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COMPLEX SYMMETRY OF WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE

PHAM VIET HAI AND LE HAI KHOI

ABSTRACT. With the techniques of weighted composition operators, we introduce a new concept of general weighted composition conjugations on the Fock space \( F^2(\mathbb{C}) \) and completely solve the following problems: the description of weighted composition operators which are conjugations; the criteria for weighted composition operators to be complex symmetric; the spectrum and the point spectrum of complex symmetric operators; and the criteria for Schatten class membership as well as for normality of such operators.

1. INTRODUCTION

Many problems appearing in analysis require much research to non-Hermitian operators. Among them, the operators, known as complex symmetric operators, have become particularly important on both theoretic and application aspects (see, e.g., [6]). In a general setting, we have the following definitions.

Definition 1.1. A mapping \( T \) acting on a separable complex Hilbert space \( \mathcal{H} \) is said to be anti-linear (also conjugate-linear), if
\[
T(ax + by) = \overline{a} T(x) + \overline{b} T(y), \quad \forall x, y \in \mathcal{H}, \forall a, b \in \mathbb{C}.
\]

Definition 1.2. An anti-linear mapping \( C : \mathcal{H} \to \mathcal{H} \) is called a conjugation, if it is

1. involutive: \( C^2 = I \), the identity operator;

2. isometric: \( ||Cx|| = ||x|| \), for all \( x \in \mathcal{H} \).

Definition 1.3. A bounded linear operator \( T \) on \( \mathcal{H} \) is called complex symmetric, if there exists a conjugation \( C \) on \( \mathcal{H} \), such that \( T = CT^*C \). In this case, \( T \) is often called a \( C \)-symmetric operator (or CSO, for short).

The general study of the complex symmetry was commenced by Gar-\c{c}ia and Putinar in [4, 5]. Thereafter, a number of the papers is devoted to the topic. The results show that the bounded complex symmetric

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operators are quite diverse. It includes the Volterra integration operators, normal operators, compressed Toeplitz operators, etc.

On the other hand, given two holomorphic (or entire) functions $\varphi$ and $\psi$, the weighted composition operator $W_{\psi, \varphi}$ is defined as $W_{\psi, \varphi}f = \psi \cdot f \circ \varphi$. The study of weighted composition operators on function spaces has been received a special attention by many authors during the past several decades (see, e.g., [1]). There is a great number of topics in the study of weighted composition operators: boundedness, compactness, compact difference, essential norm, closed range, etc.

Naturally, we can ask a question of whether there exist the complex symmetric weighted composition operators. Recently, this question was analyzed in [3, 9] for Hardy spaces in the unit disk. It is showed that such operators do exist. It is also worth to mention the work by Jung and colleagues [9] for the classical Hardy space and by Garcia and Hammond [3] for the weighted Hardy spaces. Independently, they discovered the complex symmetric structure of $W_{\psi, \varphi}$ when the conjugation is of the form $Cf(z) = \overline{f(z)}$.

It should be noted that in all works mentioned above, the complex symmetry is studied mainly on the Hardy spaces. For spaces of entire functions this is still an open topic which needs addressing. Being motivated by the recent results on Hardy spaces, we investigate the complex symmetry on the Fock spaces of entire functions.

2. Preliminaries

In this section, we present some basic results about the Fock space, which are used in the sequel. Technically speaking, the Fock space $F^2(\mathbb{C})$, we denote simply by $F^2$, is defined as follows

$$F^2 = \left\{ f \in \mathcal{O}(\mathbb{C}) : \|f\| = \left( \frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2} \, dV(z) \right)^{1/2} < \infty \right\},$$

where $dV$ is the Lebesgue measure on $\mathbb{C}$ and $\mathcal{O}(\mathbb{C})$ is the set of all entire functions on $\mathbb{C}$. It is a reproducing kernel Hilbert space, with the inner product given by

$$\langle f, g \rangle = \frac{1}{\pi} \int_{\mathbb{C}} f(z) \overline{g(z)} e^{-|z|^2} \, dV(z)$$

and the kernel $K_z(u) = e^{zu}$. More generally, we have

$$f^{(m)}(u) = \frac{1}{\pi} \int_{\mathbb{C}} f(z) \overline{z^m e^{zu} e^{-|z|^2}} \, dV(z), \quad \forall f \in F^2, \forall m \in \mathbb{N}.$$ 

Here $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of non-negative integers. Setting $K_z^{[m]}(u) = u^m K_z(u)$, the $m$-th derivative evaluation kernel, the equality becomes $f^{(m)}(u) = \langle f, K_u^{[m]} \rangle$. Taking $z = 0$, we put

$$P_m(u) := K_0^{[m]}(u) = u^m, \quad m \geq 0.$$
This sequence of monomials \((P_m)\), with \(\|P_m\| = \sqrt{m!}\), plays an important role in the sequel, as \(\left(\frac{z^m}{\sqrt{m!}}\right)_{m=0}^{\infty}\) forms an orthonormal basis for \(\mathcal{F}^2\). We note that \(\|K_z\| = e^{\|z\|^2/2}\).

In this paper, we also use the normalized kernel

\[
(2.2) \quad k_u = \frac{K_u}{\|K_u\|}.
\]

Recently, T. Le [10] gave a characterization for boundedness and compactness of \(W_{\psi,\varphi}\) on \(\mathcal{F}^2\). To state these results, for convenience, we put

\[
M_z(\psi, \varphi) := |\psi(z)|^2 e^{\|\varphi(z)\|^2 - |z|^2}, \quad z \in \mathbb{C}, \quad \text{and} \quad M(\psi, \varphi) := \sup_{z \in \mathbb{C}} M_z(\psi, \varphi).
\]

**Proposition 2.1 ([10]).** The operator \(W_{\psi,\varphi}: \mathcal{F}^2 \to \mathcal{F}^2\)

(i) is bounded if and only if \(\psi \in \mathcal{F}^2\) and \(M(\psi, \varphi) < \infty\). In this case, \(\varphi(z) = az + b\) with \(|a| \leq 1\).

(ii) is compact if and only if \(\psi \in \mathcal{F}^2\) and \(\lim_{|z| \to \infty} M_z(\psi, \varphi) = 0\). In this case, \(\varphi(z) = az + b\) with \(|a| < 1\).

It is worth to note that the key result that allows to prove Proposition 2.1 is the following.

**Lemma 2.2 ([10]).** Let \(\psi\) and \(\varphi\) be two entire functions with \(\psi \neq 0\). Suppose there is a constant \(C > 0\) such that \(M_z(\psi, \varphi) \leq C, \forall z \in \mathbb{C}\). Then \(\varphi(z) = az + b\) with \(|a| \leq 1\). In addition, if \(|a| = 1\), then \(\psi(z) = ce^{-abz}\).

The following well-known action on the kernel \(K_z\) plays an important role in the study of [10].

**Lemma 2.3.** Suppose that \(W_{\psi,\varphi}\) is bounded on \(\mathcal{F}^2\). Then, for every \(z \in \mathbb{C}\),

\[
W_{\psi,\varphi}^* K_z = \overline{\psi(z)} K_{\varphi(z)}.
\]

Furthermore, we also need the following formula of the adjoint operator acting on the sequence of polynomials \((P_m)\).

**Lemma 2.4.** Suppose that \(W_{\psi,\varphi}\) is bounded (i.e. \(\varphi(z) = az + b\), with \(|a| \leq 1\)). Then,

\[
W_{\psi,\varphi}^* P_m = \sum_{j=0}^{m} \binom{m}{j} a^j \psi^{(m-j)}(0) K_{[j]}^{b}, \quad m \in \mathbb{N}.
\]
Proof. For an arbitrary $g \in \mathcal{F}^2$, we have
\[
\langle g, W_{\psi,\varphi}^* P_m \rangle = \langle W_{\varphi}^* g, P_m \rangle = \langle \psi \cdot g \circ \varphi, K^{|m|}_0 \rangle = \langle \psi \cdot g \circ \varphi \rangle^{(m)}(0)
\]
\[
= \sum_{j=0}^{m} \binom{m}{j} a^j g^{(j)}(b) \psi^{(m-j)}(0)
\]
\[
= \langle g, \sum_{j=0}^{m} \binom{m}{j} a^j \psi^{(m-j)}(0) K_{b}^{[j]} \rangle,
\]
which implies the desired conclusion. \qed

Remark 2.5. With a different approach than that used in [10] (where the role of adjoint operators is essential), in [8] the results in Proposition 2.1 are generalized to the case $\mathcal{F}^p(\mathbb{C})$.

In [8] we also gave some illustrative examples for boundedness and compactness of $W_{\psi,\varphi}$ when $\psi$, $\varphi$ are specific functions. We recall here those results as they are used in the sequel.

Proposition 2.6 ([8]). Let $\varphi(z) = az + b$, $\psi(z) = ce^{dz}$, where $a, b, c,$ and $d$ are complex constants. Then

(i) $W_{\psi,\varphi}$ is bounded if and only if
(a) either $|a| < 1$
(b) or $|a| = 1$, $d + ab = 0$.
In the last case, we always have
\[
M_z(\psi, \varphi) = |c|^2 e^{|b|^2}, \quad \forall z \in \mathbb{C}.
\]

(ii) $W_{\psi,\varphi}$ is compact if and only if $|a| < 1$.
In this case, we have
\[
M_z(\psi, \varphi) \leq |c|^2 e^{2|b|} e^{-(1-|a|^2)|z|^2 + 2(|ab| + |d|)|z|}, \quad \forall z \in \mathbb{C}.
\]

Finally, we note the following result that gives an useful relationship between convergence in the norm of $\mathcal{F}^2$ and a point convergence.

Lemma 2.7 ([13]). For every $f$ in $\mathcal{F}^2$, we have $|f(z)| \leq e^{\frac{|z|^2}{2}} \|f\|$.

We refer the reader to the monograph [13] for more information on Fock spaces.
Throughout the paper, we always assume that $\psi$ is not identically zero.

3. Complex Symmetry on $\mathcal{F}^2$

In this section, we investigate the complex symmetry on $\mathcal{F}^2$. We start the section with some properties of $\psi$ and $\varphi$ when $W_{\psi,\varphi}$ is complex symmetric.
3.1. Some properties of complex symmetric weighted composition operators. Recall that for a bounded linear operator $T$ acting on a Hilbert space, $\sigma(T)$ and $\sigma_p(T)$ denote respectively the spectrum, point spectrum.

**Theorem 3.1.** Let $\varphi$ be a nonconstant entire function, $C$ a conjugation on $\mathcal{F}^2$, and $W_{\psi,\varphi}$ a bounded weighted composition operator on $\mathcal{F}^2$ with $\varphi(z) = az + b, |a| \leq 1$.

Suppose $W_{\psi,\varphi}$ is $C$-symmetric. Then the following assertions hold:

(i) The function $\psi(z)$ is nowhere vanished on $\mathbb{C}$.

(ii) There is the identity

$$\frac{\psi(u)}{\psi(z)} = \frac{(CK_u) \circ \varphi(z)}{CK_{\varphi(u)}(z)}, \forall z, u \in \mathbb{C}.$$ 

(iii) The kernel $\ker(W_{\psi,\varphi}) = \{0\}$ and the range $\operatorname{Im}(W_{\psi,\varphi})$ is dense in $\mathcal{F}^2$.

(iv) The point spectrum $\sigma_p(W_{\psi,\varphi}) = \{a^k\psi(\lambda) : k \in \mathbb{N}\}$, where $\lambda = \frac{b}{1-a}$, if $a \neq 1$, the fixed point of the function $\varphi$.

**Proof.** The arguments are, in general, similar to those of classical Hardy spaces.

i) Assume in contrary that $\psi(\alpha) = 0$ for some $\alpha \in \mathbb{C}$. Then there is a neighborhood $U$ of $\alpha$ such that $\psi(z) \neq 0$ for every $z \in U \setminus \{\alpha\}$. By Lemma 2.3, $W_{\psi,\varphi}^* K_\alpha = \psi(\alpha) K_\varphi(\alpha) = 0$. But $W_{\psi,\varphi}$ is $C$-symmetric, hence $W_{\psi,\varphi} C K_\alpha = C W_{\psi,\varphi}^* K_\alpha = 0$.

The last equation implies that for every $z \in \mathbb{C}$

$$\psi(z)CK_\alpha(\varphi(z)) = W_{\psi,\varphi} C K_\alpha(z) = 0,$$

which shows that $CK_\alpha \circ \varphi = 0$ on $U \setminus \{\alpha\}$. Consequently, $CK_\alpha = 0$: a contradiction.

ii) For $z, u \in \mathbb{C}$, again by Lemma 2.3, we have

$$\psi(u)CK_{\varphi(u)}(z) = C \left( \psi(u)K_{\varphi(u)}(z) \right) = C \left( W_{\psi,\varphi}^* K_u(z) \right) = W_{\psi,\varphi} C K_u(z) = \psi(z) \cdot (CK_u \circ \varphi(z)),$$

from which the desired identity follows.

iii) Let $f \in \ker(W_{\psi,\varphi})$. For every $z \in \mathbb{C}$, we have $\psi \cdot f \circ \varphi = W_{\psi,\varphi} f = 0$, which gives $f(\varphi(z)) = 0$. This implies, since $\varphi$ is linear and nonconstant, that $f$ is necessarily the zero function. Thus $\ker(W_{\psi,\varphi}) = \{0\}$.

To prove $\operatorname{Im}(W_{\psi,\varphi})$ is dense in $\mathcal{F}^2$, since $\mathcal{F}^2 = \overline{\operatorname{Im}(W_{\psi,\varphi})} \oplus \ker(W_{\psi,\varphi}^*)$, it suffices to show that $\ker(W_{\psi,\varphi}^*) = \{0\}$. Indeed, if $f \in \ker(W_{\psi,\varphi}^*)$, then $W_{\psi,\varphi} f = C W_{\psi,\varphi}^* f = 0$. By the same arguments as above, $f$ must necessarily be the zero function.

iv) Let $a \neq 1$ and $\lambda = \frac{b}{1-a}$. We first prove the inclusion $\sigma_p(W_{\psi,\varphi}) \subseteq \{a^k\psi(\lambda) : k \in \mathbb{N}\}$. Take an arbitrary $\mu \in \sigma_p(W_{\psi,\varphi})$. Then there exists
a nonzero function \( f \in \mathcal{F}^2 \), such that
\[
(3.1) \quad \psi(z)f(\varphi(z)) = W_{\psi,\varphi}f(z) = \mu f(z), \quad \text{for every } z \in \mathbb{C}.
\]
If \( f(\lambda) \neq 0 \), then \( \psi(\lambda)f(\lambda) = \psi(\lambda)f(\varphi(\lambda)) = \mu f(\lambda) \), which implies that \( \mu = \psi(\lambda) \). If \( f \) has a zero at \( \lambda \) of order \( q \in \mathbb{N} \), then differentiating (3.1) \( q \) times and evaluating it at the point \( z = \lambda \) yields 
\[
\mu f^{(q)}(\lambda) = \psi(\lambda)a^q f^{(q)}(\lambda),
\]
which gives \( \mu = \psi(\lambda)a^q \).

Now we prove the inverse inclusion \( \{ \mu f : \mu \psi(\lambda) \} \subseteq \sigma_p(W_{\psi,\varphi}) \).

- If \( a \) is not a root of unity. By induction (along \( n \)), we can easily show, using Lemma 2.3, that \( K_{\lambda}^{[n]} \) has the form

\[
K_{\lambda}^{[n]} = v_n + s_{n-1}v_{n-1} + \cdots + s_0v_0,
\]
where \( v_j \) is an eigenvector of \( W_{\varphi,\psi}^* \) corresponding to the eigenvalue \( \overline{\psi(\lambda)a^n} \), and \( s_j \) are complex numbers.

- If \( a \) is a \( p \)th root of unity, then by a similar argument,

\[
K_{\lambda}^{[n]} = v_n + s_{n-1}v_{n-1} + \cdots + s_0v_0,
\]
whenever \( 0 \leq n \leq p - 1 \). Hence, \( \overline{\psi(\lambda)a^n} \) is an eigenvalue of \( W_{\varphi,\psi}^* \), for \( n \leq p - 1 \), and hence for all \( n \).

In both cases, \( \overline{\psi(\lambda)a^n} \) are eigenvalues of \( W_{\varphi,\psi}^* \). Since \( W_{\psi,\varphi} \) is \( \mathbb{C} \)-symmetric, \( \psi(\lambda)a^n \) are eigenvalues of \( W_{\psi,\varphi} \) and the assertion (iv) is proved.

Remark 3.2. We note that for part (iv) in Theorem 3.1, the eigenvectors corresponding to the eigenvalues \( \psi(\lambda)a^n \), in general, cannot be indicated. However, as it is seen below in subsection 3.4, in case of symmetric weighted composition operators, we can give an explicit formula of those eigenvectors.

3.2. Anti-linear weighted composition operators. As noted in Introduction, a complex symmetric structure of \( W_{\psi,\varphi} \) is investigated in [3, 9] for the case when the conjugation is of the form \( \mathcal{C}f(z) = \overline{f(\bar{z})} \). In this subsection, we study a more general situation of that conjugation.

For given entire functions \( \xi \) and \( \eta \), we define an anti-linear weighted composition operator \( A_{\xi,\eta} \) acting on \( \mathcal{F}^2 \) by
\[
(A_{\xi,\eta}f)(z) = \xi(z)\overline{f(\eta(z))}.
\]

The first natural question to ask is: when does the operator \( A_{\xi,\eta} \) act from the Fock space \( \mathcal{F}^2 \) into itself? We note that by Lemma 2.7, a point evaluation is bounded on \( \mathcal{F}^2 \). Then \( \mathcal{F}^2 \) is a functional Hilbert space, and so \( A_{\xi,\eta} \) is \( \mathcal{F}^2 \)-invariant if and only if it is bounded on \( \mathcal{F}^2 \).

Following the arguments as in either [10] or [8] for linear weighted composition operators \( W_{\psi,\varphi} \), we can obtain a similar result for \( A_{\xi,\eta} \).

Proposition 3.3. The anti-linear weighted composition operator \( A_{\xi,\eta} \) is \( \mathcal{F}^2 \)-invariant (or bounded on \( \mathcal{F}^2 \)) if and only if
(1) $\xi \in \mathcal{F}^2$;
(2) $M(\xi, \eta) < \infty$.

Moreover, in this case, $\eta(z) = az + b$ with $|a| \leq 1$, and

$$\sqrt{M_z(\xi, \eta)} \leq \|A_{\xi,\eta}k_{\eta(z)}\| \leq \|A_{\xi,\eta}\|, \forall z \in \mathbb{C}. \quad (3.2)$$

Proof. • Suppose $A_{\xi,\eta}$ is bounded. The assertion (1) is true, because $\xi = A_{\xi,\eta}(1)$. Furthermore, for each $u$ in $\mathbb{C}$, since $\|k_u\| = 1$, by Lemma 2.7, we have

$$\|A_{\xi,\eta}\| \geq \|A_{\xi,\eta}(k_u)\| \geq |(A_{\xi,\eta}k_u)(z)|e^{-\frac{|z|^2}{2}}, \ z \in \mathbb{C}. \quad (3.2)$$

In particular, with $u = \eta(z)$, the last expression is precisely $\sqrt{M_z(\xi, \eta)}$, which gives (3.2). Then taking the supremum by $z$ over $\mathbb{C}$ yields (2).

• Conversely, suppose conditions (1) and (2) hold. By Lemma 2.2, $\eta(z) = az + b$ with $|a| \leq 1$.
- If $a = 0$, i.e. $\eta(z) = b$, then by Lemma 2.7 and (1), for every $f \in \mathcal{F}^2$,

$$\|A_{\xi,\eta}(f)\| = |f(\bar{b})|\|\xi\| \leq \|\xi\|e^{\frac{|b|^2}{2}}\|f\|.$$  

- If $a \neq 0$, then by (2) and linearity of $\eta$, for every $f \in \mathcal{F}^2$,

$$\|A_{\xi,\eta}(f)\| \leq \sqrt{M(\xi, \eta)} \left( \frac{1}{\pi} \int_{\mathbb{C}} |Cf(\eta(z))|^2 e^{-|\eta(z)|^2} \ dV(z) \right)^{1/2}$$

$$= \sqrt{M(\xi, \eta)} \left( \frac{1}{\pi} \int_{\mathbb{C}} |Cf(u)|^2 e^{-|u|^2} \ dV(u) \right)^{1/2}$$

$$= \frac{\sqrt{M(\xi, \eta)}}{|a|} \|Cf\| = \frac{\sqrt{M(\xi, \eta)}}{|a|} \|f\|,$$

where $Cf(z) = \overline{f(\bar{z})}$.

It turns out that in case $\eta \neq \text{const}$, condition (1) in Theorem 3.3 can be removed. More precisely, if condition (2) is satisfied, then so is condition (1). We have the following result.

**Corollary 3.4.** Let $\eta$ be a non-constant entire function. The operator $A_{\xi,\eta} : \mathcal{F}^2 \to \mathcal{F}^2$ is bounded if and only if $M(\xi, \eta) < \infty$.

Proof. Suppose that $M(\xi, \eta) < \infty$. By Lemma 2.2, $\eta(z) = az + b$ with $|a| \leq 1$. Moreover, $a \neq 0$ because $\eta$ is a non-constant entire function. By Proposition 3.3, it suffices to show that $\xi \in \mathcal{F}^2$. Indeed,

$$\|\xi\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} M_z(\xi, \eta)e^{-|az+b|^2} \ dV(z) \leq \frac{M(\xi, \eta)}{\pi} \int_{\mathbb{C}} e^{-|az+b|^2} \ dV(z).$$

Since $a \neq 0$, the right-hand side is finite. This implies $\xi \in \mathcal{F}^2$. ☐
The next question is: under what conditions, \( A_{\xi,\eta} \) is a conjugation on \( F^2 \)?

In what follows, we always assume that \( \xi \) and \( \eta \) are entire functions, \( \eta \not\equiv \text{const} \), for which \( M(\xi,\eta) < \infty \), that is, the anti-linear weighted composition operator \( A_{\xi,\eta} \) is bounded on \( F^2 \).

To study an isometry, we first note that the operator \( C f(z) = \overline{f(z)} \), being a conjugation on the Hardy space \( H^2(\mathbb{D}) \), is also a conjugation in our case.

**Lemma 3.5.** The operator \( C f(z) = \overline{f(z)} \), is a conjugation on \( F^2 \).

**Proof.** We can see that the operator \( C \) acts from \( F^2 \) to itself. Indeed, expanding each \( f \in F^2 \) into the Taylor series,

\[
f(z) = \sum_{n \geq 0} f_n z^n, \quad z \in \mathbb{C},
\]

we have \( ( Cf )(z) = \sum_{n \geq 0} \overline{f_n} z^n \). Since \( ( P_n ) \) is an orthogonal basis, we get

\[
\| Cf \|^2 = \sum_{n \geq 0} | \overline{f_n} |^2 \| P_n \|^2 = \| f \|^2,
\]

which shows that \( Cf \in F^2 \).

Furthermore, it is easy to verify that \( C \) has properties of anti-linearity, involution, and isometry. \( \square \)

Now for each \( b \in \mathbb{C} \), we define the operator \( A_b : F^2 \to F^2 \) by

\[
A_b f(z) = k_{-b}(z) \overline{f(z+b)},
\]

where \( k_u \) is defined in (2.2).

We prove the following property of \( A_b \).

**Lemma 3.6.** For every \( b \in \mathbb{C} \), \( A_b \) is isometric on \( F^2 \).

**Proof.** We note the following equality

\[
|k_{-\beta}(\alpha - \beta)|^2 e^{-|\alpha-\beta|^2} = e^{-|\alpha|^2}, \quad \text{for any } \alpha, \beta \in \mathbb{C}.
\]

Indeed, we have

\[
|k_{-\beta}(\alpha - \beta)|^2 e^{-|\alpha-\beta|^2} = |e^{-2\mathbb{R}(\alpha+|\beta|^2)}| \cdot e^{-|\alpha-\beta|^2} = |e^{-2\mathbb{R}(\alpha+|\beta|^2)}| \cdot e^{-|\alpha|^2 - |\beta|^2 + 2\mathbb{R}(\mathbb{I}(\alpha))} = e^{-|\alpha|^2}.
\]
Now take an arbitrary \( f \in \mathcal{F}_2 \). By changing the variable \( u = z + b \), we have
\[
\|A_b f\|^2 = \frac{1}{\pi} \int_C |k_{-b}(z)|^2 \cdot |Cf(z + b)|^2 e^{-|z|^2} \, dV(z)
\]
\[
= \frac{1}{\pi} \int_C |k_{-b}(u - b)|^2 |Cf(u)|^2 e^{-|u - b|^2} \, dV(u)
\]
\[
= \frac{1}{\pi} \int_C |Cf(u)|^2 e^{-|u|^2} \, dV(u) = \|Cf\|^2 = \|f\|^2.
\]
Here the last equality is followed by (3.3).

It turns out that for an isometry of \( A_{\xi,\eta} \), the functions \( \eta \) and \( \xi \) must be of special forms. Namely, we have the following characterization of isometry for anti-linear weighted composition operators on \( \mathcal{F}_2 \).

**Proposition 3.7.** Suppose that \( A_{\xi,\eta} : \mathcal{F}_2 \to \mathcal{F}_2 \) is bounded. Then \( A_{\xi,\eta} \) is isometric on \( \mathcal{F}_2 \) if and only if the functions \( \xi \) and \( \eta \) are of the following forms
\[
(3.4) \quad \eta(z) = az + b, \quad \xi(z) = ce^{-a\overline{b}z},
\]
where \( a, b, \) and \( c \) are complex numbers satisfying the conditions
\[
(3.5) \quad |a| = 1, \quad |c|^2 e^{|b|^2} = 1.
\]

**Proof.** Suppose that \( A_{\xi,\eta} \) is isometric on \( \mathcal{F}_2 \). By Proposition 3.3, \( \eta(z) = az + b \) for some constants \( a \) and \( b \) with \( |a| \leq 1 \). For every \( f \in \mathcal{F}_2 \), we have
\[
A_{\xi,\eta} f(z) = k_{\overline{\eta}(z)} f(z - b),
\]
and hence \( A_{\xi,\eta} f(\eta(z)) = k_{\overline{\eta}(az + b)} f(az) \). From this, it follows that
\[
A_{\xi,\eta} \overline{A_{\xi,\eta}} f(z) = \xi(z) \overline{A_{\xi,\eta} f(\eta(z))}
\]
\[
= \xi(z) e^{a\overline{b}z + |b|^2} f(az) = \xi(z) k_b(az + b) f(az), \quad z \in \mathbb{C}.
\]

Obviously, \( A_{\xi,\eta} \overline{A_{\xi,\eta}} \) is linear. Moreover, since \( A_{\xi,\eta} \) is isometric, by Lemma 3.6, we have
\[
\|A_{\xi,\eta} \overline{A_{\xi,\eta}} f\| = \|A_{\xi,\eta} f\| = \|f\|, \quad \text{for every } f \in \mathcal{F}_2,
\]
and so \( A_{\xi,\eta} A_{\xi,\eta} \) is isometric.

Setting \( \zeta(z) = \xi(z) k_b(az + b) \) and \( \chi(z) = az \), we see that
\[
A_{\xi,\eta} A_{\xi,\eta} = W_{\zeta,\chi},
\]
which is the linear weighted composition operator on \( \mathcal{F}_2 \), induced by symbols \( \zeta \) and \( \chi \).

Furthermore, since \( W_{\zeta,\chi} \) is isometric on \( \mathcal{F}_2 \), by [10, Proposition 4.3], \( |a| = 1 \) and \( \zeta \) must be a constant of modulus 1. Consequently, denoting \( \varsigma = \xi(z) k_b(az + b) \), we have \( |\varsigma| = 1 \) and \( \xi(z) = \varsigma e^{-|b|^2/2} e^{-a\overline{b}z} \). Then for
\( c = \varsigma e^{-|b|^2/2} \), we get \( \varsigma = ce^{-|b|^2/2} \). Since \( |\varsigma| = 1 \), we have \( |ce^{-|b|^2/2}| = 1 \). Thus, (3.4) and (3.5) hold.

- Conversely, suppose that \( \eta, \xi \) satisfy (3.4) and (3.5). Take \( f \in \mathcal{F}^2 \), by (3.4), we have

\[
\|A_{\xi,\eta}f\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} |ce^{-|b|z}|^2 |Cf(az + b)|^2 e^{-|z|^2} \, dV(z).
\]

Furthermore, since \( |\varsigma| = 1 \), the changing the variables \( u = az \) yields

\[
\|A_{\xi,\eta}f\|^2 = \frac{1}{\pi} \int_{\mathbb{C}} |ce^{-|b|u}|^2 |Cf(u + b)|^2 e^{-|u|^2} \, dV(u)
\]

\[
= \frac{1}{\pi} \int_{\mathbb{C}} |c|^2 e^{|b|^2} |e^{-|\bar{b}|u/2}|^2 |Cf(u + b)|^2 e^{-|u|^2} \, dV(u)
\]

\[
= \frac{1}{\pi} \int_{\mathbb{C}} k_-(u)^2 |Cf(u + b)|^2 e^{-|u|^2} \, dV(u) = \|A_{\xi,\eta}f\|^2 = \|f\|^2.
\]

Here we note that the third equality holds by (3.5) and the formula \( k_-(u) = e^{-|\bar{b}|u/2} \), while the last equality is followed by Lemma 3.6.

Now we study an involution of \( A_{\xi,\eta} \). Surprisingly, an involutive \( A_{\xi,\eta} \) must necessarily be isometric. We have the following characterization of involution for anti-linear weighted composition operators on \( \mathcal{F}^2 \).

**Proposition 3.8.** Suppose that \( A_{\xi,\eta} : \mathcal{F}^2 \to \mathcal{F}^2 \) is bounded. Then \( A_{\xi,\eta} \) is involutive if and only if \( \xi \) and \( \eta \) are of the forms (3.4), i.e.

\[
\eta(z) = az + b, \quad \xi(z) = ce^{-abz},
\]

where \( a, b, \) and \( c \) are complex numbers satisfying condition (3.5) with \( \bar{a}b + \bar{b} = 0 \), i.e.

\[
(3.6) \quad |a| = 1, \quad \bar{a}b + \bar{b} = 0, \quad |c|^2 e^{|b|^2} = 1.
\]

**Proof.** Since \( A_{\xi,\eta} \) is bounded on \( \mathcal{F}^2 \), \( \eta(z) = az + b \) with \( 0 < |a| \leq 1 \), because \( \eta \) is nonconstant.

- Suppose that \( A_{\xi,\eta} \) is an involution, this means

\[
(3.7) \quad \xi(z)\overline{\eta(\eta(z))}f\left(\overline{\eta(\eta(z))}\right) = f(z), \quad \forall f \in \mathcal{F}^2.
\]

In particular, for \( f \equiv 1 \in \mathcal{F}^2 \), we get

\[
(3.8) \quad \xi(z)\overline{\eta(\eta(z))} = 1,
\]

and hence equation (3.7) is reduced to

\[
(3.9) \quad f\left(\overline{\eta(\eta(z))}\right) = f(z), \quad \forall f \in \mathcal{F}^2.
\]
Now taking \( f(z) \equiv z \) in (3.9), we have
\[ \eta(\eta(z)) = z, \quad \forall z \in \mathbb{C}, \]
which means, as \( \eta(z) = az + b \), that
\[ |a|^2 z + (\bar{a}b + \bar{b}) = z, \quad \forall z \in \mathbb{C}, \]
and so \( |a| = 1 \) and \( \bar{a}b + \bar{b} = 0 \).

Since \( |a| = 1 \), by Lemma 2.2, we have \( \xi(z) = ce^{-abz}, \ c \in \mathbb{C} \). But \( \bar{a}b + \bar{b} = 0 \), and so we can write \( \xi(z) = ce^{bz} \). Then equation (3.8) gives \( |c|^2 e^{|b|^2} = 1 \).

- Conversely, suppose that \( \eta, \xi \) satisfy (3.4) and (3.6). Then we have
\[ \xi(z)\xi(\eta(z)) = |c|^2 e^{(a\bar{b}+b)z+|b|^2} = 1, \]
and
\[ \eta(\eta(z)) = |a|^2 z + \bar{a}b + \bar{b} = z, \]
from which (3.7) follows. \( \square \)

As an immediate consequence of Propositions 3.7 and 3.8, we obtain the following result.

**Corollary 3.9.** Suppose that \( A_{\xi,\eta} : F^2 \to F^2 \) is bounded. If \( A_{\xi,\eta} \) is involutive then it is isometric.

**Remark 3.10.** The inverse direction of Corollary 3.9 fails to hold, as there are infinitely many triples \( \{a, b, c\} \), for which (3.5) holds \( |a| = |c|^2 e^{|b|^2} = 1 \), but (3.6) is not satisfied \( (ab + b \neq 0) \).

Combining Propositions 3.7 – 3.8 yields the following important criterion.

**Theorem 3.11.** Let \( \xi \) and \( \eta \) be entire functions, with \( \eta \neq \text{const} \), such that the anti-linear weighted composition operator \( A_{\xi,\eta} \) is bounded on \( F^2 \). Then this \( A_{\xi,\eta} \) represents a conjugation on \( F^2 \) if and only if
\[ \eta(z) = az + b, \quad \xi(z) = ce^{-abz}, \]
where \( a, b, \) and \( c \) are complex numbers satisfying the conditions
\( |a| = 1, \quad \bar{a}b + \bar{b} = 0, \quad |c|^2 e^{|b|^2} = 1 \).

**Proof.** Suppose that \( A_{\xi,\eta} \) is a conjugation on \( F^2 \). Since it is involutive (and isometric), by Proposition 3.8, the functions \( \eta \) and \( \xi \) are of the form (3.4) with conditions (3.6).

Conversely, suppose \( \eta \) and \( \xi \) satisfy conditions (3.4) and (3.6). Again by Proposition 3.8, \( A_{\xi,\eta} \) is involutive, which is also isometric, by Proposition 3.7. So \( A_{\xi,\eta} \) is a conjugation. \( \square \)

**Remark 3.12.** It is worth to mention that by the condition \( \bar{a}b + \bar{b} = 0 \), which is equivalent to \( ab + b = 0 \), the formula for the symbol \( \xi \) can also be written as
\[ \xi(z) = ce^{-abz} = ce^{bz}. \]
In what follows, the weighted composition conjugation on $\mathcal{F}^2$ is denoted by $C_{a,b,c}$, that is

$$C_{a,b,c}f(z) := ce^{bz} f(az+b),$$

where $a$, $b$, $c$ are constants satisfying (3.6).

**Remark 3.13.** It is clear that with $b = 0$, $a = c = 1$, the conjugation $C_{1,0,1}$ is precisely the one considered in [3, 9].

### 3.3. $C_{a,b,c}$-symmetric weighted composition operators.

In this subsection, we consider bounded weighted composition operators and study under what conditions, these operators are $C_{a,b,c}$-symmetric. As we see below, in this case, the function $\psi$ can be precisely computed.

Note the set of all polynomials is dense in $\mathcal{F}^2$, and hence if the operator is $C_{a,b,c}$-symmetric on polynomials, then it is $C_{a,b,c}$-symmetric on the whole $\mathcal{F}^2$.

To study necessary conditions for $C_{a,b,c}$-symmetry, we apply the symmetric condition to polynomials ($P_m$) defined in (2.1). It turns out that the action on only $P_0$ can already give the sufficient conditions.

**Proposition 3.14.** Let $C_{a,b,c}$ be a weighted composition conjugation on $\mathcal{F}^2$, $\psi$ and $\varphi$ be two entire functions with $\psi \not\equiv 0$. Suppose that $W_{\psi,\varphi}$ is bounded on $\mathcal{F}^2$, that is, $\varphi(z) = Az + B$ with $|A| \leq 1$.

1. If $W_{\psi,\varphi}$ is $C_{a,b,c}$-symmetric w.r.t. $P_0$, i.e.

$$W_{\psi,\varphi}C_{a,b,c}P_0 = C_{a,b,c}W_{\psi,\varphi}P_0,$$

then the function $\psi$ has the form

$$\psi(z) = Ce^{Dz}, \quad \text{with } C \neq 0, \quad D = ab - bA + b.$$

2. Conversely, if the function $\psi$ satisfies the form above, then

$$W_{\psi,\varphi}C_{a,b,c}P_m = C_{a,b,c}W_{\psi,\varphi}P_m, \quad \forall m \in \mathbb{N}.$$

**Proof.**

1. Since $P_0(z) \equiv 1$, we have $C_{a,b,c}P_0(z) = ce^{bz}$ and hence

$$W_{\psi,\varphi}C_{a,b,c}P_0(z) = \psi(z)ce^{bz} = \psi(z)ce^{b(Az+B)}, \quad z \in \mathbb{C}.$$

On the other hand, by Lemma 2.4, $W_{\psi,\varphi}^*P_0 = \psi(0)K_{\varphi(0)}$, which gives

$$C_{a,b,c}W_{\psi,\varphi}^*P_0(z) = ce^{bz}\psi(0)e^{\varphi(0)(az+b)} = ce^{bz}\psi(0)e^{B(az+b)}, \quad z \in \mathbb{C}.$$

Therefore, the equality $W_{\psi,\varphi}C_{a,b,c}P_0 = C_{a,b,c}W_{\psi,\varphi}^*P_0$ is reduced to

$$\psi(z) = \psi(0)e^{(ab - bA + b)z} = Ce^{Dz},$$

where $C = \psi(0) \neq 0$ and $D = ab - bA + b$.

2. Conversely, for each $m \in \mathbb{N}$, by Lemma 2.4, we have

$$W_{\psi,\varphi}^*P_m(z) = \sum_{j=0}^{m} \binom{m}{j} A^j C D^{m-j} z^j e^{Bz}.$$
Then
\[
C_{a,b,c} W^*_{\psi,\varphi} P_m(z) = ce^{b} \sum_{j=0}^{m} \binom{m}{j} A^j C D^{m-j}(az + b) \ e^{B(az+b)} \\
= Cce^{(b+aB)z+bB} \sum_{j=0}^{m} \binom{m}{j} A^j D^{m-j}(az + b)^j \\
= Cce^{(b+aB)z+bB}[A(az + b) + D]^m \\
= C e^{Dz} \cdot ce^{b(Az+B)}[a(Az+B) + b]^m \\
= \psi(z) \cdot ce^{b\varphi(z)}[a\varphi(z) + b]^m = W_{\psi,\varphi} C_{a,b,c} P_m(z).
\]

\[\square\]

Now with the help of illustrative examples in Section 2, we can establish the following criterion for bounded weighted composition operators to be symmetric.

**Theorem 3.15.** Let \(C_{a,b,c}\) be a weighted composition conjugation on \(\mathcal{F}^2\), \(\psi\) and \(\varphi\) be two entire functions with \(\psi \neq 0\). Let further, \(W_{\psi,\varphi}\) be bounded on \(\mathcal{F}^2\) (i.e., \(\varphi(z) = Az + B\) with \(|A| \leq 1\)). Then \(W_{\psi,\varphi}\) is \(C_{a,b,c}\)-symmetric if and only if the following conditions hold:

(3.10) \(\psi(z) = Ce^{Dz}\), with \(C \neq 0\), \(D = aB - bA + b\).

(3.11) \[\left\{\begin{array}{ll}
\text{either } |A| < 1 \\
\text{or } |A| = 1, \ D + AB = 0.
\end{array}\right.\]

**Proof.** Note that since the conjugation is involutive, the equality \(T = CT^* C\) (complex symmetry) can be rewritten as

\[CT = T^* C.\]

- Suppose \(W_{\psi,\varphi}\) is \(C_{a,b,c}\)-symmetric on \(\mathcal{F}^2\). This implies, in particular, that \(C_{a,b,c} W^*_{\psi,\varphi} P_0 = W_{\psi,\varphi} C_{a,b,c} P_0\). Hence, by Proposition 3.14 (1), \(\psi\) must be of the form (3.10).

  Furthermore, since \(W_{\psi,\varphi}\) is bounded and \(\psi(z) = Ce^{Dz}\), by Proposition 2.6 (i), we get (3.11).

- Conversely, suppose \(\psi\) and \(\varphi\) satisfy (3.10)–(3.11). By Proposition 3.14, we have

\[W_{\psi,\varphi} C_{a,b,c} P_m = C_{a,b,c} W^*_{\psi,\varphi} P_m.\]

Since the set of all polynomials is dense in \(\mathcal{F}^2\) and \(W_{\psi,\varphi}\) is bounded, we get

\[W_{\psi,\varphi} C_{a,b,c} f = C_{a,b,c} W^*_{\psi,\varphi} f, \ \forall f \in \mathcal{F}^2.\]

That this, \(W_{\psi,\varphi}\) is \(C_{a,b,c}\)-symmetric. \[\square\]
3.4. Spectral properties of $C_{a,b,c}$-symmetric operators. In this subsection we provide the explicit formula for eigenvectors of $W_{\psi,\varphi}$ when $\varphi$ has one fixed point (compare with Remark 3.2). This happens if and only if $A \neq 1$, and the fixed point is $\lambda = \frac{B}{1-A}$.

**Theorem 3.16.** Let $W_{\psi,\varphi}$ be a $C_{a,b,c}$-symmetric operator on $F^2$, induced by entire functions $\varphi, \psi$ (that is $\varphi(z) = Az + B$, with $|A| \leq 1$ and (3.10)–(3.11) hold). Then
\[
g_k(z) = (z - \lambda)^k e^{Dz/(1-A)} \quad (k = 0, 1, 2, \ldots)
\]
are eigenvectors of $W_{\psi,\varphi}$ corresponding to the eigenvalues $A^k \psi(\lambda)$.

**Proof.** As we have proved in Theorem 3.1, the point spectrum of $W_{\psi,\varphi}$ is $\sigma_p(W_{\psi,\varphi}) = \{A^k \psi(\lambda) : k \in \mathbb{N}\}$. To prove the theorem, we first note that $g_k \in F^2$, for each $k \geq 0$. Then by Theorem 3.15, we have
\[
W_{\psi,\varphi}g_k(z) = Ce^{Dz} \cdot (Az + B - \lambda)^k e^{Dz/(1-A)} = Ce^{Dz} \cdot (Az + B - \lambda)^k e^{-Dz + \frac{Dz}{1-A} + \frac{DB}{1-A}} = \psi(\lambda) \cdot (Az + B - \lambda)^k e^{\frac{DB}{1-A}}.
\]
Since $\lambda = A\lambda + B$, the last expression is
\[
\psi(\lambda)(Az + B - A\lambda - B)^k e^{\frac{DB}{1-A}} = A^k \psi(\lambda)g_k(z).
\]

Our next task is to determine when a $C_{a,b,c}$-symmetric $W_{\psi,\varphi}$ belongs to a Schatten class. Recall that a bounded linear operator $T$ acting on a Hilbert space $H$ is said to be in the Schatten class $S_p$ ($0 < p < \infty$), if its singular value sequence $\{\lambda_n\}$ belonging to $\ell^p$. The *singular values* or *s-numbers* of a bounded linear operator $T$ on $H$ are defined by
\[
\lambda_n = \inf \|T - S\|, \quad n \geq 1,
\]
where the infimum is taken over all operators $S$ with rank at most $n-1$. If $T$ happens to be compact, then $\{\lambda_n\}$ is the sequence of eigenvalues of the positive compact operator $|T| = \sqrt{T^*T}$. Note that $S_1$ is the space of trace class operators, while $S_2$ is the Hilbert space of Hilbert-Schmidt operators (see, e.g., [7, 11]).

The following result gives a characterization for a Schatten class membership of a $C_{a,b,c}$-symmetric weighted composition operator.

**Theorem 3.17.** Let $W_{\psi,\varphi}$ be a $C_{a,b,c}$-symmetric operator on $F^2$, induced by entire functions $\varphi, \psi$ (that is $\varphi(z) = Az + B$, with $|A| \leq 1$ and (3.10)–(3.11) hold). The following assertions are equivalent.

1. $W_{\psi,\varphi}$ is compact.
2. $W_{\psi,\varphi}$ belongs to the Schatten class $S_p$ for every $p > 0$. 


Proof. • (1) $\implies$ (2). Suppose $W_{\psi,\varphi}$ is compact. By Proposition 2.6 (ii), $|A| < 1$ and for each $z \in \mathbb{C}$,

$$M_z(\psi, \varphi) = |\psi(z)|^2e^{p|z|^2-|z|^2} \leq |C|^2e^{|B|^2}e^{-(1-|A|^2)|z|^2+2(|AB|+|D|)|z|}.$$  

From this, it follows that

$$(3.12) \quad \int M_z(\psi, \varphi)^{p/2} \, dV(z) < \infty, \forall p > 0,$$

which, by [2, Theorem 2.3], is equivalent to the fact that $W_{\psi,\varphi} \in S_p$ for every $p > 0$.

• (2) $\implies$ (1). Suppose $W_{\psi,\varphi}$ belongs to the Schatten class $S_p$ for every $p > 0$. Assume in a contrary that $|A| = 1$. By (3.11), $D + AB = 0$, which in turns, by Proposition 2.6 (i), implies that $M_z(\psi, \varphi) = |C|^2e^{|B|^2}$. But in this case, condition (3.12) does not hold: a contradiction. \hfill \Box

Our next task is to compute the spectrum of the symmetric $W_{\psi,\varphi}$. Remind, by Theorem 3.1, that the point spectrum of $W_{\psi,\varphi}$ is $\sigma_p(W_{\psi,\varphi}) = \{A^k\psi(\lambda) : k \in \mathbb{N}\}$.

**Theorem 3.18.** Let $W_{\psi,\varphi}$ be a $C_{\alpha,\beta,\gamma}$-symmetric operator on $\mathcal{F}^2$, induced by entire functions $\varphi, \psi$ (that is $\varphi(z) = Az + B$, with $|A| \leq 1$ and (3.10)–(3.11) hold). The following assertions are true.

1. If $|A| < 1$ then $\sigma(W_{\psi,\varphi}) = \sigma_p(W_{\psi,\varphi}) \cup \{0\}$.
2. If $|A| = 1$ and $A \neq 1$, then $\sigma(W_{\psi,\varphi}) = \{A^k\psi(\lambda) : k \in \mathbb{N}\}$.
3. If $A = 1$, then $\sigma(W_{\psi,\varphi}) = C e^{|B|^2/2}\mathbb{T}$, where $\mathbb{T}$ is the unit circle.

Here, $\lambda = \frac{B}{1-A}$, with $A \neq 1$.

**Proof.** First note that (1) is true, because due to $|A| < 1$, $W_{\psi,\varphi}$ is compact.

For (2) and (3), since $|A| = 1$, from (3.11) it follows that that $D = -AB$. Then,

$$W_{\psi,\varphi}f(z) = Ce^{-AB\overline{z}}f(Az + B), \quad f \in \mathcal{F}^2.$$  

Set $\phi(z) = e^{-AB\overline{z}-|B|^2/2}$. Then $W_{\psi,\varphi} = Ce^{|B|^2/2}W_{\phi,\varphi}$, and this gives

$$\sigma(W_{\psi,\varphi}) = Ce^{|B|^2/2}\sigma(W_{\phi,\varphi}).$$

By [12, Corollary 1.4], $W_{\phi,\varphi}$ is unitary and

$$\sigma(W_{\phi,\varphi}) = \begin{cases} \{A^k e^{\frac{|B|^2}{2} \frac{A^k+1}{A^k-1}} : k \in \mathbb{N}\}, & A \neq 1, \\ \mathbb{T}, & A = 1. \end{cases}$$

Consequently, if $A = 1$, then $\sigma(W_{\phi,\varphi}) = Ce^{|B|^2/2}\mathbb{T}$, while if $A \neq 1$, then

$$\sigma(W_{\phi,\varphi}) = Ce^{|B|^2/2}\left\{A^k e^{\frac{|B|^2}{2} \frac{A^k+1}{A^k-1}} : k \in \mathbb{N}\right\} = \{A^k\psi(\lambda) : k \in \mathbb{N}\},$$

because $e^{|B|^2/2}e^{\frac{|B|^2}{2} \frac{A^k+1}{A^k-1}} = e^{\frac{|B|^2}{2} \frac{A^k+1}{A^k-1}} = e^{\frac{2\pi i}{A-1}} = e^{-A\overline{\lambda}} = e^{D\lambda}$. \hfill \Box
We end up the present paper with normality problem. As mentioned in Introduction, every normal operator is complex symmetric with some conjugation. A natural question to ask is whether $C_{a,b,c}$-symmetric weighted composition operators are normal.

We have the following criterion for normality of such operators.

**Proposition 3.19.** Let $W_{\psi,\varphi}$ be a $C_{a,b,c}$-symmetric operator on $F^2$, induced by entire functions $\varphi, \psi$ (that is $\varphi(z) = Az + B$, with $|A| \leq 1$ and (3.10)–(3.11) hold). Then $W_{\psi,\varphi}$ is normal if and only if

\[
D = \begin{cases} 
\frac{1-A}{1-\overline{A}}, & A \neq 1, \\
-\overline{B}, & A = 1.
\end{cases}
\]

**Proof.** By definition, the operator $W_{\psi,\varphi}$ is normal if and only if

\[
W_{\psi,\varphi}W_{\psi,\varphi}^* f = W_{\psi,\varphi}^* W_{\psi,\varphi} f, \quad \text{for all } f \in F^2.
\]

Since $W_{\psi,\varphi}$ is bounded, and the span of the kernel functions is dense in $F^2$, the equality above occurs if and only if

\[
W_{\psi,\varphi}W_{\psi,\varphi}^* K_u(z) = W_{\psi,\varphi}^* W_{\psi,\varphi} K_u(z), \quad \text{for all } u, z \in \mathbb{C}.
\]

On one hand, by Lemma 2.3, we have

\[
W_{\psi,\varphi}W_{\psi,\varphi}^* K_u(z) = W_{\psi,\varphi} (\overline{\psi(u)} K_{\varphi(u)})(z)
= \overline{\psi(u)} \psi(z) e^{\overline{\varphi(u)} \varphi(z)}
= |C|^2 e^{B^2} e^{(\overline{D} + \overline{A}B)\overline{\pi} + |A|^2 \overline{\pi} z + (\overline{D} + \overline{A}B) z}.
\]

On the other hand, since

\[
W_{\psi,\varphi} K_u(z) = C e^{Dz} e^{\pi(Az + B)} = C e^{B\pi} e^{(D + A\pi)z} = C e^{B\pi} K_{\overline{D} + \overline{A}u}(z),
\]

by Lemma 2.3, we also have

\[
W_{\psi,\varphi}^* W_{\psi,\varphi} K_u(z) = C e^{B\overline{\psi}} e^{(\overline{D} + \overline{A}u) K_{\varphi(D + A\overline{u})}(z)}
= |C|^2 e^{|D|^2} e^{(B + A\overline{D})\overline{\pi} |A|^2 + (B + A\overline{D}) + \overline{B} z}
= |C|^2 e^{|D|^2} e^{(\overline{D}A + B)\overline{\pi} + |A|^2 \overline{\pi} z + (\overline{D} + \overline{B}) z}.
\]

So the equation (3.14) is reduced to

\[
\begin{cases} 
|B|^2 = |D|^2, \\
\overline{D} + \overline{A}B = \overline{DA} + B \\
D + AB = \overline{AD} + \overline{B}
\end{cases} \iff \begin{cases} 
|B| = |D| \\
AD + B = D + A\overline{B}.
\end{cases}
\]

The last system, due to condition (3.11), is equivalent to (3.13).
References


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