigh-Ritz theorem can be used to conclude that

$$\lambda_{\min}(P) \|Z\|^2 \leq V(t, Z) \leq \lambda_{\max}(P) \|Z\|^2. \quad (15)$$

Taking the derivative of  $V$  gives

$$\begin{aligned}\dot{V} &= Z^T(t)(P \otimes I_N)\dot{Z}(t) \\ &= [sig^T(E), E^T, Y^T](P \otimes I_N) \begin{bmatrix} \frac{1}{2}diag(|E|^{-\frac{1}{2}})\dot{E} \\ \dot{E} \\ \dot{Y} \end{bmatrix}.\end{aligned}$$

Based on (10), we have

$$\begin{aligned}\dot{V} &= -[sig^T(E), E^T, Y^T](\Pi_2 \otimes I_N) \begin{bmatrix} sig(E) \\ E \\ Y \end{bmatrix} \\ &\quad -[sig^T(E), E^T, Y^T](\Pi_1 \otimes diag(|E|^{-\frac{1}{2}})) \\ &\quad \begin{bmatrix} sig(E) \\ E \\ Y \end{bmatrix} + [sig^T(E), E^T, Y^T] \begin{bmatrix} -k_1\Phi \\ -k_2\Phi \\ 2\Phi \end{bmatrix} \\ &= -\sum_{i=1}^N \frac{1}{|e_i|^{\frac{1}{2}}} [sig(e_i), e_i, y_i] \Pi_1 \begin{bmatrix} sig(e_i) \\ e_i \\ y_i \end{bmatrix} \\ &\quad - \sum_{i=1}^N [sig(e_i), e_i, y_i] \Pi_2 \begin{bmatrix} sig(e_i) \\ e_i \\ y_i \end{bmatrix} \\ &\quad + \sum_{i=1}^N [sig(e_i), e_i, y_i] \begin{bmatrix} -k_1\psi_i \\ -k_2\psi_i \\ 2\psi_i \end{bmatrix},\end{aligned}$$

where  $diag(|E|^{-\frac{1}{2}}) = diag(|e_1|^{-\frac{1}{2}}, \dots, |e_N|^{-\frac{1}{2}})$ , and

$$\begin{aligned}\Pi_1 &= \begin{bmatrix} k_1k_3 + \frac{1}{2}k_1^3 & 0 & -\frac{1}{2}k_1^2 \\ 0 & \frac{5}{2}k_1k_2^2 + k_1k_4 & -\frac{3}{2}k_1k_2 \\ -\frac{1}{2}k_1^2 & -\frac{3}{2}k_1k_2 & \frac{1}{2}k_1 \end{bmatrix}, \\ \Pi_2 &= \begin{bmatrix} 2k_1^2k_2 + k_2k_3 & 0 & 0 \\ 0 & k_2^3 + k_2k_4 & -k_2^2 \\ 0 & -k_2^2 & k_2 \end{bmatrix}.\end{aligned}$$

Since

$$\begin{aligned}& [sig(e_i), e_i, y_i] \begin{bmatrix} -k_1\psi_i \\ -k_2\psi_i \\ 2\psi_i \end{bmatrix} \\ &= \frac{1}{|e_i|^{\frac{1}{2}}} [sig(e_i), e_i, y_i] \Delta_1 \begin{bmatrix} sig(e_i) \\ e_i \\ y_i \end{bmatrix} \\ &\quad + [sig(e_i), e_i, y_i] \Delta_2 \begin{bmatrix} sig(e_i) \\ e_i \\ y_i \end{bmatrix},\end{aligned}$$

where

$$\begin{aligned}\Delta_1 &= \begin{bmatrix} -k_1\psi_i sgn(e_i) & 0 & \psi_i sgn(e_i) \\ 0 & 0 & 0 \\ \psi_i sgn(e_i) & 0 & 0 \end{bmatrix}, \\ \Delta_2 &= \begin{bmatrix} -k_2\psi_i sgn(e_i) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\end{aligned}$$

we have

$$\begin{aligned}\dot{V} &= -\sum_{i=1}^N \frac{1}{|e_i|^{\frac{1}{2}}} [sig(e_i), e_i, y_i] \Theta \begin{bmatrix} sig(e_i) \\ e_i \\ y_i \end{bmatrix} \\ &\quad - \sum_{i=1}^N [sig(e_i), e_i, y_i] Q \begin{bmatrix} sig(e_i) \\ e_i \\ y_i \end{bmatrix},\end{aligned}\quad (16)$$

where

$$\begin{aligned}\Theta &= \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}, \\ \Theta_{11} &= k_1k_3 + \frac{1}{2}k_1^3 - k_1\psi_i sgn(e_i), \\ \Theta_{12} &= \Theta_{21}^T = \begin{bmatrix} 0 & -\frac{1}{2}k_1^2 + \psi_i sgn(e_i) \end{bmatrix}, \\ \Theta_{22} &= \begin{bmatrix} \frac{5}{2}k_1k_2^2 + k_1k_4 & -\frac{3}{2}k_1k_2 \\ -\frac{3}{2}k_1k_2 & \frac{1}{2}k_1 \end{bmatrix}, \\ Q_{11} &= 2k_1^2k_2 + k_2k_3 - k_2\psi_i sgn(e_i), \\ Q_{22} &= \begin{bmatrix} k_2^3 + k_2k_4 & -k_2^2 \\ -k_2^2 & k_2 \end{bmatrix}.\end{aligned}$$

It is easy to check that  $\Theta$  and  $Q$  are both positive definite if the gains  $k_1, k_2, k_3, k_4$  satisfy the conditions in (13). Noting that  $\Theta$  and  $Q$  are both time varying,  $\psi_i$  is bounded as in (12), and  $sgn(e_i)$  has only three possible values (i.e.,  $-1, 0, 1$ ), we define  $\lambda_{\min}(\Theta)$  and  $\lambda_{\min}(Q)$  as the minimum eigenvalue of  $\Theta$  and  $Q$ , respectively.<sup>1</sup>

Since

$$\begin{aligned}V &\leq \lambda_{\max}(P) \sum_{i=1}^N (|e_i| + |e_i|^2 + y_i^2) \\ &\leq \lambda_{\max}(P) \sum_{i=1}^N \frac{\max_i\{|e_i|^{\frac{1}{2}}\}}{|e_i|^{\frac{1}{2}}} (|e_i| + |e_i|^2 + y_i^2) \\ &\leq \lambda_{\max}(P) \max_i\{|e_i|^{\frac{1}{2}}\} \sum_{i=1}^N \frac{1}{|e_i|^{\frac{1}{2}}} (|e_i| + |e_i|^2 + y_i^2) \\ &\leq \lambda_{\max}(P) \frac{V^{\frac{1}{2}}}{\sqrt{\lambda_{\min}(P)}} \sum_{i=1}^N \frac{1}{|e_i|^{\frac{1}{2}}} (|e_i| + |e_i|^2 + y_i^2),\end{aligned}$$

we get

$$\sum_{i=1}^N \frac{1}{|e_i|^{\frac{1}{2}}} (|e_i| + |e_i|^2 + y_i^2) \geq \frac{\sqrt{\lambda_{\min}(P)} V^{\frac{1}{2}}}{\lambda_{\max}(P)}. \quad (17)$$

Based on (15), we have

$$V \leq \lambda_{\max}(P) \sum_{i=1}^N (|e_i| + |e_i|^2 + y_i^2). \quad (18)$$

<sup>1</sup>The matrices  $\Theta$  and  $Q$  can be approximated as time-invariant matrices by selecting  $k_1^2 \gg \sigma$ .

Applying the inequalities (17) and (18) into (16), we get

$$\begin{aligned} \dot{V} &\leq -\sum_{i=1}^N \frac{1}{|e_i|^{\frac{1}{2}}} \lambda_{\min}(\Theta) (|e_i| + |e_i|^2 + y_i^2) \\ &\quad - \sum_{i=1}^N \lambda_{\min}(Q) (|e_i| + |e_i|^2 + y_i^2) \\ &\leq -\gamma_1 V^{\frac{1}{2}} - \gamma_2 V, \end{aligned} \quad (19)$$

where

$$\begin{aligned} \gamma_1 &= \frac{\lambda_{\min}(\Theta) \sqrt{\lambda_{\min}(P)}}{\lambda_{\max}(P)}, \\ \gamma_2 &= \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}. \end{aligned}$$

According to Lemma 2,  $V(x)$  will approach 0 in finite time. This ensures that the error function  $E$  converges to zero in finite time, and equivalently,  $[x_1, \dots, x_N]^T \rightarrow x_d(t) \mathbf{1}$ . Hence, fast finite-time consensus is achieved. ■

**Remark 4:** As mentioned in Remark 3, the derivative of  $sig(e_i)$  can be calculated in the following way:

$$\begin{aligned} \frac{d[sig(e_i)]}{dt} &= \frac{d[sgn(e_i) |e_i|^{\frac{1}{2}}]}{dt} \\ &= \frac{1}{2} sgn(e_i) |e_i|^{-\frac{1}{2}} sgn(e_i) \dot{e}_i \\ &= \frac{1}{2} |e_i|^{-\frac{1}{2}} \dot{e}_i. \end{aligned}$$

**Remark 5:** The term  $|e_i|^{-\frac{1}{2}}$  exists in many papers dealing with the finite-time problem (e.g., [17]). It can be seen that it may cause a singularity to occur when  $e_i = 0$ . In the future, we will try to address the singularity problem existing in this method.

#### IV. SIMULATION

Consider a multi-agent system with 4 agents. The system dynamics of each agent is described by

$$\dot{x}_i = u_i + f_i(x_i) + d_i(t), \quad (20)$$

where

$$\begin{aligned} f_1 &= 0.1 \sin x_1, \quad f_2 = 0.2 \sin(2x_2), \\ f_3 &= 0.3 \sin(3x_3), \quad f_4 = 0.4 \sin(4x_4), \end{aligned}$$

and

$$\begin{aligned} d_1 &= 0.1 \sin(t), \quad d_2 = 0.2 \sin(2t), \\ d_3 &= 0.3 \sin(t), \quad d_4 = 0.4 \sin(2t), \end{aligned}$$

are the unmodelled dynamics and unknown disturbances, respectively.

The initial states are given by

$$x(0) = [2, 4, -6, 8]^T.$$

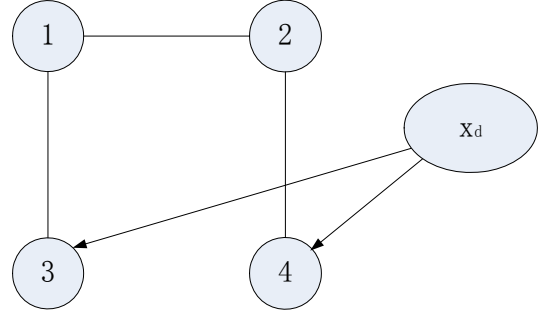


Fig. 1. Communication graph

The dynamics of the desired tracking signal is given by

$$\dot{x}_d = \sin(2t), \quad x_d(0) = 0. \quad (21)$$

and the desired signals  $x_d(t)$  are only provided to the agent 3 and 4.

The communication topology of the 4 agents are shown in Fig. 1. The adjacency matrix  $A$  associated with this group of agents is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The diagonal matrix which represents the access of the 4 agents to the desired trajectory (21) is given by  $B = \text{diag}\{0, 0, 1, 1\}$ .

Based on Theorem 1, the control gains are chosen as

$$\begin{aligned} k_1 &= 1, \quad k_2 = 1, \\ k_3 &= 40, \quad k_4 = 15. \end{aligned}$$

Fig. 2 and Fig. 3 show the performance of the proposed algorithm.

#### V. CONCLUSION

In this paper, we consider the fast finite-time consensus tracking problem for a class of first-order multi-agent systems with unmodelled dynamics and unknown disturbances. We propose a continuous and distributed control law to guarantee fast finite-time consensus tracking of a desired signal despite of uncertainties and disturbances in the system dynamics. By using Lyapunov analysis methods, sufficient conditions for finite-time consensus tracking are established. An example is provided to show the effectiveness of the developed algorithm.

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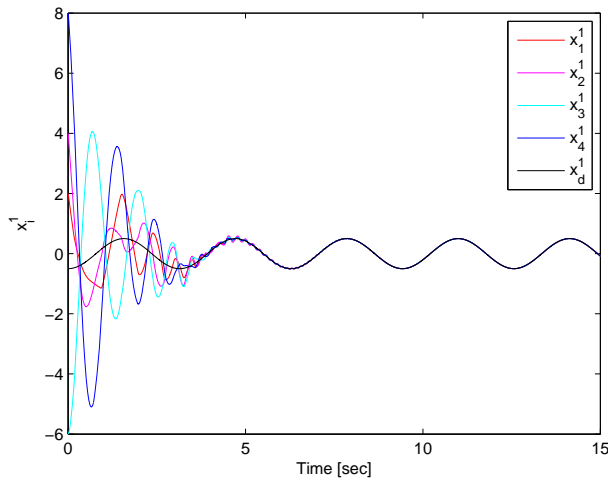


Fig. 2. Under the proposed control protocol given in (6) and (7), the position states  $x_i(t)$ ,  $i \in \{1, \dots, 4\}$  of the group of agents reach consensus in finite time and track the desired position information  $x_d(t)$ .

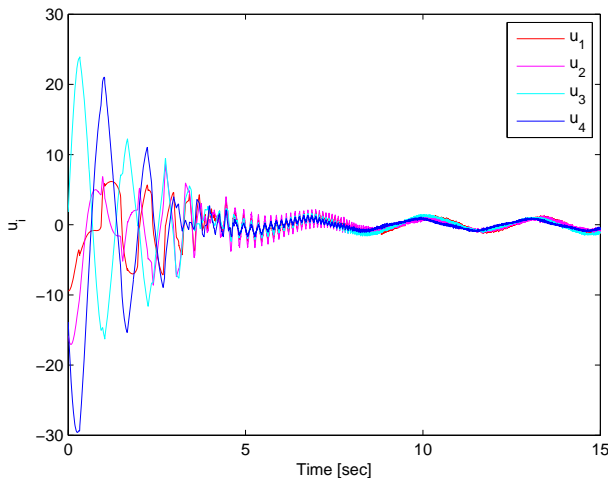


Fig. 3. Trajectories of the control inputs

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