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On Polynomial Pairs of Integers

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Abstract

The reversal of a positive integer A is the number obtained by reading A backwards in its decimal representation. A pair (A, B) of positive integers is said to be palindromic if the reversal of the product $A \times B$ is equal to the product of the reversals of A and B. A pair (A, B) of positive integers is said to be polynomial if the product $A \times B$ can be performed without carry.

In this paper, we use polynomial pairs in constructing and in studying the properties of palindromic pairs. It is shown that polynomial pairs are always palindromic. It is further conjectured that, provided that neither A nor B is itself a palindrome, all palindromic pairs are polynomial. A connection is made with classical topics in recreational mathematics such as reversal multiplication, palindromic squares, and repunits.

1 Introduction

On March 13, 2012 the following identity appeared on K. T. Arasu's Facebook posting[1]:

> Notice that
> $$25986 = 213 \times 122.$$
> Now, read the expression above in reverse order and observe that
> $$221 \times 312 = 68952.$$

The point here is, of course, that the second equality still holds in first order arithmetic!

For $A \in \mathbb{N}$ let the *reversal* of A, denoted by A^*, be the integer obtained by reading A backwards in base 10. We say that A is a *palindrome* if $A = A^*$. In this paper, we investigate how to determine which pairs (A, B) of positive integers satisfy the property

$$C = A \times B \text{ and } C^* = A^* \times B^*. \tag{1}$$

In words, the product of the reversals is the reversal of the product. We shall call such a pair a *palindromic pair*.

Note that there are integers C with more than one corresponding pair (A, B) satisfying Eq. (1). For example, we have

$$2448 = 12 \times 204 = 24 \times 102.$$

Upon reversal, one has

$$8442 = 21 \times 402 = 42 \times 201.$$

It is easy to see that palindromic pairs always occur in distinct pairs (A, B) and (A^*, B^*) unless both A and B are palindromes. The pair $(12, 13)$, for instance, comes with the pair $(21, 31)$ upon reversal.

The question of how to characterize palindromic pairs had appeared in Ball and Coxeter [1, p. 14] where the pair $(122, 213)$ was given, yet this matter has hardly been looked at more closely. In this short note we introduce the notion of *polynomial pairs* as a tool to study palindromic pairs. We show that all the examples of palindromic pairs presented below can be explained in terms of polynomial pairs. We conjecture, but cannot yet prove that, when neither A nor B is a palindrome, all palindromic pairs (A, B) are polynomial pairs.

The concept of polynomial pairs intersects many classical topics in recreational mathematics. An interesting topic concerns the *repunits* which are numbers all of whose digits are 1 [2, Ch. 11]. Another topic deals with a known technique to produce palindromes usually referred to as *reversal multiplication* [4]. The integers A with the property that $A \times A^*$ is a palindrome form sequence A062936 in the *Online Encyclopedia of Integer Sequences*

[1]His account has since been deactivated for personal reasons.

(henceforth, OEIS) [3]. Whenever (A, A^*) form a polynomial pair, we learn that reversal multiplication always produces a palindrome.

The material is arranged as follows. Section 2 introduces, and Section 3 develops, the concept of palindromic pairs. Section 4 explores the multiplicity of the representation of a repunit as a product of a palindromic pair of integers. Section 5 covers the special case where the two members of a palindromic pair are reversals of each other. Section 6 is dedicated to palindromes that are perfect squares. Section 7 replaces multiplication by addition in the definition of a palindromic pair. The paper ends with a summary.

2 Formalization

To formalize the problem mathematically, some notation is in order. Let

$$A := \sum_{i=0}^{a} a_i 10^i$$

be an integer expressed in base 10. Its reversal is

$$A^* := \sum_{i=0}^{a} a_i 10^{a-i}.$$

Define the polynomial $P(A, x) := \sum_{i=0}^{a} a_i x^i \in \mathbb{Q}[x]$ so that $P(A, 10) = A$. Then its *reciprocal*

$$P^*(A, x) := \sum_{i=0}^{a} a_i x^{a-i} = x^a P(A, 1/x)$$

satisfies $P^*(A, 10) = A^*$. A pair (A, B) of not necessarily distinct positive integers is said to be a *palindromic pair* if

$$P^*(A, 10) P^*(B, 10) = P^*(A \times B, 10).$$

We shall say that the pair (A, B) is *polynomial* if

$$P(A, x) P(B, x) = P(A \times B, x).$$

The pair $(12, 21)$, for instance, is a polynomial pair since

$$P(12 \times 21, x) = 2x^2 + 5x + 2 = (x + 2)(2x + 1),$$

but $(13, 15)$ is not a polynomial pair because

$$(x + 3)(x + 5) = x^2 + 8x + 15 \text{ while } P(13 \times 15, x) = x^2 + 9x + 5.$$

The following characterization of polynomial pairs will be used repeatedly.

Proposition 1. *The following assertions are equivalent.*

1) (A, B) *is a polynomial pair.*

2) The multiplication of A *by* B *can be performed without carry.*

3) The coefficients of the polynomial $P(A, x)P(B, x)$ *are bounded above by* 9*.*

Proof. Let j be the smallest integer such that $c_j := \sum_{j=i+k} a_i b_k > 9$. Then c_j is the coefficient of x^j in $P(A, x)P(B, x)$, while the coefficient of x^j in $P(A \times B, x)$ is $c_j \pmod{10} \neq c_j$. Thus, 1) implies 2) by contrapositive argument.

Now, assume that there is some j such that $c_j := \sum_{j=i+k} a_i b_k > 9$. Then, in the multiplication $A \times B$, the term $y := \left\lfloor \frac{c_j}{10} \right\rfloor$ is carried over to the coefficient of 10^{j+1}. This establishes that 2) implies 3).

Lastly, to show that 3) implies 1), we begin by substituting $x = 10$. Hence,

$$A \times B = P(A, 10)P(B, 10) = \sum_{k=0}^{a+b} \left(\sum_{i+j=k} a_i b_j \right) 10^k.$$

The coefficient of x^k in $P(A \times B, x)$ is $\sum_{k=i+j} a_i b_j$, which is assumed to be ≤ 9. This means that (A, B) is indeed a polynomial pair. The proof is therefore complete. □

Polynomial pairs are palindromic pairs as the next result shows.

Proposition 2. *A polynomial pair* (A, B) *is palindromic.*

Proof. If $P(A, x)P(B, x) = P(A \times B, x)$, then, by taking reciprocals, we get

$$P^*(A, x)P^*(B, x) = P^*(A \times B, x).$$

Using $x = 10$ completes the proof. □

This observation raises an initial question:

Problem 3. Are there palindromic pairs that are not polynomial?

Our investigation quickly reveals that the answer is yes. If we allow either A or B to be palindromes then there are palindromic pairs which are not polynomial pairs.

The test that a pair (A, B) is palindromic is done simply by checking if the definition is satisfied. We record A, B, and C whenever we have $A \times B = C$ and $A^* \times B^* = C^*$. We then perform a check if the multiplication of A by B can be performed without carry. The pair (A, B) that fails to pass this check is not a polynomial pair by Proposition 1. Table 1 provides the list of all such pairs (A, B) with $A \leq B$ and $A \times B = C \leq 10^7$ generated by exhaustive search. We require $A \leq B$ to avoid duplication of pairing.

Table 1: Palindromic but not polynomial pairs (A, B) with $A \leq B$ and $A \times B \leq 10^7$

(A, B)	$A \times B$	(A, B)	$A \times B$	(A, B)	$A \times B$	(A, B)	$A \times B$
$(7, 88)$	616	$(555, 979)$	543345	$(737, 888)$	654456	$(707, 8558)$	6050506
$(8, 77)$		$(55, 9999)$	549945	$(777, 858)$	666666	$(7, 880088)$	6160616
$(55, 99)$	5445	$(99, 5555)$		$(969, 5335)$	5169615	$(8, 770077)$	
$(7, 858)$	6006	$(707, 858)$	606606	$(575, 9119)$	5243425	$(77, 80008)$	
$(77, 88)$	6776	$(7, 88088)$	616616	$(979, 5555)$		$(88, 70007)$	
$(55, 999)$	54945	$(8, 77077)$		$(55, 99999)$	5499945	$(77, 80088)$	**6166776**
$(99, 555)$		$(77, 8008)$		$(99, 55555)$		$(88, 70077)$	
$(77, 858)$	66066	$(88, 7007)$		$(7, 858088)$	**6006616**	$(898, 7227)$	6489846

In addition to providing a positive answer to Problem 3, the table reveals some interesting facts. Except for the two values of C printed in boldface, all other Cs are themselves palindromes in which case *both* A and B are palindromes. The pairs $(7, 858088)$, yielding $C = 6006616$, and $(77, 80088)$ and $(88, 70077)$, giving $C = 6166776$, contain a palindrome A. On the other hand, up to $C \leq 10^7$, no palindromic pairs were found, with neither A nor B being a palindrome, that was not polynomial. Computational evidence strongly suggests the following conjecture.

Conjecture 4. If (A, B) is a palindromic pair, with neither A nor B a palindrome, then (A, B) is a polynomial pair.

In attempting to answer the conjecture, we begin by establishing properties of polynomial pairs in the next section.

3 Some properties of polynomial pairs

For $A \in \mathbb{N}$, let A_∞ denote the maximum of the coefficients of $P(A, x)$.

Proposition 5. *If (A, B) is a polynomial pair, then $A_\infty B_\infty \leq 9$. If, moreover, $A_\infty \geq 5$, then $B_\infty = 1$.*

Proof. This proposition is a direct consequence of Proposition 1. Let j and l be, respectively, the smallest index such that $a_j = A_\infty$ and $b_l = B_\infty$. Then $A_\infty B_\infty > 9$ would imply that the coefficient of x^{j+l} in the multiplication $P(A, x)P(B, x)$ is > 9, violating Proposition 1 Part 3). □

To derive a sufficient condition for (A, B) to be a palindromic pair we define the norm of an integer by the formula $A_1 = P(A, 1)$.

Proposition 6. *For $A, B \in \mathbb{N}$, $(AB)_\infty \leq A_\infty B_1$.*

Proof. Write $P(A,x)=\sum_{i=0}^{a} a_i x^i$, and $P(B,x)=\sum_{i=0}^{b} b_i x^i$. Then, the coefficient of x^k in $P(AB,x)$ is

$$\sum_{i+j=k} a_i b_j \leq A_\infty \sum_{j=0}^{b} b_j = A_\infty B_1.$$

□

A construction of polynomial pairs can be deduced from Proposition 6.

Proposition 7. *Let $A, B \in \mathbb{N}$ with $A_\infty B_1 \leq 9$. Then (A,B) is a polynomial pair.*

Proof. Combine Proposition 1 Part 3) and Proposition 6. □

Table 2 list downs all polynomial pairs (A,B) with $A \times B = C$, $A \leq B$, and $C \leq C^* \leq 10^4$. Neither $A = A^*$ nor $B = B^*$ is allowed although $C = C^*$ is allowed. For each pair, there is a corresponding pair (A^*, B^*) with $A^* \times B^* = C^*$. When C is a palindrome, it is written in bold.

Table 2: The list of polynomial pairs (A,B) with $A \leq B$, neither $A = A^*$ nor $B = B^*$ is allowed, and $A \times B = C \leq C^* \leq 10^4$

C	(A,B)	C	(A,B)	C	(A,B)	C	(A,B)	C	(A,B)
144	(12, 12)	1356	(12, 113)	2352	(21, 112)		(24, 112)	3624	(12, 302)
156	(12, 13)	1368	(12, 114)	2369	(23, 103)	2743	(13, 211)	3648	(12, 304)
168	(12, 14)	1428	(14, 102)	2373	(21, 113)	2769	(13, 213)	3744	(12, 312)
169	(13, 13)	1456	(13, 112)	2394	(21, 114)	**2772**	(12, 231)	3768	(12, 314)
252	(12, 21)	1464	(12, 122)	2436	(12, 203)		(21, 132)	3864	(12, 322)
273	(13, 21)	1469	(13, 113)	2448	(12, 204)	2793	(21, 133)	3888	(12, 324)
276	(12, 23)	1476	(12, 123)		(24, 102)	2796	(12, 233)	3926	(13, 302)
288	(12, 24)	1488	(12, 124)	2556	(12, 213)	2814	(14, 201)	3984	(12, 332)
294	(14, 21)	1568	(14, 112)	2562	(21, 122)	2873	(13, 221)	4284	(21, 204)
299	(13, 23)	1584	(12, 132)	2568	(12, 214)	2892	(12, 241)		(42, 102)
384	(12, 32)	1586	(13, 122)	2576	(23, 112)	2899	(13, 223)	4386	(43, 102)
1224	(12, 102)	1596	(12, 133)	2583	(21, 123)	2954	(14, 211)	4494	(21, 214)
1236	(12, 103)	1599	(13, 123)	2599	(23, 113)	3193	(31, 103)	4669	(23, 203)
1248	(12, 104)	2142	(21, 102)	2613	(13, 201)	3264	(32, 102)	4836	(12, 403)
1326	(13, 102)	2163	(21, 103)	2639	(13, 203)	3296	(32, 103)	4899	(23, 213)
1339	(13, 103)	2184	(21, 104)	2676	(12, 223)	3468	(34, 102)	4956	(12, 413)
1344	(12, 112)	2346	(23, 102)	2688	(12, 224)	3584	(32, 112)	6496	(32, 203)

4 Integers with many polynomial pairs

From Tables 1 and 2 we notice that some $C \in \mathbb{N}$ can be the product of the elements of distinct polynomial pairs. Here we give a construction to show that some numbers can be the product of the elements of an arbitrarily large number of distinct polynomial pairs.

First, let us define a *repunit* $R(n)$ as the n-digit number whose digits are ones. The term, which abbreviates *repeated unit*, first appeared in Beiler [2, Ch. 11]. More formally,

$$R(n) = \sum_{i=0}^{n-1} 10^i = \frac{10^n - 1}{9}.$$

Sequence A004023 in the OEIS [3] records the known values of n for which $R(n)$ is prime.

Theorem 8. *The repunit $R(2^n)$ is the product of n pairwise distinct positive integers. It can be expressed as the product $A \times B$ of $2^{n-1} - 1$ pairwise distinct polynomial pairs (A, B).*

Proof. It is clear that $R(2) = 10^1 + 1$ and $R(4) = R(2)(100 + 1) = (10^1 + 1)(10^2 + 1)$. Using the difference of squares, we can inductively write

$$R(2^n) = \frac{1}{9}(10^{2^n} - 1) = \frac{1}{9}(10^{2^{n-1}} - 1)(10^{2^{n-1}} + 1) = R(2^{n-1})(10^{2^{n-1}} + 1) = \prod_{i=0}^{n-1}(10^{2^i} + 1).$$

This establishes the first assertion. Moreover, all of the multiplications can be performed without carry since $R(a)_\infty = 1$, for all integers $a \geq 1$.

Since there are n distinct factors, we can group them into two disjoint nontrivial sets $\mathcal{A}$ and $\mathcal{B}$. Let A be the product of the elements in $\mathcal{A}$. Let B be defined analogously based on $\mathcal{B}$. Since we want to avoid repetition, two cases based on the parity of n need to be considered.

When $n = 2k + 1$, the set $\mathcal{A}$ has j elements with $1 \leq j \leq k = (n - 1)/2$. The remaining $n - j$ elements not chosen for $\mathcal{A}$ automatically form the set $\mathcal{B}$. Thus, there are $\sum_{j=1}^{k} \binom{n}{j} = 2^{n-1} - 1$ pairwise distinct pairs (A, B).

When $n = 2k$, we can do similarly for $1 \leq |\mathcal{A}| \leq k - 1$ but we need to treat the case of $|\mathcal{A}| = |\mathcal{B}| = k$ with more care. To avoid forming repetitive pairs, we halve the count. In total, the number of pairwise distinct pairs (A, B) formed is $\sum_{j=1}^{k-1} \binom{n}{j} + \frac{1}{2}\binom{n}{k} = 2^{n-1} - 1$. □

5 Reversal multiplication

A popular way to obtain palindromes is to multiply a number by its reversal. This is called *reversal multiplication* in [4] and the numbers that give palindromes in that way form Sequence A062936 in the OEIS [3]. This recipe always works with polynomial pairs as the next result shows.

Proposition 9. *If (A, A^*) is a polynomial pair, then $A \times A^*$ is always a palindrome.*

Proof. It suffices to confirm that $P(A^*, x) = P^*(A, x)$ and that $P(A, x)P^*(A, x)$ is a self-reciprocal polynomial. □

De Geest [4] observes that all elements > 3 in sequence A062936 have only the digits $0, 1,$ and 2. This is easy to show in the polynomial pair case, and correlates with an observation made by David Wilson A062936 on July 6, 2001 stating that said sequence includes positive integers not ending in 0 whose sum of squares of the digits is ≤ 9.

Proposition 10. *If (A, A^*) is a polynomial pair, then the sum of the squares of the digits of A is ≤ 9. In particular, if $A > 9$, we have $A_\infty \leq 2$. Conversely, if the sum of the squares of the digits of A is ≤ 9, then (A, A^*) is a polynomial pair.*

Proof. Write $P(A, x) = \sum_{i=0}^{d} a_i x^i$. By the Cauchy-Schwarz inequality, for $0 \leq k \leq d$,

$$\sum_{l=0}^{k} a_l a_{d-k+l} \leq \sum_{i=0}^{d} a_i^2.$$

The left-hand side is the coefficient of x^k while the right-hand side is the coefficient of x^d in $P(A \times A^*, x)$. Thus, by Proposition 9, we have

$$(A \times A^*)_\infty = \sum_{i=0}^{d} a_i^2, \tag{2}$$

which is the sum of the squares of the digits of A. This establishes the first statement from which follows that if A has at least two nonzero digits, then none can be ≥ 3.

The converse follows from Eq. (2) and Proposition 1. □

We have generated a list of elements $A < 10^9$ of sequence A062936 and verified that the sum of the squares of the digits of A is bounded above by 9. Applying the converse part of Proposition 10, we are led to the following conjecture.

Conjecture 11. If $A \times A^*$ is a palindrome, then (A, A^*) is a polynomial pair.

To emphasize that we make Conjecture 11 for base $b = 10$ only, we observe the following counterexamples for $b \neq 10$.

In base 2, we have $11 \times 11 = 1001$. More generally, for any integer $l \geq 2$,

$$\begin{aligned}&11\overbrace{00\cdots0}^{2l}10101\overbrace{00\cdots0}^{2l-1}11 \times 11\overbrace{00\cdots0}^{2l-1}10101\overbrace{00\cdots0}^{2l}11\\ &= 1001 \times 2^{8l+12} + 10111101 \times 2^{6l+7} + 1001001001 \times 2^{4l+3} + 10111101 \times 2^{2l+1} + 1001,\end{aligned} \tag{3}$$

which can be seen to be a palindrome. Using $2l + 1$, instead of $2l - 1$, also works. Our computation reveals that there are no other counterexamples with A having less than 20-digit base 2 representation.

In base 4, the only counterexample with A having less than 10-digit representation is $2232213 \times 3122322 = 21111033011112$. The next counter example, if exists, must be a considerably large number.

Table 3 gives the counterexamples we found for base $b \in \{3, 4, 5, 7, 8, 9, 11\}$. We have not been able to find counterexamples in either base 6 or base 10.

Table 3: Examples of A and $A \times A^*$ where (A, A^*) is not a polynomial pair for $A \leq A^*$ in bases $b \in \{3, 4, 5, 7, 8, 9, 11\}$. In base 11, a stands for 10

Base	A	$A \times A^*$	Base	A	$A \times A^*$
3	2	11	8	47	4444
	202	112211		303	112211
	2002	11022011		306	225522
	20002	1100220011		333	135531
	200002	110002200011		3003	11022011
	201102	111221122111		3006	22055022
	2000002	11000022000011		3033	12244221
	20000002	1100000220000011		3116	23300332
	20011002	1101211111121011		3306	24377342
	200000002	110000002200000011		3333	13577531
	2000000002	11000000022000000011		30003	1100220011
	2000110002	11001210111101210011		30006	2200550022
4	2232213	21111033011112		30033	1211441121
5	314	242242		30303	1122332211
	22033	1334334331	9	516	350053
	220033	133133331331		44055	2667557662
	2200033	13310333301331		440055	266255552662
	2301123	14000211200041		2403555	14682744728641
	22000033	1331003333001331		4400055	26620555502662
	23410123	1423123003213241	11	6	33
	24200303	1344343003434431		66	3993
7	4	22		77	5335
	44	2662		374	161161
	55	4444		419	350053
	404	224422		606	336633
	4004	22044022		6006	33066033
	4114	23300332		21896	139a00a931
	25124	1456446541		33088	1669999662
	40004	2200440022		60006	3300660033
	400404	222426624222		60606	3366996633
	403304	223652256322		63328	4779559774
	404004	222426624222		283306	156695596651
	4000004	22000044000022		330088	266279972662
8	3	11		391744	15a484484a51
	6	44		441739	379373373973
	33	1331		600006	330006600033
	36	2772		600606	333639936333

Proposition 12. *There are infinitely many values of b for which the analogue of Conjecture 11 in base b is false.*

Proof. First, consider the base b such that $b = r^2 - 1$ for $2 \le r \in \mathbb{N}$. Writing in base b, $r \times r = 11$, and for any non-negative integer j,

$$r\overbrace{00\cdots0}^{j+1}r \times r\overbrace{00\cdots0}^{j+1}r = (r \times b^{j+2} + r)^2 = (r^2 \times b^{2j+4}) + (2 \times r^2 \times b^{j+2}) + r^2$$
$$= 11\overbrace{00\cdots0}^{j}22\overbrace{00\cdots0}^{j}11.$$

There are obviously infinitely many such bases b.

For any base b of the form $b = 4k - 1$,

$$(2k) \times (2k) = 4k^2 = (k \times b) + k = kk \tag{4}$$

in base b. More generally, in the said base, one can easily verify, by using Eq. (4), that

$$((2k)\overbrace{00\cdots0}^{j+1}(2k))^2 = ((2k \times b^{j+2}) + 2k)^2 = (4k^2 \times b^{2j+4}) + (2 \times 4k^2 \times b^{j+2}) + 4k^2$$
$$= kk\overbrace{00\cdots0}^{j}(2k)(2k)\overbrace{00\cdots0}^{j}kk.$$

When the base b is of the form $b = 4k + 1$, we can write

$$(2k) \times (2k+1) = k \times (4k+1) + k = (k \times b) + k = kk. \tag{5}$$

Using Eq. (5), one gets

$$\begin{aligned}
&[(2k)(2k)00(2k+1)(2k+1)] \times [(2k+1)(2k+1)00(2k)(2k)] \\
&= [b^4 \times 2k \times (b+1) + (2k+1) \times (b+1)] \times [b^4 \times (2k+1) \times (b+1) + 2k \times (b+1)] \\
&= kk + [(2k)(2k) \times b] + [kk \times b^2] + [(2k)(2k+1) \times b^4] + [101 \times b^5] + [(2k)(2k+1) \times b^6] \\
&+ [kk \times b^8] + [(2k)(2k) \times b^9] + [kk \times b^{10}] \\
&= k(3k)(3k)k(2k+1)(2k+1)(2k+1)(2k+1)k(3k)(3k)k.
\end{aligned}$$

In fact, one can obtain a slightly more general result since, for any non-negative integer j,

$$(2k)(2k)\overbrace{00\cdots0}^{j+2}(2k+1)(2k+1) \times (2k+1)(2k+1)\overbrace{00\cdots0}^{j+2}(2k)(2k)$$
$$= k(3k)(3k)k\overbrace{00\cdots0}^{j}(2k+1)(2k+1)(2k+1)(2k+1)\overbrace{00\cdots0}^{j}k(3k)(3k)k.$$

□

Thus, a necessary but insufficient condition for the analogue of Conjecture 11 in base b to hold is for b to be even and for $b+1$ to be square-free.

Remark 13. Let (A, B) be a palindromic pair. If either A or B is itself a palindrome, then we cannot conclude immediately that (A, B) is a polynomial pair. Indeed, in many cases, for example, when $A = 121$ and $B = A^* = A$, the pair (A, B) is both palindromic and polynomial. Yet, as shown by the pairs listed in Table 1, a palindromic pair may fail to be polynomial when either A or B is a palindrome.

Conjecture 11 posits that, regardless of whether A itself is a palindrome, so long as $A \times A^*$ is a palindrome, then (A, A^*) is polynomial. Thus, this conjecture does not follow from Conjecture 4. If, however, we add the condition that $A \neq A^*$, then a positive answer to Conjecture 4 settles this modified version of Conjecture 11 since, if $A \times A^*$ is a palindrome, then (A, A^*) is of course a palindromic pair.

Note that Proposition 12 still holds if we use the base b analogue for the modified version of Conjecture 11 using only bases $b = 4k + 1$ in the proof. In this case, removing all entries in Table 3 having $A = A^*$ provides analogous examples.

To end this section we prove a special case of Conjecture 11.

Proposition 14. *If A is an $n-$digit number and $A \times A^*$ is a $(2n-1)$-digit palindrome then (A, A^*) is a polynomial pair.*

Proof. Let A be an n-digit number such that $A \times A^*$ is a $(2n-1)$-digits palindrome, with the notation $P(A, x) = \sum_{i=0}^{n-1} a_i x^i$. Let $c_0, c_1, \ldots, c_{2n-2}$ be the digits of $A \times A^*$. We now make completely explicit how the digits are manipulated when the multiplication is performed.

Let γ_i be the carry that is propagated on the i-th digits and σ_i be the sum of the products of digits that appear in the i-th position. Hence,

$$\gamma_0 = 0$$

and, for all $0 \leq i \leq 2n-1$,

$$\begin{aligned} \sigma_i &= \gamma_i + \sum_{k=\max(0,i+1-n)}^{\min(n-1,i)} a_k\, a_{n-1-i+k}, \\ c_i &= \sigma_i \pmod{10}, \\ \gamma_{i+1} &= (\sigma_i - c_i)/10. \end{aligned}$$

Note that (A, A^*) is a polynomial pair if and only if $\gamma_i = 0$ for all $i \leq 2n-1$. We prove this fact by induction.

Since $A \times A^*$ has only $2n-1$ digits, we have $c_{2n-1} = 0$, and thus $\gamma_{2n-1} = 0$. Suppose that for a certain integer ℓ we have proven that $\gamma_\ell = 0$ and $\gamma_{2n-\ell-1} = 0$. Since $\gamma_{2n-\ell-1} = 0$, we must have

$$\sigma_{2n-\ell-2} = c_{2n-\ell-2} \leq 9.$$

Now,

$$\sigma_\ell = \sigma_{2n-2-\ell} - \gamma_{2n-\ell-2} + \gamma_\ell = \sigma_{2n-\ell-2} - \gamma_{2n-\ell-2}$$

must be ≤ 9 too. So $\gamma_{\ell+1} = 0$ and $c_\ell = \sigma_\ell$. Since $A \times A^*$ is a palindrome, we have $c_\ell = c_{2n-\ell-2}$. So we also have $\sigma_{2n-\ell-2} = \sigma_\ell$. Now we can compute that

$$\gamma_{2n-\ell-2} = \sigma_{2n-\ell-2} - \sigma_\ell + \gamma_\ell = 0,$$

which concludes the induction step. □

6 Squares and palindromes

In this short section we show that some results established above shed light on several connections between palindromes and squares.

There are two sequences in the OEIS [3] concerning palindromes and squares. Sequence A002779 lists palindromic perfect squares, while sequence A002778 contains integers whose squares are palindromes. The next result, which is a direct consequence of Proposition 9, gives a sufficient but not a necessary condition for an integer A to belong to sequence A002778.

Proposition 15. *If (A, A) is a polynomial pair with A a palindrome, then A^2 is a palindrome.*

Each entry of sequence A156317 in the OEIS [3] is a perfect square that forms either an equal or a larger perfect square when reversed. Here is a technique to produce examples of such integers.

Proposition 16. *If (A, A) is a polynomial pair then so is (A^*, A^*). Moreover, $(A^2)^* = (A^*)^2$.*

Proof. It suffices to verify that $P((A^2)^*, x) = P^*(A^2, x) = (P^*(A, x))^2 = (P(A^*, x))^2$. □

7 Additive pairs

It is natural to consider as well the additive analogue of polynomial pairs. The pair (A, B) of positive integers is said to be an *additive pair* if

$$P(A, x) + P(B, x) = P(A + B, x).$$

The counterpart of Proposition 1 can then be established.

Proposition 17. *The following assertions are equivalent.*

1) The pair (A, B) is an additive pair.

2) The addition of A by B can be performed without carry.

3) The coefficients of the polynomial $P(A,x)+P(B,x)$ are bounded above by 9.

Proof. We use the same representation of $P(A,x)$ and $P(B,x)$ as in the proof of Proposition 6. Let j be the smallest integer such that $c_j = a_j + b_j > 9$. Then c_j is the coefficient of x^j in $P(A,x)+P(B,x)$ while $c_j \pmod{10} \neq c_j$ is the coefficient of x^j in $P(A+B,x)$. By contrapositive argument, 1) implies 2).

It is clear by definition of polynomial addition that 2) implies 3). To verify that 3) implies 1) note that for $0 \leq j \leq \max(a,b)$ we have $c_j = a_j + b_j \leq 9$, which leads immediately to the desired conclusion since c_j is the coefficient of x^j in both $P(A+B,x)$ and $P(A,x)+P(B,x)$. □

A sufficient condition for (A,B) to be an additive pair is $A_\infty + B_\infty \leq 9$. Additive pairs can be used to generates palindromes.

Proposition 18. *If (A,A^*) is an additive pair, then $A+A^*$ is a palindrome.*

Proof. It is straightforward to verify that $P(A^*,x) = P^*(A,x)$ and that $P(A+A^*,x) = P(A,x)+P^*(A,x)$ is a self-reciprocal polynomial. □

There are, however, integers A such that $A+A^*$ is a palindrome yet (A,A^*) is not an additive pair. The numbers 56 and 506 are some easy examples of such A.

8 Summary

In this note we have shown how to use polynomial pairs to study the properties of palindromic pairs. Furthermore, a large number of palindromic pairs can be constructed by using polynomial pairs. Connections to well-known numbers and integer sequences in the OEIS have also been explicated.

It is of interest to either find counterexamples to or to prove the validity of the conjectures mentioned here for future investigations. As an added incentive, we offer a ripe durian for a correct proof of, or a valid counterexample to, any of the conjectures.

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