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POLYHEDRAL GAUSS SUMS, AND POLYTOPES WITH SYMMETRY

Romanos-Diogenes Malikiosis,^{*} Sinai Robins,[†] and Yichi Zhang,[‡]

ABSTRACT. We define certain natural finite sums of n 'th roots of unity, called $G_P(n)$, that are associated to each convex integer polytope P , and which generalize the classical 1-dimensional Gauss sum $G(n)$ defined over $\mathbb{Z}/n\mathbb{Z}$, to higher dimensional abelian groups and integer polytopes. We consider the finite Weyl group \mathcal{W} , generated by the reflections with respect to the coordinate hyperplanes, as well as all permutations of the coordinates; further, we let \mathcal{G} be the group generated by \mathcal{W} as well as all integer translations in \mathbb{Z}^d . We prove that if P multi-tiles \mathbb{R}^d under the action of \mathcal{G} , then we have the closed form $G_P(n) = \text{vol}(P)G(n)^d$. Conversely, we also prove that if P is a lattice tetrahedron in \mathbb{R}^3 , of volume $1/6$, such that $G_P(n) = \text{vol}(P)G(n)^d$, for $n \in \{1, 2, 3, 4\}$, then there is an element g in \mathcal{G} such that $g(P)$ is the fundamental tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$.

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1 Introduction

Our goal is to define certain finite sums of roots of unity, associated to a convex lattice polytope P , in order to help us determine whether P has certain symmetries and in fact whether P is a fundamental domain of a certain Weyl group. For 3-dimensional integer tetrahedra P , we discover that certain natural generalizations of the classical 1-dimensional Gauss sums, which we call polyhedral Gauss sums, collapse to a closed form over P if and only if P is a fundamental domain of a Weyl group.

Intuitively, we are projecting the structure of P onto the 2-dimensional complex plane, and seeing what a closed form of its associated Gauss sum of roots of unity in the complex plane tells us about the question of whether or not P is a fundamental domain for some group acting on P . It is much easier to handle 2-dimensional computations directly than

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d -dimensional geometric computations, and surprisingly we can discern the geometry of P in a very detailed way by sufficiently many of these computations with roots of unity. From a number-theoretic perspective, these computations generalize the classical 1-dimensional results of Gauss to d -dimensional integer polytopes.

Gauss sums over finite abelian groups have been studied by [7] and [2, 8], and they can be viewed as the study of Gauss sums over integer parallelepipeds, because when we quotient \mathbb{Z}^d by the discrete subgroup generated by the edge vectors of an integer parallelepiped, we get a finite abelian group. Here we extend the closed form results in the existing literature on Gauss sums over parallelepipeds, to more general Gauss sums over integer polytopes.

In one direction, if we assume that P is any d -dimensional integer polytope that tiles or multi-tiles Euclidean space by a special Weyl group, then we can show that its corresponding polyhedral Gauss sum always achieves a nice closed form, proportional to the volume of P . In the other direction, for $d = 3$, if we assume that the polyhedral Gauss sum of certain integer tetrahedra P achieve a closed form proportional to their volume, then we show P must be a fundamental domain for a certain Weyl group.

The Weyl group considered herein is the finite group generated by reflections with respect to the coordinate hyperplanes, as well as permutations of coordinates. We denote it by \mathcal{W} , and its cardinality is $2^d d!$. In this note, we will deal with sets that multi-tile the space under the action of \mathcal{G} , the group of operators generated by \mathcal{W} and all lattice translations. Clearly, $\mathcal{G} \cong \mathcal{W} \times \mathbb{Z}^d$.

In order to precisely define our generalized Gauss sums, we first need the notion of a solid angle at any point $x \in \mathbb{R}^d$, relative to a fixed polytope P . We let 1_P be the indicator function of P , and we define the solid angle at any point $x \in \mathbb{R}^d$ by

$$\omega_P(x) := \frac{\text{vol}(B(x, r) \cap P)}{\text{vol}(B(x, r))}, \quad (1.1)$$

for all sufficiently small values of $r > 0$, where $B(x, r)$ is the Euclidean ball centered at x with radius r . Some obvious but noteworthy properties of ω_P are the following: $\omega_P(x) = 1$ if $x \in \text{int } P$ and $\omega_P(x) = 0$ if $x \notin P$. For the non-trivial case that $x \in \partial P$ (the boundary of P), $\omega_P(x)$ is equal to the solid angle of the smallest cone containing P with apex at x .

Definition 1.1. The polyhedral Gauss sum over P is defined by

$$G_P(n) = \sum_{x \in \mathbb{Z}^d} \omega_{nP}(x) e\left(\frac{\|x\|^2}{n}\right),$$

for $n \in \mathbb{N}$, where nP denotes the dilation of P by n , and as usual, $e(x) := e^{2\pi i x}$.

The classical 1-dimensional Gauss sum, for example, is the case of the 1-dimensional polytope $P = [0, 1]$, and for this important case we define

$$G(n) = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} e\left(\frac{k^2}{n}\right).$$

Gauss discovered a closed form for this 1-dimensional Gauss sum [5], given by:

$$G(n) := \sum_{k=0}^{n-1} e\left(\frac{k^2}{n}\right) = \begin{cases} (1+i)\sqrt{n} & n \equiv 0 \pmod{4} \\ \sqrt{n} & n \equiv 1 \pmod{4} \\ 0 & n \equiv 2 \pmod{4} \\ i\sqrt{n} & n \equiv 3 \pmod{4} \end{cases} \quad (1.2)$$

It is natural to wonder what geometric properties an integer polytope must possess in order to achieve similar closed forms in higher dimensions. To this end we have the following result.

Theorem 1.2. *If P multi-tiles the space \mathbb{R}^d with the group \mathcal{G} , then*

$$G_P(n) = \text{vol}(P)G(n)^d.$$

In general, the converse question of whether such a closed form for a polyhedral Gauss sum over an integer polytope P implies that P must tile or multi-tile Euclidean space seems to be out of reach for general polytopes in dimension $d \geq 3$. However, we discovered a partial converse for $d = 3$ and in the case that P belongs to a class of integer simplices.

Theorem 1.3. *Let T be a lattice tetrahedron of volume $1/6$, such that $G_T(n) = \text{vol}(T)G(n)^3$ for $n \in \{1, 2, 3, 4\}$. Then there is an element g in the group \mathcal{G} such that $g(T)$ is the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(1, 1, 1)$.*

2 Preliminaries

The orbit of any point x under the action of \mathcal{G} is denoted by $\mathcal{G}(x)$, and the stabilizer of any x is denoted by \mathcal{G}_x (and similarly for \mathcal{W}). Obviously, \mathcal{G}_x is finite for all x , as x cannot remain invariant under any lattice translation, and almost all x have full orbit, i.e. $|\mathcal{G}_x| = 1$ except for a set of Lebesgue measure zero. Furthermore, the action of \mathcal{W} can be restricted to $[0, 1]^d = \mathbb{T}^d$, a fundamental domain for the action of the group \mathbb{Z}^d , acting by translations on \mathbb{R}^d ; usually we will treat elements of \mathbb{T}^d as elements of \mathbb{R}^d . Then, it is not hard to verify that $|\mathcal{G}_x| = |\mathcal{W}_x|$.

There are many choices of fundamental domains for \mathcal{G} , and a natural choice for such a fundamental domain is the simplex

$$T = \left\{ (x_1, \dots, x_d) \in \mathbb{R}^d \mid 0 \leq x_1 \leq \dots \leq x_d \leq 1/2 \right\},$$

which is also a fundamental domain of \mathcal{W} acting on \mathbb{T}^d .

Definition 2.1. We say that P multi-tiles \mathbb{R}^d , with multiplicity m , if $\sum_{g \in \mathcal{G}} 1_P(gx) = m$ for almost all $x \in \mathbb{R}^d$.

Equivalently, we may also say that P multi-tiles with multiplicity m if $|\mathcal{G}(x) \cap P| = m$, for almost all x . It is clear from definition 2.1 that this m must be a positive integer.

Next, define the functions f_P and g_P on T as follows:

$$f_P(x) = \sum_{g \in \mathcal{G}} \omega_P(gx), \quad g_P(x) = \sum_{y \in \mathcal{G}(x)} \omega_P(y).$$

Obviously, $f_P = g_P$ almost everywhere; in particular

$$g_P(x) = \frac{1}{|\mathcal{W}_x|} f_P(x),$$

so they differ only on the boundary of T .

Proposition 2.2. *If P multi-tiles the space, then f_P is constant, equal to $|\mathcal{W}| \operatorname{vol}(P)$.*

Proof. By definition, $|\mathcal{G}(x) \cap P| = m$ for almost all x and some positive integer m . Then, for all $x \in T$ we have

$$\begin{aligned} f_P(x) &= \sum_{g \in \mathcal{G}} \omega_P(gx) = \sum_{g \in \mathcal{G}} \lim_{r \rightarrow 0} \frac{1}{\operatorname{vol}(B(gx, r))} \int_{B(gx, r)} 1_P(y) dy \\ &= \lim_{r \rightarrow 0} \frac{1}{\operatorname{vol}(B(x, r))} \sum_{g \in \mathcal{G}} \int_{B(x, r)} 1_P(gy) dy \\ &= \lim_{r \rightarrow 0} \frac{1}{\operatorname{vol}(B(x, r))} \int_{B(x, r)} \sum_{g \in \mathcal{G}} 1_P(gy) dy \\ &= m. \end{aligned}$$

The above sum commutes with the limit and the integral, because it is finite. For the second part,

$$\begin{aligned} \frac{m}{|\mathcal{W}|} &= \int_T f_P(x) dx = \int_T \sum_{g \in \mathcal{G}} \omega_P(gx) dx \\ &= \sum_{g \in \mathcal{G}} \int_T \omega_P(gx) dx \\ &= \sum_{g \in \mathcal{G}} \int_T 1_P(gx) dx \\ &= \sum_{g \in \mathcal{G}} \operatorname{vol}(g(T) \cap P) \\ &= \operatorname{vol}(P), \end{aligned}$$

where again, interchanging summation and integration is justified by the fact that the sum is finite. \square

3 Gauss sums

The Weyl group satisfies the following properties:

- it preserves both the Lebesgue and discrete volumes; in particular, it consists of invertible linear transformations that preserve the lattice \mathbb{Z}^d .
- it preserves norms, so it also preserves Gauss sums.

It easily follows that the full group \mathcal{G} also preserves Lebesgue and discrete measures, as well as Gauss sums.

Lemma 3.1. *With notation as above, we have*

$$G_P(n) = \sum_{x \in T \cap \frac{1}{n}\mathbb{Z}^d} g_P(x) e(n\|x\|^2).$$

Proof. Replacing x by nx in the definition of a Gauss sum, we get

$$\begin{aligned} G_P(n) &= \sum_{x \in \frac{1}{n}\mathbb{Z}^d} \omega_P(x) e(n\|x\|^2) \\ &= \sum_{x \in T \cap \frac{1}{n}\mathbb{Z}^d} \sum_{y \in \mathcal{G}(x)} \omega_P(y) e(n\|x\|^2) \\ &= \sum_{x \in T \cap \frac{1}{n}\mathbb{Z}^d} g_P(x) e(n\|x\|^2), \end{aligned}$$

since $n\|gx\|^2 \equiv n\|x\|^2 \pmod{1}$; indeed, if $gx = wx + \lambda$, where $w \in \mathcal{W}$, $\lambda \in \mathbb{Z}^d$, then

$$n\|wx + \lambda\|^2 = n\|wx\|^2 + n\|\lambda\|^2 + 2n\langle wx, \lambda \rangle \equiv n\|x\|^2 + 2\langle w(nx), \lambda \rangle \equiv n\|x\|^2 \pmod{1},$$

for all $x \in \frac{1}{n}\mathbb{Z}^d$. □

Proof. (of Theorem 1.2) By Proposition 2.2, the function f_P is constant and equal to $|\mathcal{W}| \text{vol}(P)$. So,

$$\begin{aligned} \text{vol}(P)G(n)^d &= \text{vol}(P) \sum_{x \in \frac{1}{n}\mathbb{Z}^d / \mathbb{Z}^d} e(n\|x\|^2) \\ &= \text{vol}(P) \sum_{x \in T \cap \frac{1}{n}\mathbb{Z}^d} \frac{1}{|\mathcal{W}_x|} \sum_{g \in \mathcal{W}} e(n\|x\|^2) \\ &= \sum_{x \in T \cap \frac{1}{n}\mathbb{Z}^d} \frac{|\mathcal{W}| \text{vol}(P)}{|\mathcal{G}_x|} e(n\|x\|^2) \\ &= \sum_{x \in T \cap \frac{1}{n}\mathbb{Z}^d} g_P(x) e(n\|x\|^2) \\ &= G_P(n), \end{aligned}$$

by Lemma 3.1 and the fact that

$$g_P(x) = \frac{f_P(x)}{|\mathcal{G}_x|} = \frac{|\mathcal{W}| \operatorname{vol}(P)}{|\mathcal{G}_x|}. \quad \square$$

Question. Is the converse true? That is, if $G_P(n) = \operatorname{vol}(P)G(n)^d$ for all n , then is it true that P multi-tiles the space by \mathcal{G} ?

The converse is indeed true for dimensions $d = 1, 2$. We have nothing to prove when $d = 1$, as any convex lattice polytope in \mathbb{R} has the form $[a, b]$, where $a, b \in \mathbb{Z}$, and hence multi-tiles \mathbb{R} $b - a$ times.

The case $d = 2$ is quite easy, too. As P can be triangulated, it suffices to prove the converse for lattice triangles. But any lattice triangle multi-tiles the plane under \mathcal{G} ; indeed, suppose that $T = \operatorname{conv}\{0, v_1, v_2\}$, where $v_1, v_2 \in \mathbb{Z}^2$ are linearly independent. The union $T \cup (-T + v_1 + v_2)$ is a parallelogram, in particular the closure of a fundamental domain of the sublattice of \mathbb{Z}^2 generated by v_1 and v_2 , which shows that T multi-tiles the plane, therefore any lattice polygon satisfies the Gauss sum formula and there is nothing else to prove.

4 Solid and dihedral angles of a tetrahedron

Before proceeding to the first 3-dimensional case, it would be useful to revise a couple of things related to the geometry of the tetrahedron, as well as the basic tools. Consider the tetrahedron T in \mathbb{R}^3 with vertices v_0, v_1, v_2 , and v_3 . The solid angle at vertex v_i is denoted by ω_i and the dihedral angle at the edge connecting v_i and v_j is denoted by ω_{ij} . Here, and throughout the paper, we normalize everything by considering the angles corresponding to both S^1 and S^2 to be equal to 1 (not 2π and 4π , respectively). Under this normalization, we have the *Gram relations* [3, 4], which are equalities connecting the solid with the dihedral angles of a tetrahedron:

$$\omega_i = \frac{1}{2} \sum_{j \neq i} \omega_{ij} - \frac{1}{4}, \quad (4.1)$$

which yield

$$1 + \sum_{i=0}^3 \omega_i = \sum_{0 \leq i < j \leq 3} \omega_{ij}.$$

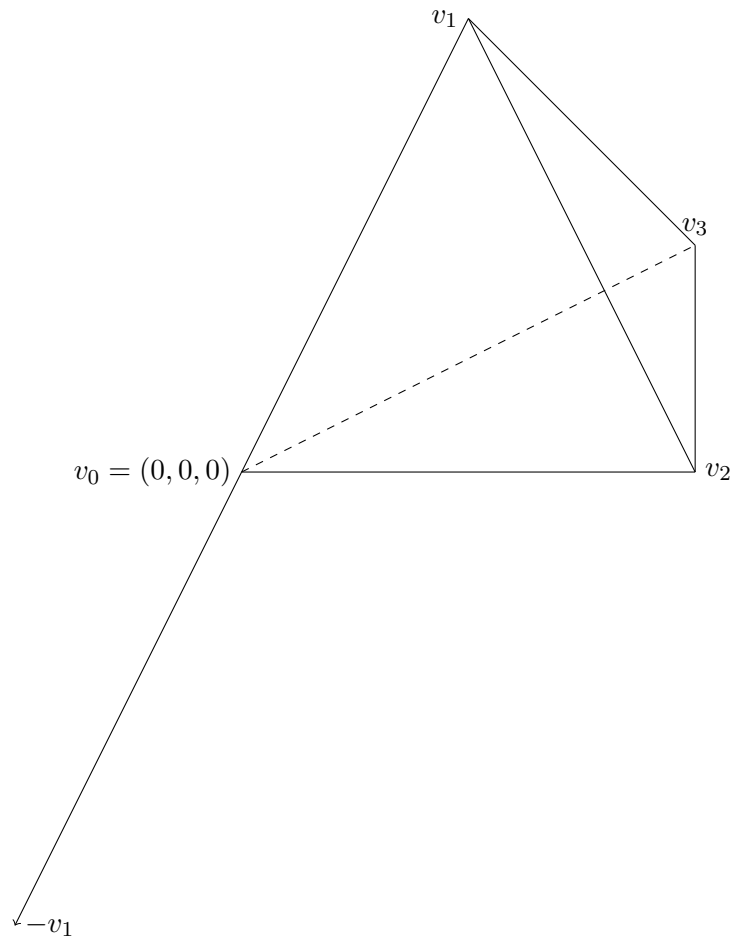
We also denote by $n_{ij} = \|v_i - v_j\|^2$ the squared lengths of the edges. Now let $\{0, 1, 2, 3\} = \{i, j, k, l\}$. Oosterom and Strackee [6] had proved the following formula for the solid angle:

$$\cot 2\pi\omega_i = \frac{A}{B}, \quad (4.2)$$

where

$$A = \sqrt{n_{ij}n_{ik}n_{il}} + \langle v_k - v_i, v_l - v_i \rangle \sqrt{n_{ij}} + \langle v_l - v_i, v_j - v_i \rangle \sqrt{n_{ik}} + \langle v_j - v_i, v_k - v_i \rangle \sqrt{n_{il}},$$

and $B = |\det(v_j - v_i, v_k - v_i, v_l - v_i)|$. Next, we will focus on the *external solid angles* of a tetrahedron. Unlike the 2-dimensional case, there isn't a unique external angle, but three; every external solid angle is determined by a vertex and an adjacent edge. So, the external solid angle at v_i with respect to the edge $v_j - v_i$ is the solid angle spanned by the vectors $v_i - v_j, v_k - v_i, v_l - v_i$. The figure below shows us the external solid angle at v_0 with respect to the edge $v_1 - v_0$ (for convenience we put $v_0 = (0, 0, 0)$):



We denote the external solid angle at v_i along $v_j - v_i$ by φ_{ij} . A basic relation is

$$\omega_{ij} = \omega_i + \varphi_{ij}. \quad (4.3)$$

The solid angle φ_{ij} is defined by the vectors $v_i - v_j, v_k - v_i, v_l - v_i$, and hence

$$\cot 2\pi\varphi_{ij} = \frac{C}{D}, \quad (4.4)$$

where

$$C = \sqrt{n_{ij}n_{ik}n_{il}} + \langle v_k - v_i, v_l - v_i \rangle \sqrt{n_{ij}} - \langle v_l - v_i, v_j - v_i \rangle \sqrt{n_{ik}} - \langle v_j - v_i, v_k - v_i \rangle \sqrt{n_{il}},$$

and $D = |\det(v_j - v_i, v_k - v_i, v_l - v_i)|$. Next, we will make the following assumptions:

- (a) $v_0 = (0, 0, 0)$.
 (b) $v_i \in \mathbb{Z}^3$, for all i .
 (c) T has minimal volume, i. e. $\text{vol}(T) = 1/6$, or equivalently, v_1, v_2, v_3 is a basis of \mathbb{Z}^3 .

Then (4.2) and (4.4) become

$$\begin{aligned} \cot 2\pi\omega_i &= \sqrt{n_{ij}n_{ik}n_{il}} + \langle v_k - v_i, v_l - v_i \rangle \sqrt{n_{ij}} \\ &\quad + \langle v_l - v_i, v_j - v_i \rangle \sqrt{n_{ik}} + \langle v_j - v_i, v_k - v_i \rangle \sqrt{n_{il}}, \end{aligned} \quad (4.5)$$

and

$$\begin{aligned} \cot 2\pi\varphi_{ij} &= \sqrt{n_{ij}n_{ik}n_{il}} + \langle v_k - v_i, v_l - v_i \rangle \sqrt{n_{ij}} \\ &\quad - \langle v_l - v_i, v_j - v_i \rangle \sqrt{n_{ik}} - \langle v_j - v_i, v_k - v_i \rangle \sqrt{n_{il}}, \end{aligned} \quad (4.6)$$

respectively. Apparently, $\cot 2\pi\omega_i$ and $\cot 2\pi\varphi_{ij}$ are both algebraic integers, belonging both to the multiquadratic field $\mathbb{Q}(\sqrt{n_{ij}}, \sqrt{n_{ik}}, \sqrt{n_{il}})$, which we denote by K_i . Between these two numbers there is a simple algebraic relation.

Proposition 4.1. *Suppose that $\sqrt{n_{ij}} \notin \mathbb{Q}(\sqrt{n_{ik}}, \sqrt{n_{il}})$ and τ is the unique nontrivial $\mathbb{Q}(\sqrt{n_{ik}}, \sqrt{n_{il}})$ -automorphism of K_i (i.e. it fixes $\mathbb{Q}(\sqrt{n_{ik}}, \sqrt{n_{il}})$, but $\tau(\sqrt{n_{ij}}) = -\sqrt{n_{ij}}$), then $\cot 2\pi\varphi_{ij} = -\tau(\cot 2\pi\omega_i)$ and $\sqrt{n_{ij}} \cot 2\pi\omega_{ij} \in \mathbb{Q}(\sqrt{n_{ik}}, \sqrt{n_{il}})$.*

Proof. The first conclusion is an immediate consequence of (4.5) and (4.6). The second follows from (4.3) and the formula for the cotangent of a sum:

$$\cot 2\pi\omega_{ij} = \frac{\cot 2\pi\omega_i \cot 2\pi\varphi_{ij} - 1}{\cot 2\pi\omega_i + \cot 2\pi\varphi_{ij}} = \frac{-N(\cot 2\pi\omega_i) - 1}{2\sqrt{n_{ij}}(\sqrt{n_{ik}n_{il}} + \langle v_k - v_i, v_l - v_i \rangle)},$$

hence

$$\sqrt{n_{ij}} \cot 2\pi\omega_{ij} = \frac{-N(\cot 2\pi\omega_i) - 1}{2(\sqrt{n_{ik}n_{il}} + \langle v_k - v_i, v_l - v_i \rangle)} \in \mathbb{Q}(\sqrt{n_{ik}}, \sqrt{n_{il}}),$$

where N is the number theoretic norm of the quadratic extension $K_i/\mathbb{Q}(\sqrt{n_{ik}}, \sqrt{n_{il}})$. \square

5 A converse for 3-dimensional tetrahedra of volume 1/6

Assume that

$$G_T(n) = \text{vol}(T)G(n)^d$$

holds for all n , for a convex lattice polytope, T . Any convex polytope is a union of simplices, so it is natural to check whether the converse holds for simplices first. This is the first nontrivial case as there are lattice tetrahedra that do not satisfy the Gauss sum formula, such as $\text{conv}\{0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where \mathbf{e}_i are the vectors of the standard basis of \mathbb{R}^3 .

So, we assume that $T = \text{conv}\{v_0 = 0, v_1, v_2, v_3\}$ with the additional condition that T has minimal volume. This means that $\text{vol}(T) = 1/6$ and v_1, v_2, v_3 is a basis of \mathbb{Z}^3 . Let ω_i be the solid angle of T at the vertex v_i and ω_{ij} be the dihedral angle at the edge $v_j - v_i$.

Now let's consider the Gauss sum relations, which for T take the form

$$\sum_{x \in \mathbb{Z}^3} \omega_{nT}(x) e\left(\frac{\|x\|^2}{n}\right) = \frac{G(n)^3}{6}. \quad (5.1)$$

The only lattice points in T are v_i for $0 \leq i \leq 3$ and their contribution to the Gauss sum is precisely ω_i for each i , so for $n = 1$, (5.1) becomes

$$\sum_{i=0}^3 \omega_i = \frac{1}{6}. \quad (5.2)$$

In general, the lattice points of nT that lie on the vertices or the edges have the form $av_i + bv_j$ for all $i \neq j$ where $a + b = n$ with $a, b \geq 0$ integers. So, the contribution of these points to $G_T(n)$ is

$$\begin{aligned} & \sum_{0 \leq i \leq 3} \omega_{nT}(nv_i) e\left(\frac{\|nv_i\|^2}{n}\right) + \sum_{0 \leq i < j \leq 3} \sum_{\substack{a+b=n \\ a,b > 0}} \omega_{nT}(av_i + bv_j) e\left(\frac{\|av_i + bv_j\|^2}{n}\right) \\ &= \sum_{0 \leq i \leq 3} \omega_i + \sum_{0 \leq i < j \leq 3} \sum_{a=1}^{n-1} \omega_{ij} e\left(\frac{\|nv_j + a(v_i - v_j)\|^2}{n}\right) \\ &= \sum_{0 \leq i \leq 3} \omega_i + \sum_{0 \leq i < j \leq 3} \sum_{a=1}^{n-1} \omega_{ij} e\left(\frac{n^2\|v_j\|^2 + 2n\langle v_j, a(v_i - v_j) \rangle + a^2\|v_i - v_j\|^2}{n}\right) \\ &= \sum_{0 \leq i \leq 3} \omega_i + \sum_{0 \leq i < j \leq 3} \sum_{a=1}^{n-1} \omega_{ij} e\left(\frac{a^2\|v_i - v_j\|^2}{n}\right) \\ &= \sum_{0 \leq i \leq 3} \omega_i + \sum_{0 \leq i < j \leq 3} \omega_{ij} [G(n_{ij}, n) - 1] \\ &= -1 + \sum_{0 \leq i < j \leq 3} \omega_{ij} G(n_{ij}, n), \end{aligned}$$

using (4.1), where we put $n_{ij} = \|v_j - v_i\|^2$, the squared lengths of the edges, and $G(a, b)$ is the quadratic Gauss sum given by

$$G(a, b) = \sum_{n=0}^{b-1} e\left(\frac{an^2}{b}\right).$$

The following formula by Gauss [5] for $\gcd(a, b) = 1$ will be very useful:

$$G(a, b) = \begin{cases} 0, & b \equiv 2 \pmod{4} \\ \varepsilon_b \sqrt{b} \left(\frac{a}{b}\right), & b \text{ odd} \\ (1+i)\varepsilon_a^{-1} \sqrt{b} \left(\frac{b}{a}\right), & 4|b \end{cases} \quad (5.3)$$

where

$$\varepsilon_m = \begin{cases} 1, & m \equiv 1 \pmod{4} \\ i, & m \equiv 3 \pmod{4} \end{cases}$$

and $\left(\frac{a}{b}\right)$ is the Jacobi symbol. For $\gcd(a, b) = d > 1$ we simply have $G(a, b) = dG(a/d, b/d)$. If x is any other lattice point in nT , then we have $\omega_{nT}(x) = 1/2$ when x is in the relative interior of one facet, and $\omega_{nT}(x) = 1$ when $x \in \text{int}(T)$. This yields:

Proposition 5.1. *Let T be a lattice tetrahedron with vertices v_i , $0 \leq i \leq 3$. Let ω_{ij} be the dihedral angle at the edge $v_i - v_j$ and let $n_{ij} = \|v_i - v_j\|^2$. Then*

$$G_T(n) = -1 + \sum_{0 \leq i < j \leq 3} \omega_{ij} G(n_{ij}, n) + \kappa(n),$$

where $\kappa(n) \in \mathbb{Q}(e(1/n))$.

Remark. The above holds for all lattice tetrahedra, not just the ones with minimal volume. However, if $\text{vol}(T) = 1/6$, then the only lattice points of $2T$ are the vertices and the midpoints of the edges, therefore $\kappa(1) = \kappa(2) = 0$. The explicit formula for $\kappa(n)$ is

$$\begin{aligned} \kappa(n) = \frac{1}{2} \sum_{0 \leq i < j < k \leq 3} \sum_{\substack{a+b+c=n \\ a,b,c > 0}} e\left(\frac{\|av_i + bv_j + cv_k\|^2}{n}\right) \\ + \sum_{\substack{a+b+c+d=n \\ a,b,c,d > 0}} e\left(\frac{\|av_0 + bv_1 + cv_2 + dv_3\|^2}{n}\right). \end{aligned}$$

In particular,

$$\kappa(3) = \frac{1}{2} \sum_{0 \leq i < j < k \leq 3} e\left(\frac{\|v_i + v_j + v_k\|^2}{3}\right)$$

and

$$\begin{aligned} \kappa(4) &= \frac{1}{2} \sum_{\substack{0 \leq i < j \leq 3 \\ k \neq i, j}} e\left(\frac{\|v_i + v_j + 2v_k\|^2}{4}\right) + e\left(\frac{\|v_0 + v_1 + v_2 + v_3\|^2}{4}\right) \\ &= \sum_{0 \leq i < j \leq 3} e\left(\frac{\|v_i + v_j\|^2}{4}\right) + e\left(\frac{\|v_0 + v_1 + v_2 + v_3\|^2}{4}\right) \end{aligned}$$

Next, we will investigate the parity of n_{ij} . Since $\text{vol}(T) = 1/6$, any three vectors corresponding to edges at a common vertex of T form a basis of \mathbb{Z}^3 . Furthermore, if $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ then $\|x\|^2 \equiv x_1 + x_2 + x_3 \pmod{2}$, so if $\|u\|^2$ and $\|v\|^2$ have the same parity, then $\|u - v\|^2$ is even. This means that at any face of T , either all or exactly one edge has even squared length. Moreover, not all three squared lengths of edges with a common vertex can be even, otherwise these vectors would span a proper even sublattice of \mathbb{Z}^3 . Thus, we have one of the following two situations for the edges with even squared lengths of T : either they form a triangle, or they are opposite, having no vertex in common. By an appropriate lattice translation of T , we may assume that $v_0 = 0$, $n_{01} = \|v_1\|^2$ and $n_{03} = \|v_3\|^2$ are odd.

$n_{02} = \|v_2\|^2$ is odd. Then n_{ij} for $1 \leq i < j \leq 3$ are even. Then by Proposition 5.1 and (5.3) we get

$$G_T(2) = -1 + 2(\omega_{12} + \omega_{13} + \omega_{23}).$$

From (4.1) and (5.2) we get

$$\sum_{0 \leq i < j \leq 3} \omega_{ij} = \frac{7}{6}, \quad (5.4)$$

and since $G_T(2) = 0$, as T satisfies the Gauss sum formula for all n , we get

$$\omega_{12} + \omega_{13} + \omega_{23} = \frac{1}{2},$$

and

$$\omega_{01} + \omega_{02} + \omega_{03} = \frac{2}{3}, \quad (5.5)$$

hence

$$\omega_0 = \frac{1}{12}, \quad (5.6)$$

by virtue of (4.1).

Next, we wish to examine the possible values of $n_{ij} \pmod 4$. For the even n_{ij} , it is not hard to see that $n_{ij} \equiv 2 \pmod 4$, because the edges $v_i - v_j$ correspond to primitive vectors in \mathbb{Z}^3 ; if $4|x_1^2 + x_2^2 + x_3^2$ then all x_i must be even. The residue $n_{0i} \pmod 4$ depends on the parity of the coordinates of v_i . First we notice that no two of the n_{0i} can be $3 \pmod 4$; if, for example, $n_{01} \equiv n_{02} \equiv 3 \pmod 4$, then all coordinates of v_1 and v_2 must be odd, which yields $\frac{1}{2}(v_1 + v_2) \in \mathbb{Z}^3$, a contradiction, because v_1, v_2, v_3 is a basis of \mathbb{Z}^3 . We have thus proven:

Proposition 5.2. *Let v_1, v_2, v_3 be a basis of \mathbb{Z}^3 such that all $\|v_i\|^2$ are odd. Then at most one of the $\|v_i\|^2$ is $3 \pmod 4$.*

We will show that if $n_{0i} \equiv 1 \pmod 4$ for all i , then T cannot satisfy the Gauss sum relation for $n = 4$. In this case, each v_i has exactly one odd coordinate and two even. Since $\frac{1}{2}(v_i + v_j) \notin \mathbb{Z}^3$, different coordinates in the vectors v_i are odd (or in simple terms, the entries mod 2 of the matrix whose columns are v_i is equal to the identity matrix). This shows that the coordinates of $v_1 + v_2 + v_3$ are all odd. Therefore,

$$\kappa(4) = \sum_{0 \leq i < j \leq 3} e\left(\frac{n_{ij}}{4}\right) + e\left(\frac{3}{4}\right) = -3 + 2i.$$

Since $n_{ij} \equiv 2 \pmod 4$ for $1 \leq i < j \leq 3$, we have $G(n_{ij}, 4) = 2G(n_{ij}/2, 2) = 0$ by (5.3) and 5.1 we get

$$G_T(4) = -1 + \sum_{i=1}^3 \omega_{0i} G(4) + \kappa(4) = -1 + \frac{2}{3} \cdot 2(1+i) - 3 + 2i = -\frac{8}{3} + \frac{10}{3}i,$$

while by (5.3) again we have

$$\text{vol}(T)G(4)^3 = \frac{1}{6}[2(1+i)]^3 = \frac{8}{3}(-1+i) \neq G_T(4). \quad (5.7)$$

Hence, we may assume that $n_{03} \equiv 3 \pmod{4}$, while $n_{01} \equiv n_{02} \equiv 1 \pmod{4}$. It is not hard to see that $\|v_1 + v_2 + v_3\|^2 \equiv 1 \pmod{4}$. Therefore,

$$\kappa(4) = \sum_{0 \leq i < j \leq 3} e\left(\frac{n_{ij}}{4}\right) + e\left(\frac{1}{4}\right) = -3 + 2i$$

and

$$\begin{aligned} G_T(4) &= -1 + (\omega_{01} + \omega_{02})G(4) + \omega_{03}G(3, 4) - 3 + 2i \\ &= [2(\omega_{01} + \omega_{02} + \omega_{03}) - 4] + [2(\omega_{01} + \omega_{02} - \omega_{03}) + 2]i \\ &= -\frac{8}{3} + \left[\frac{10}{3} - 4\omega_{03}\right]i, \end{aligned}$$

by (5.5), therefore $\omega_{03} = 1/6$ since $\text{vol}(T)G(4)^3 = \frac{8}{3}(-1 + i)$. We also get $\omega_{01} + \omega_{02} = 1/2$ from (5.5).

Applying (4.5) for $i = 0$ we get

$$\cot 2\pi\omega_0 = \sqrt{n_{01}n_{02}n_{03}} + \langle v_1, v_2 \rangle \sqrt{n_{03}} + \langle v_2, v_3 \rangle \sqrt{n_{01}} + \langle v_3, v_1 \rangle \sqrt{n_{02}}, \quad (5.8)$$

so by (5.6) we get

$$\sqrt{3} = \sqrt{n_{01}n_{02}n_{03}} + \langle v_1, v_2 \rangle \sqrt{n_{03}} + \langle v_2, v_3 \rangle \sqrt{n_{01}} + \langle v_3, v_1 \rangle \sqrt{n_{02}}. \quad (5.9)$$

Let $K = \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{02}})$. Since $n_{01} \equiv n_{02} \equiv 1 \pmod{4}$, we have $\sqrt{q} \notin K$ for any $q \equiv 3 \pmod{4}$. This is trivial if $K = \mathbb{Q}$, as q cannot be a square. If $\sqrt{q} \in K \neq \mathbb{Q}$, then $\mathbb{Q}(\sqrt{q})$ is a quadratic subfield of K . The quadratic subfields are exactly $\mathbb{Q}(\sqrt{n_{01}})$, $\mathbb{Q}(\sqrt{n_{02}})$, and $\mathbb{Q}(\sqrt{n_{01}n_{02}})$ (they coincide if $[K : \mathbb{Q}] = 2$), which yields that q has the same square-free part with one of n_{01} , n_{02} , $n_{01}n_{02}$, but this is impossible as $q \equiv 3 \pmod{4}$ while $n_{01} \equiv n_{02} \equiv n_{01}n_{02} \equiv 1 \pmod{4}$. Therefore, $[K(\sqrt{n_{03}}) : K] = 2$, and $1, \sqrt{n_{03}}$ is a K -basis of $K(\sqrt{n_{03}})$. As $\sqrt{3} \in K(\sqrt{n_{03}}) \setminus K$ by (5.9), we get $\sqrt{3} = a + b\sqrt{n_{03}}$ for some $a, b \in K$ with $b \neq 0$. Squaring both sides we obtain $3 = a^2 + b^2n_{03} + 2ab\sqrt{n_{03}}$, so we must have $a = 0$. Again, by (5.9) we get

$$\langle v_2, v_3 \rangle \sqrt{n_{01}} + \langle v_3, v_1 \rangle \sqrt{n_{02}} = 0.$$

If n_{01} and n_{02} do not have the same square-free part, then $\sqrt{n_{01}}$ and $\sqrt{n_{02}}$ are linearly independent over \mathbb{Q} , so we must have

$$\langle v_2, v_3 \rangle = \langle v_3, v_1 \rangle = 0,$$

a contradiction, since

$$2\langle v_2, v_3 \rangle = n_{02} + n_{03} - n_{23} \equiv 2 \pmod{4}.$$

So n_{01} and n_{02} have the same square-free part, hence $\sqrt{n_{01}n_{02}} \in \mathbb{Z}$, and by (5.9) we obtain

$$\sqrt{3} = (\sqrt{n_{01}n_{02}} + \langle v_1, v_2 \rangle) \sqrt{n_{03}}.$$

Since $\sqrt{n_{03}} \geq 3$ and $\sqrt{n_{01}n_{02}} + \langle v_1, v_2 \rangle \geq 1$ (as an integer), we must have equality in both cases, which yields $n_{03} = 3$.

Proposition 5.3. *With notation as above, let $n_{03} = 3$, and assume that $\omega_{03} = 1/6$. Then, up to an appropriate action of \mathcal{W} , we may assume that*

$$v_1 = (k + 1, k, k), \quad v_2 = (l, l, l - 1), \quad v_3 = (1, 1, 1).$$

Proof. Applying an appropriate reflection from the group \mathcal{W} , we may assume without loss of generality that

$$v_3 = (1, 1, 1).$$

Now consider the hyperplane $H = v_3^\perp$, and let Λ be the orthogonal projection of \mathbb{Z}^3 onto H . It is not hard to see that Λ is isomorphic to the hexagonal lattice, and the vectors of smallest length are $\pi(\pm \mathbf{e}_i)$, where $\pi : \mathbb{R}^3 \rightarrow H$ is the orthogonal projection. By hypothesis, $\pi(v_1)$ and $\pi(v_2)$ is a basis of Λ and the angle between these two vectors is $\pi/3$ by $\omega_{03} = 1/6$, therefore they must be of smallest length. Permutations of coordinates of \mathbb{R}^3 correspond to rotations of H by multiples of $\pi/3$ or reflections along $\pi(\mathbf{e}_i)$, so without loss of generality we may assume that $\pi(v_1) = \pi(\mathbf{e}_1)$ and $\pi(v_2) = \pi(-\mathbf{e}_3)$, hence

$$v_1 = (k + 1, k, k), \quad v_2 = (l, l, l - 1). \quad \square$$

From Proposition 5.3 and the fact that n_{01} and n_{02} are odd, follows that k and l are even in our case. Since $\omega_{01} + \omega_{02} = 1/2$, we will have

$$\cos 2\pi\omega_{01} + \cos 2\pi\omega_{02} = 0.$$

But

$$\begin{aligned} \cos 2\pi\omega_{01} &= \frac{\langle v_1 \times v_2, v_1 \times v_3 \rangle}{\|v_1 \times v_2\| \cdot \|v_1 \times v_3\|} \\ &= \frac{\langle (-k, k - l + 1, l), (0, -1, 1) \rangle}{\sqrt{2}\sqrt{k^2 + l^2 + (k - l + 1)^2}} = \frac{-k + 2l - 1}{\sqrt{2k^2 + 2l^2 + 2(k - l + 1)^2}}, \end{aligned}$$

while

$$\begin{aligned} \cos 2\pi\omega_{02} &= \frac{\langle v_2 \times v_1, v_2 \times v_3 \rangle}{\|v_2 \times v_1\| \cdot \|v_2 \times v_3\|} \\ &= \frac{\langle (k, -k + l - 1, -l), (1, -1, 0) \rangle}{\sqrt{2}\sqrt{k^2 + l^2 + (k - l + 1)^2}} = \frac{2k - l + 1}{\sqrt{2k^2 + 2l^2 + 2(k - l + 1)^2}}, \end{aligned}$$

therefore we must have $k = l$, hence

$$v_2 = (k, k, k - 1).$$

As we've seen above, $n_{01} = 3k^2 + 2k + 1$ and $n_{02} = 3k^2 - 2k + 1$ must have the same square free part, say d . But this d is odd and also a common divisor of n_{01} and n_{02} , therefore $d|n_{01} - n_{02} = 4k$, so $d|k$. Since $\gcd(k, n_{01}) = 1$, we must have $d = 1$, so n_{01} and n_{02} are both perfect (odd) squares. Let $m, n \geq 0$ be such that

$$\begin{aligned} 3k^2 + 2k + 1 &= (2m + 1)^2 \\ 3k^2 - 2k + 1 &= (2n + 1)^2, \end{aligned} \tag{5.10}$$

which yields

$$k = (m - n)(m + n + 1).$$

Adding the equations (5.10) we get

$$3k^2 = 2(m^2 + n^2 + m + n).$$

If $m \neq n$, we obtain

$$3k^2 \geq 3(m + n + 1)^2 = 3(m^2 + n^2 + 1 + 2mn + 2m + 2n) > 2(m^2 + n^2 + m + n),$$

so we must have $m = n$ and $k = 0$. Therefore,

$$v_1 = (1, 0, 0), \quad v_2 = (0, 0, -1).$$

Next, we will verify that the Gauss sum relation for $n = 3$ fails. We have

$$n_{01} = n_{02} = 1, n_{03} = 2, n_{12} = n_{13} = 2, n_{23} = 6,$$

and

$$\text{vol}(T)(G(3))^3 = -i\frac{\sqrt{3}}{2},$$

but

$$\begin{aligned} G_T(3) &= -1 + \sum_{0 \leq i < j \leq 3} \omega_{ij} G(n_{ij}, 3) + \kappa(3) \\ &= -1 + (\omega_{01} + \omega_{02})G(3) + (\omega_{12} + \omega_{13})G(2, 3) + 3(\omega_{03} + \omega_{23}) + \frac{1}{2}[1 + 3e(2/3)] \\ &= -1 + i\frac{\sqrt{3}}{2} - (1/2 - \omega_{23})i\sqrt{3} + 3(1/6 + \omega_{23}) + \frac{1}{2} \left[-2 - i\frac{3\sqrt{3}}{2} \right] \\ &= 3(\omega_{23} - 1/2) + (\omega_{23} - 5/4)i\sqrt{3}. \end{aligned}$$

Taking real and imaginary parts, if $G_T(3) = \text{vol}(T)(G(3))^3$ then we should simultaneously have $\omega_{23} = 1/2$ and $\omega_{23} = 3/4$, an absurdity. We thus conclude that:

Proposition 5.4. *Let $T = \text{conv}(0, v_1, v_2, v_3)$ with v_1, v_2, v_3 basis of \mathbb{Z}^3 , such that all $\|v_i\|^2$ are odd. Then T cannot satisfy the Gauss sum relations. In particular, $G_T(n) = \text{vol}(T)(G(n))^3$ fails for some $n \leq 4$.*

$n_{02} = \|v_2\|^2$ is even Then, $n_{02} \equiv n_{13} \equiv 2 \pmod{4}$, and all other n_{ij} are odd. As we have already seen, two adjacent edges cannot have both squared length $3 \pmod{4}$, so there are at most two of them in T . So, we may assume that $v_1 \equiv (1, 0, 0) \pmod{2\mathbb{Z}^3}$. If $n_{03} \equiv 1 \pmod{4}$, then, up to the action of group \mathcal{W} and possibly interchanging v_1 and v_3 we will have

$$M \equiv \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

if we consider the entries of $M = (v_1^T \ v_2^T \ v_3^T)$ taken mod 2. Then, it is clear that exactly one edge satisfies $n_{ij} \equiv 3 \pmod 4$, in particular n_{23} . If $n_{03} \equiv 3 \pmod 4$, then again, up to the action of \mathcal{W} we will have

$$M \equiv \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad (5.11)$$

and again, only one edge satisfies $n_{ij} \equiv 3 \pmod 4$, this time n_{03} . So, in any case, there is exactly one edge satisfying $n_{ij} \equiv 3 \pmod 4$, and after an appropriate lattice translation, we can always take n_{03} to be that edge. Without loss of generality, M satisfies (5.11) and we have

$$n_{01} \equiv n_{12} \equiv n_{23} \equiv 1 \pmod 4, \quad n_{02} \equiv n_{13} \equiv 2 \pmod 4, \quad n_{03} \equiv 3 \pmod 4, \quad (5.12)$$

or more succinctly,

$$n_{ij} \equiv j - i \pmod 4.$$

Also from (5.11) we get that

$$\langle v_1, v_2 \rangle \text{ and } \langle v_1, v_3 \rangle \text{ are odd, while } \langle v_2, v_3 \rangle \text{ is even.} \quad (5.13)$$

By Proposition 5.1 and (5.12), the Gauss sum relation for $n = 2$ becomes

$$0 = G_T(2) = -1 + 2(\omega_{02} + \omega_{13}),$$

therefore,

$$\omega_{02} + \omega_{13} = \frac{1}{2}, \quad (5.14)$$

and

$$\omega_{01} + \omega_{12} + \omega_{23} + \omega_{03} = \frac{2}{3}, \quad (5.15)$$

because of (5.4). By (5.11) and (5.12) we get

$$\kappa(4) = \sum_{0 \leq i < j \leq 3} e\left(\frac{n_{ij}}{4}\right) + e\left(\frac{\|v_1 + v_2 + v_3\|^2}{4}\right) = -3 + 2i,$$

hence Proposition 5.1 for $n = 4$ yields

$$\begin{aligned} G_T(4) &= -1 + (\omega_{01} + \omega_{12} + \omega_{23})G(4) + \omega_{03}G(3, 4) - 3 + 2i \\ &= -4 + 2(\omega_{01} + \omega_{12} + \omega_{23} + \omega_{03}) + 2(\omega_{01} + \omega_{12} + \omega_{23} - \omega_{03})i + 2i \\ &= -\frac{8}{3} + \left(\frac{10}{3} - 4\omega_{03}\right)i, \end{aligned}$$

while $\text{vol}(T)(G(4))^3 = \frac{8}{3}(-1 + i)$, so if the Gauss sum relation holds for $n = 4$, then we get

$$\omega_{03} = \frac{1}{6}, \quad (5.16)$$

and

$$\omega_{01} + \omega_{12} + \omega_{23} = \frac{1}{2}, \quad (5.17)$$

by (5.15). Next, we consider again the orthogonal projection $\pi : \mathbb{R}^3 \rightarrow H$, where $H = v_3^\perp$ and put $\Lambda = \pi(\mathbb{Z}^3)$. The vectors $\pi(v_1)$ and $\pi(v_2)$ is a basis of Λ and the angle between them is equal to the dihedral angle ω_{03} . However, the lattice Λ contains also vectors orthogonal to $\pi(v_1)$, namely $v_3 \times v_1$, so let $a\pi(v_1) + b\pi(v_2)$ be orthogonal to $\pi(v_1)$, with $a, b \in \mathbb{Z}$ nonzero. Hence, the orthogonal projection of $b\pi(v_2)$ on $\mathbb{R}\pi(v_1)$ is equal to $-a\pi(v_1)$, therefore $\|-a\pi(v_1)\| = \frac{1}{2}\|b\pi(v_2)\|$ or

$$\|\pi(v_2)\| = \left| \frac{a}{2b} \right| \|\pi(v_1)\|.$$

Since v_1, v_2, v_3 is a basis of \mathbb{Z}^3 we have

$$\begin{aligned} 1 &= |\langle v_3, v_1 \times v_2 \rangle| = |\langle v_3, \pi(v_1) \times \pi(v_2) \rangle| = \sqrt{n_{03}} \|\pi(v_1) \times \pi(v_2)\| \\ &= \sqrt{n_{03}} \|\pi(v_1)\| \|\pi(v_2)\| \sin 2\pi\omega_{03} = \sqrt{3n_{03}} \left| \frac{a}{4b} \right| \|\pi(v_1)\|^2, \end{aligned}$$

and since $\left| \frac{a}{4b} \right| \|\pi(v_1)\|^2 \in \mathbb{Q}$ we must have

$$n_{03} = 3m^2, \tag{5.18}$$

for some $m \in \mathbb{Z}$.

Next, the Gram relations (4.1) along with (5.14) and (5.17) form a system of six linear equations in terms of the dihedral angles ω_{ij} . This system has a unique solution, namely,

$$\omega_{01} = \frac{1}{2}\omega_0 - \frac{1}{2}\omega_1 - \frac{3}{2}\omega_2 - \frac{1}{2}\omega_3 + \frac{1}{4} \tag{5.19}$$

$$\omega_{02} = \frac{1}{2}\omega_0 - \frac{1}{2}\omega_1 + \frac{1}{2}\omega_2 - \frac{1}{2}\omega_3 + \frac{1}{4} \tag{5.20}$$

$$\omega_{03} = \omega_0 + \omega_1 + \omega_2 + \omega_3 \tag{5.21}$$

$$\omega_{12} = 2\omega_1 + 2\omega_2 \tag{5.22}$$

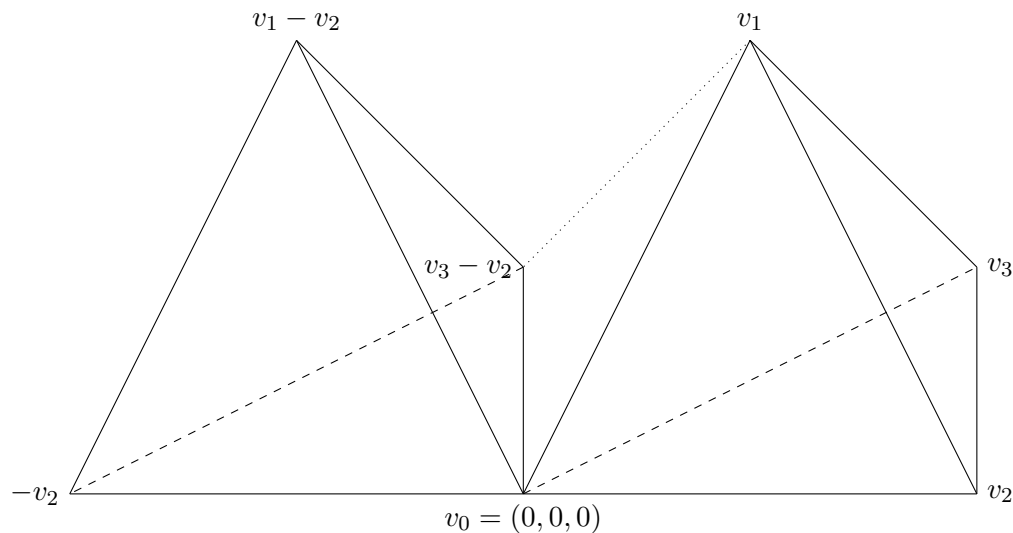
$$\omega_{13} = -\frac{1}{2}\omega_0 + \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{1}{4} \tag{5.23}$$

$$\omega_{23} = -\frac{1}{2}\omega_0 - \frac{3}{2}\omega_1 - \frac{1}{2}\omega_2 + \frac{1}{2}\omega_3 + \frac{1}{4}. \tag{5.24}$$

Formulae (5.20) and (5.23) along with (5.2) yield

$$\omega_{02} - \omega_0 - \omega_2 = \omega_{13} - \omega_1 - \omega_3 = 1/6. \tag{5.25}$$

In order to visualize $\omega_{02} - \omega_0 - \omega_2$, we consider T and its translate $T - v_2$, as in the figure below.



As can be seen, $\omega_{02} - \omega_0 - \omega_2$ is the solid angle of the cone with vectors $v_1, v_3, v_3 - v_2, v_1 - v_2$, which we divide into two simplicial cones, one with vectors $v_3, v_3 - v_2, v_1$, and one with $v_3 - v_2, v_1 - v_2, v_1$. We denote the solid angles by Ω_1, Ω_2 , respectively. Then, from (5.25) we get

$$\Omega_1 + \Omega_2 = \frac{1}{6}. \tag{5.26}$$

By (4.5) we get

$$\cot 2\pi\Omega_1 = \sqrt{n_{01}n_{23}n_{03}} + \langle v_3, v_3 - v_2 \rangle \sqrt{n_{01}} + \langle v_3 - v_2, v_1 \rangle \sqrt{n_{03}} + \langle v_1, v_3 \rangle \sqrt{n_{23}} \tag{5.27}$$

and

$$\cot 2\pi\Omega_2 = \sqrt{n_{01}n_{12}n_{23}} + \langle v_3 - v_2, v_1 - v_2 \rangle \sqrt{n_{01}} + \langle v_1 - v_2, v_1 \rangle \sqrt{n_{23}} + \langle v_1, v_3 - v_2 \rangle \sqrt{n_{12}}. \tag{5.28}$$

Put $K = \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{12}}, \sqrt{n_{23}})$. By (5.12) we have $\sqrt{n_{03}} \notin K$. We observe that $\cot 2\pi\Omega_2 \in K$ and by (5.26) we have

$$\frac{1}{\sqrt{3}} = \frac{\cot 2\pi\Omega_1 \cot 2\pi\Omega_2 - 1}{\cot 2\pi\Omega_1 + \cot 2\pi\Omega_2} \tag{5.29}$$

or equivalently

$$\cot 2\pi\Omega_1 + \cot 2\pi\Omega_2 = \sqrt{3} \cot 2\pi\Omega_1 \cot 2\pi\Omega_2 - \sqrt{3}. \tag{5.30}$$

As 1 and $\sqrt{3}$ are K -linearly independent we get

$$\cot 2\pi\Omega_2 = \frac{\langle v_3, v_3 - v_2 \rangle \sqrt{n_{01}} + \langle v_1, v_3 \rangle \sqrt{n_{23}}}{3m(\sqrt{n_{01}n_{23}} + \langle v_3 - v_2, v_1 \rangle) - 1} = \frac{m(\sqrt{n_{01}n_{23}} + \langle v_3 - v_2, v_1 \rangle) + 1}{\langle v_3, v_3 - v_2 \rangle \sqrt{n_{01}} + \langle v_1, v_3 \rangle \sqrt{n_{23}}}, \tag{5.31}$$

by (5.30) and (5.18). (5.31) yields $\cot 2\pi\Omega_2 \in \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{23}})$, and then by (5.28) we get

$$\sqrt{n_{12}} \in \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{23}}). \tag{5.32}$$

Indeed, if $\sqrt{n_{12}} \notin \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{23}})$, then 1 and $\sqrt{n_{12}}$ are $\mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{23}})$ -linearly independent, and the coefficient of $\sqrt{n_{12}}$ in (5.28) is $\sqrt{n_{01}n_{23}} + \langle v_1, v_3 - v_2 \rangle$ which is nonzero, since v_1 and $v_3 - v_2$ are not parallel. This would yield $\cot 2\pi\Omega_2 \notin \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{23}})$, a contradiction.

Combining the two equations in (5.31) we get

$$\cot^2 2\pi\Omega_2 = \frac{m(\sqrt{n_{01}n_{23}} + \langle v_3 - v_2, v_1 \rangle) + 1}{3m(\sqrt{n_{01}n_{23}} + \langle v_3 - v_2, v_1 \rangle) - 1}, \quad (5.33)$$

so $\cot^2 2\pi\Omega_2 \in \mathbb{Q}(\sqrt{n_{01}n_{23}})$ and by (5.28) $\cot^2 2\pi\Omega_2$ is an algebraic integer.

Proposition 5.5. *If $\sqrt{n_{01}n_{23}} \in \mathbb{Q}$ then $m = 1$, hence $n_{03} = 3$ and $\cot 2\pi\Omega_2 = 1$, hence $\Omega_2 = 1/8$ and $\Omega_1 = 1/24$.*

Proof. If $\sqrt{n_{01}n_{23}} \in \mathbb{Q}$ then $\cot^2 2\pi\Omega_2 \in \mathbb{Z}$. Put $z = \sqrt{n_{01}n_{23}} + \langle v_3 - v_2, v_1 \rangle$. By Cauchy-Schwarz inequality we have $z > 0$, and since $z \in \mathbb{Z}$ we must have $z \geq 1$. Then

$$\frac{mz + 1}{3mz - 1} \geq 1,$$

whence $mz \leq 1$, thus $m = z = 1$, which proves that $n_{03} = 3$ and $\cot 2\pi\Omega_2 = 1$, hence $\Omega_2 = 1/8$. Finally, by (5.26) we get $\Omega_1 = 1/24$. \square

Our goal is to show that the hypothesis of this Proposition is true. The next equation that we'll investigate is

$$\omega_{02} - 2\omega_2 = \omega_{01}, \quad (5.34)$$

which follows from (5.19) and (5.20). From (4.3) we then get

$$\omega_{02} - 2\omega_2 = \varphi_{20} - \omega_2, \quad (5.35)$$

and

$$\omega_{01} = \omega_0 + \varphi_{01}, \quad (5.36)$$

hence

$$\cot 2\pi(\varphi_{20} - \omega_2) = \cot 2\pi(\omega_0 + \varphi_{01}). \quad (5.37)$$

Applying (4.5) and (4.6) accordingly we have

$$\cot 2\pi\varphi_{01} = \sqrt{n_{01}n_{02}n_{03}} + \langle v_2, v_3 \rangle\sqrt{n_{01}} - \langle v_3, v_1 \rangle\sqrt{n_{02}} - \langle v_1, v_2 \rangle\sqrt{n_{03}} \quad (5.38)$$

$$\begin{aligned} \cot 2\pi\varphi_{20} &= \sqrt{n_{12}n_{02}n_{23}} + \langle v_2, v_3 - v_2 \rangle\sqrt{n_{12}} \\ &\quad + \langle v_3 - v_2, v_1 - v_2 \rangle\sqrt{n_{02}} + \langle v_1 - v_2, v_2 \rangle\sqrt{n_{23}} \end{aligned} \quad (5.39)$$

$$\begin{aligned} \cot 2\pi\omega_2 &= \sqrt{n_{12}n_{02}n_{23}} - \langle v_2, v_3 - v_2 \rangle\sqrt{n_{12}} \\ &\quad + \langle v_3 - v_2, v_1 - v_2 \rangle\sqrt{n_{02}} - \langle v_1 - v_2, v_2 \rangle\sqrt{n_{23}}. \end{aligned} \quad (5.40)$$

Then by (5.35), (5.36), (5.8), (5.38), (5.39), (5.28), and the formulae for the cotangent of a sum we get:

$$\cot 2\pi\omega_{01} = \frac{n_{01}(\sqrt{n_{02}n_{03}} + \langle v_2, v_3 \rangle)^2 - (\langle v_3, v_1 \rangle \sqrt{n_{02}} + \langle v_1, v_2 \rangle \sqrt{n_{03}})^2 - 1}{2\sqrt{n_{01}}(\sqrt{n_{02}n_{03}} + \langle v_2, v_3 \rangle)} \quad (5.41)$$

and

$$\cot 2\pi(\omega_{02} - 2\omega_2) = \frac{(\langle v_2, v_3 - v_2 \rangle \sqrt{n_{12}} + \langle v_1 - v_2, v_2 \rangle \sqrt{n_{23}})^2 - n_{02}(\sqrt{n_{12}n_{23}} + \langle v_3 - v_2, v_1 - v_2 \rangle)^2 - 1}{2(\langle v_2, v_3 - v_2 \rangle \sqrt{n_{12}} + \langle v_1 - v_2, v_2 \rangle \sqrt{n_{23}})}. \quad (5.42)$$

Rewriting (5.41) we get

$$2\sqrt{n_{01}} \cot 2\pi\omega_{01} = \frac{E}{F}, \quad (5.43)$$

where

$$E = (2n_{01}\langle v_2, v_3 \rangle - 2\langle v_3, v_1 \rangle \langle v_1, v_2 \rangle) \sqrt{n_{02}n_{03}} \\ + (n_{01}n_{02}n_{03} + n_{01}\langle v_2, v_3 \rangle^2 - n_{02}\langle v_3, v_1 \rangle^2 - n_{03}\langle v_1, v_2 \rangle^2 - 1),$$

and $F = \sqrt{n_{02}n_{03}} + \langle v_2, v_3 \rangle$. By (5.12) we have $n_{02}n_{03} \equiv 2 \pmod{4}$, hence $\sqrt{n_{02}n_{03}} \notin \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{12}}, \sqrt{n_{23}})$, therefore

$$2\sqrt{n_{01}} \cot 2\pi\omega_{01} \in \mathbb{Q}(\sqrt{n_{02}n_{03}}) \cap \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{12}}, \sqrt{n_{23}}) = \mathbb{Q}. \quad (5.44)$$

This shows that the numerator and denominator at (5.43) are \mathbb{Q} -linearly dependent, hence $2\sqrt{n_{01}} \cot 2\pi\omega_{01}$ is equal to the ratio of the corresponding coefficients of $\sqrt{n_{02}n_{03}}$, thus

$$\sqrt{n_{01}} \cot 2\pi\omega_{01} = n_{01}\langle v_2, v_3 \rangle - \langle v_3, v_1 \rangle \langle v_1, v_2 \rangle \in \mathbb{Z}. \quad (5.45)$$

Furthermore, this number is also nonzero, because it is odd, as follows from (5.12) and (5.13), hence

$$\sqrt{n_{01}} \in \mathbb{Q}(\sqrt{n_{12}}, \sqrt{n_{23}}). \quad (5.46)$$

Now we will show that $\mathbb{Q}(\sqrt{n_{01}}) = \mathbb{Q}(\sqrt{n_{12}})$. If $\mathbb{Q}(\sqrt{n_{12}}, \sqrt{n_{23}})$ is equal to \mathbb{Q} , it is trivial. If it is equal to a quadratic extension, then $\mathbb{Q}(\sqrt{n_{12}}) = \mathbb{Q}(\sqrt{n_{23}}) \neq \mathbb{Q}$. If $\sqrt{n_{01}} \in \mathbb{Q}$, then from (5.45) and (5.34) we have that $\cot 2\pi(\omega_{02} - 2\omega_2) \in \mathbb{Q}$. But since $\mathbb{Q}(\sqrt{n_{12}}) = \mathbb{Q}(\sqrt{n_{23}}) \neq \mathbb{Q}$, the numerator from (5.42) is nonzero rational, while the denominator is a rational multiple of $\sqrt{n_{12}}$, a contradiction. Hence, in this case, $\mathbb{Q}(\sqrt{n_{01}}) = \mathbb{Q}(\sqrt{n_{12}})$.

It remains to examine the case where $\mathbb{Q}(\sqrt{n_{12}}, \sqrt{n_{23}})$ is a biquadratic extension. If $\mathbb{Q}(\sqrt{n_{01}}) \neq \mathbb{Q}(\sqrt{n_{12}})$, then from (5.32) and (5.46) follows that $\mathbb{Q}(\sqrt{n_{01}}) = \mathbb{Q}(\sqrt{n_{12}n_{23}})$. Recall that by (5.34) and (5.45) we have $\sqrt{n_{01}} \cot 2\pi(\omega_{02} - 2\omega_2) \in \mathbb{Z}$. However, by (5.42), the numerator of $\sqrt{n_{01}} \cot 2\pi(\omega_{02} - 2\omega_2)$ is a rational linear combination of 1 and $\sqrt{n_{12}n_{23}}$ and is nonzero, while the denominator is a rational linear combination of $\sqrt{n_{12}}$ and $\sqrt{n_{23}}$, therefore, they are \mathbb{Q} -linearly independent, and as such their ratio cannot be rational. This contradicts the hypothesis $\mathbb{Q}(\sqrt{n_{01}}) \neq \mathbb{Q}(\sqrt{n_{12}})$, hence at all cases we have

$$\mathbb{Q}(\sqrt{n_{01}}) = \mathbb{Q}(\sqrt{n_{12}}). \quad (5.47)$$

Next, (5.23) and (5.24) yield

$$\omega_{13} - 2\omega_1 = \omega_{23}, \quad (5.48)$$

which in turn yields similar formulae to (5.41) and (5.42), where the indices 0 and 1 are interchanged with 3 and 2, respectively (*notice that this symmetry is obeyed by the formulae which follow from (5.14) and (5.17)*). Then, similar arguments to those that were used in order to obtain (5.47) can be used in order to get

$$\mathbb{Q}(\sqrt{n_{23}}) = \mathbb{Q}(\sqrt{n_{12}}),$$

and thus establish

$$\mathbb{Q}(\sqrt{n_{01}}) = \mathbb{Q}(\sqrt{n_{12}}) = \mathbb{Q}(\sqrt{n_{23}}). \quad (5.49)$$

Therefore, $\sqrt{n_{01}n_{23}} \in \mathbb{Q}$, hence by Proposition 5.5 we have $n_{03} = 3$, $\Omega_1 = 1/24$, and $\Omega_2 = 1/8$. Then, (5.28) and (5.49) yield $1 = \cot 2\pi\Omega_2 = d\sqrt{n_{01}}$, for some $d \in \mathbb{Q}$, thus,

$$\mathbb{Q}(\sqrt{n_{01}}) = \mathbb{Q}(\sqrt{n_{12}}) = \mathbb{Q}(\sqrt{n_{23}}) = \mathbb{Q}. \quad (5.50)$$

Proposition 5.3 gives us once more

$$v_1 = (k+1, k, k), \quad v_2 = (l, l, l-1), \quad v_3 = (1, 1, 1), \quad (5.51)$$

up to an action of \mathcal{W} . In our case, we have k even and l odd from (5.12). (5.12) also gives $\sqrt{n_{02}} \notin \mathbb{Q}(\sqrt{n_{12}}, \sqrt{n_{23}})$ and $\sqrt{n_{13}} \notin \mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{12}})$, hence by (5.50) and Proposition 4.1 we get

$$\sqrt{n_{02}} \cot 2\pi\omega_{02}, \sqrt{n_{13}} \cot 2\pi\omega_{13} \in \mathbb{Q}. \quad (5.52)$$

Now consider τ to be the nontrivial automorphism of $\mathbb{Q}(\sqrt{n_{01}}, \sqrt{n_{02}}, \sqrt{n_{03}}) = \mathbb{Q}(\sqrt{n_{02}}, \sqrt{3})$ that fixes $\mathbb{Q}(\sqrt{3})$ and σ be the nontrivial automorphism of $\mathbb{Q}(\sqrt{n_{03}}, \sqrt{n_{13}}, \sqrt{n_{23}}) = \mathbb{Q}(\sqrt{n_{13}}, \sqrt{3})$ that fixes $\mathbb{Q}(\sqrt{3})$, i. e.

$$\tau(\sqrt{n_{02}}) = -\sqrt{n_{02}}, \quad \sigma(\sqrt{n_{13}}) = -\sqrt{n_{13}}, \quad \tau(\sqrt{3}) = \sigma(\sqrt{3}) = \sqrt{3}. \quad (5.53)$$

Finally, let N_1 and N_2 be the number theoretic norms of the quadratic extensions $\mathbb{Q}(\sqrt{n_{02}}, \sqrt{3})/\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{n_{13}}, \sqrt{3})/\mathbb{Q}(\sqrt{3})$, respectively. By Proposition 4.1 we have

$$\sqrt{n_{02}} \cot 2\pi\omega_{02} = \frac{-N_1(\cot 2\pi\omega_0) - 1}{2(\sqrt{n_{01}n_{03}} + \langle v_1, v_3 \rangle)} \quad (5.54)$$

and

$$\sqrt{n_{13}} \cot 2\pi\omega_{13} = \frac{-N_2(\cot 2\pi\omega_3) - 1}{2(\sqrt{n_{03}n_{23}} + \langle -v_3, v_2 - v_3 \rangle)}. \quad (5.55)$$

Both numerators and denominators of the fractions in (5.54) and (5.55) belong to $\mathbb{Q}(\sqrt{3})$, hence by (5.52), the left-hand sides of these equations are also equal to the ratio of the coefficients of $\sqrt{3}$ of the numerator and the denominator, when they are written as \mathbb{Q} -linear combinations of 1 and $\sqrt{3}$. We have

$$-N_1(\cot 2\pi\omega_0) - 1 = n_{02}(\sqrt{n_{01}n_{03}} + \langle v_1, v_3 \rangle)^2 - (\langle v_2, v_3 \rangle\sqrt{n_{01}} + \langle v_1, v_2 \rangle\sqrt{n_{03}})^2 - 1, \quad (5.56)$$

hence the coefficient of $\sqrt{3}$ is

$$2n_{02}\sqrt{n_{01}}\langle v_1, v_3 \rangle - 2\sqrt{n_{01}}\langle v_1, v_2 \rangle\langle v_2, v_3 \rangle, \quad (5.57)$$

while the coefficient of $\sqrt{3}$ of the denominator in (5.54) is just $2\sqrt{n_{01}}$, which yields

$$\cot 2\pi\omega_{02} = \frac{n_{02}\langle v_1, v_3 \rangle - \langle v_1, v_2 \rangle \langle v_2, v_3 \rangle}{\sqrt{n_{02}}} = \frac{2k - l + 1}{\sqrt{3l^2 - 2l + 1}}, \quad (5.58)$$

by (5.51). Similarly,

$$-N_2(\cot 2\pi\omega_3) - 1 = n_{13}(\sqrt{n_{03}n_{23}} + \langle -v_3, v_2 - v_3 \rangle)^2 - (\langle -v_3, v_1 - v_3 \rangle \sqrt{n_{23}} + \langle v_2 - v_3, v_1 - v_3 \rangle \sqrt{n_{03}})^2 - 1, \quad (5.59)$$

hence the coefficient of $\sqrt{3}$ is

$$2n_{13}\sqrt{n_{23}}\langle -v_3, v_2 - v_3 \rangle - 2\sqrt{n_{23}}\langle -v_3, v_1 - v_3 \rangle \langle v_2 - v_3, v_1 - v_3 \rangle, \quad (5.60)$$

while the coefficient of $\sqrt{3}$ of the denominator in (5.55) is just $2\sqrt{n_{23}}$, which yields

$$\cot 2\pi\omega_{13} = \frac{n_{13}\langle -v_3, v_2 - v_3 \rangle - \langle -v_3, v_1 - v_3 \rangle \langle v_2 - v_3, v_1 - v_3 \rangle}{\sqrt{n_{13}}} = \frac{k - 2l + 2}{\sqrt{3k^2 - 4k + 2}}. \quad (5.61)$$

Equations (5.14), (5.58), and (5.61) yield

$$\frac{2k - l + 1}{\sqrt{3l^2 - 2l + 1}} = \frac{-k + 2l - 2}{\sqrt{3k^2 - 4k + 2}}. \quad (5.62)$$

Putting $x = k$, $y = -l + 1$, the above becomes

$$\frac{2x + y}{\sqrt{3y^2 - 4y + 2}} = \frac{-x - 2y}{\sqrt{3x^2 - 4x + 2}}. \quad (5.63)$$

The rest follows from:

Proposition 5.6. *The only integer solution of the equation (5.63) is $x = y = 0$.*

Proof. If $x = y$, then we can easily see that we can only have $x = y = 0$, so we may assume that $x \neq y$. Square both sides of (5.63) to obtain

$$\frac{(2x + y)^2}{3y^2 - 4y + 2} = \frac{(x + 2y)^2}{3x^2 - 4x + 2}. \quad (5.64)$$

Both sides are nonnegative and equal to

$$\frac{(2x + y)^2 - (x + 2y)^2}{(3y^2 - 4y + 2) - (3x^2 - 4x + 2)} = \frac{3(x + y)}{4 - 3(x + y)},$$

hence we must have $x + y = 0$ or 1. If $x + y = 1$, then both sides of (5.64) must be equal to 3, hence

$$\frac{(x + 1)^2}{3x^2 - 2x + 1} = 3,$$

whose only solution is $x = 1/2$. Thus, $x + y = 0$, hence by (5.63) we have

$$\frac{x}{\sqrt{3x^2 + 4x + 2}} = \frac{x}{\sqrt{3x^2 - 4x + 2}},$$

which yields either $x = 0$ or $3x^2 + 4x + 2 = 3x^2 - 4x + 2$. It is clear, that the only solution is $x = y = 0$, as desired. \square

Proposition 5.6 and (5.51) give

$$v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1), \quad (5.65)$$

which finally proves Theorem 1.3.

All such tetrahedra multi-tile \mathbb{R}^3 by the action of the group \mathcal{G} , hence the converse is true in this special case.

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