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16-QAM Almost-Complementary Sequences with Low PMEPR

Zilong Liu, Yong Liang Guan

Abstract—A pair of sequences is said to be an almost-complementary pair (ACP) if they have zero aperiodic autocorrelation sums except at only one position over all the possible/negative time-shifts. Having correlation property very close to that of Golay complementary pairs (GCPs), ACPs may be used as an alternative to GCPs in many applications in communications and radar. For high-rate code-keying OFDM communication, we construct novel 16-QAM ACPs from three new classes of quadratic offsets, leading to three large sets of 16-QAM almost-complementary sequences with maximum peak-to-mean envelope power ratio (PMEPR) of 2.4.

Index Terms—Golay complementary pair sequence (GCP/GCS), near-complementary pair sequence (NCP/NCS), QAM, orthogonal frequency-division multiplexing (OFDM), peak-to-mean envelope power ratio (PMEPR), Reed-Muller code.

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) is a modulation technique that simultaneously transmits multiple data streams over a number of sub-carriers [1]. With robustness to radio channel impairments such as multipath fading, OFDM has attracted intensive research attention over the past decades. A major drawback of uncoded OFDM signals is the high peak-to-mean envelope power ratio (PMEPR) problem which could degrade the transmission power efficiency and thus drastically decrease the range of OFDM communications [2].

A coding approach for PMEPR reduction is to construct a codebook consisting of codewords with low PMEPR and then apply it to a code-keying OFDM system. A remarkable work has been done by Davis and Jedwab [3] by constructing codebooks from polyphase Golay complementary sequences (GCSs) [4] using the algebraic tool of generalized Boolean functions. Here, every polyphase GCS, together with another (properly selected) polyphase one, form a Golay complementary pair (GCP) with zero out-of-phase aperiodic autocorrelation sums (AASs), thus producing an OFDM waveform with a PMEPR of at most 2 when this GCS is spread over the frequency domain [5]. Apart from their application in PMEPR reduction, GCSs and GCPs have been used in many other scenarios such as Doppler resilient radar waveform design [6], optimal channel estimation [7], [8], and interference-free multicarrier CDMA [9]–[11], etc.

In contrast to polyphase GCSs, QAM GCSs have sequence elements drawn from a QAM constellation set. The extension from polyphase to QAM is mainly motivated by the demand of high-rate transmission for code-keying OFDM systems because QAM GCSs have larger set size (compared to polyphase ones), although with slight (but tolerable) increase of PMEPR. In the literature, Rößing and Tarokh pioneered the construction of 16-QAM GCSs with maximum PMEPR of 3.6 from weighted sum of two quaternary GCSs [12]. Subsequently, Chong, Venkataramani and Tarokh developed an algebraic construction of 16-QAM GCSs also using generalized Boolean functions [13]. An interesting observation made in [13] is that an OFDM system with 16-QAM GCSs indeed can achieve a higher code-rate than that with only binary or quaternary GCSs, given the same PMEPR constraint. Later on, there have also been constructions of 64-QAM QAM GCSs [14], [15]. Generalized Case I-III 4\(^{q}\)-QAM (q \geq 1) complementary sequences are reported in [16]. Recently, Liu, Li and Guan constructed Generalized Case IV-V 4\(^{q}\)-QAM (q \geq 3) complementary sequences using selected Gaussian integer pairs, each consisting of two distinct Gaussian integers with identical magnitude and which are not conjugate with each other [17].

Another approach for code-rate increase is to construct near-complementary sequences (NCSs) from sequence pairs called near-complementary pairs (NCPs), each having low (but nonzero) out-of-phase AASs. By relaxing the strict correlation property imposed by GCPs, this approach allows us to increase the number of sequences with low PMEPRs because more design flexibility can be achieved. Parker and Tellambura obtained NCSs with low PMEPR by applying generalized Rudin-Shapiro constructions to certain suitable kernels with short lengths [18], [19]. Then, Schmidt proposed polyphase NCSs with PMEPRs no greater than 4 from cosets of the generalized first-order Reed-Muller codes [20]. This work was followed by Yu and Gong using a generic framework with matrix structure for NCSs with various lengths other than 2\(^m\) [21]. In addition, construction of 16-QAM NCSs with PMEPR of at most 2 + \(\frac{2}{\sqrt{5}}\) \approx 2.8944\(^1\) was proposed by Lee and Golomb in 2010 [22].

We remark that the afore-mentioned constructions of NCSs are in general not concerned with the exact value of AAS at any particular non-zero time-shift when they are arranged into pairs of sequences. In practice, however, characterizing the exact AASs is useful for the search of sequences (or

\(^1\)Although it was claimed in [22] that the maximum PMEPR is 2.4, we found the actual PMEPR upper bound is 2 + \(\frac{2}{\sqrt{5}}\) \approx 2.8944.
pairs), for instance, that are able to perform synchronization tasks or support asynchronous communications (which deal with asynchronous arrivals of signals), in addition to their PMEPR reduction capability. Examples of such sequence pairs are zero-complementary pairs (ZCPs), each having zero out-of-phase AASs except at \( \tau \), where \( \tau \) is a certain positive time-shift. By applying new classes of quadratic offsets to selected 16-QAM ACPs, we construct three disjoint sets of 16-QAM almost-complementary sequences (ACSs) with PMEPR of at most 2, which is very close to the minimum possible upper bound of 2. The characterization of the maximum PMEPR is done by identifying the exact values of \( \tau \) as well as the corresponding non-zero AAS through a proper decomposition of AASs over several disjoint sets of \( (i,j) \), where \( j = i + \tau \). Our proposed ACPs are not only capable of performing PMEPR reduction (i.e., NCs), but also supporting asynchronous communications (i.e., ZCPs) because each pair has only one non-zero (but small) AAS at a known time-shift of \( \tau \), where the largest value of \( \tau \) is \( 2^{m-1} + 2^{m-2} \) for sequence length of \( 2^m \).

The main contribution of this paper is a novel construction of 16-QAM almost-complementary pairs (ACPs), each having zero out-of-phase AASs except at \( \pm \tau \), where \( \tau \) is a certain positive time-shift. By applying new classes of quadratic offsets to selected 16-QAM ACPs, we construct three disjoint sets of 16-QAM almost-complementary sequences (ACSs) with PMEPR of at most 2, which is very close to the minimum possible upper bound of 2. The characterization of the maximum PMEPR is done by identifying the exact values of \( \tau \) as well as the corresponding non-zero AAS through a proper decomposition of AASs over several disjoint sets of \( (i,j) \), where \( j = i + \tau \). Our proposed ACPs are not only capable of performing PMEPR reduction (i.e., NCs), but also supporting asynchronous communications (i.e., ZCPs) because each pair has only one non-zero (but small) AAS at a known time-shift of \( \tau \), where the largest value of \( \tau \) is \( 2^{m-1} + 2^{m-2} \) for sequence length of \( 2^m \).

The outline of this paper is as follows. In Section II, we first define GCPs, NCPs and ACPs; then we introduce the PMEPR problem in code-keying OFDM systems as well as the algebraic tool of generalized Boolean functions. In Section III, we present three classes of 16-QAM ACPs followed by a discussion of their PMEPRs. In the end, we summarize this work in Section IV.

### II. PRELIMINARIES

#### A. Definitions

For two length-\( L \) sequences \( A = [A_0, A_1, \cdots, A_{L-1}] \) and \( B = [B_0, B_1, \cdots, B_{L-1}] \), denote by \( C_{A,B}(\tau) \) the aperiodic cross-correlation of time-shift \( \tau \) between A and B:

\[
C_{A,B}(\tau) = \begin{cases} 
\sum_{i=0}^{L-1-u} A_i B_{i+u}^*, & 0 \leq u \leq (L-1); \\
\sum_{i=0}^{L-1-u} A_{i-u} B_i^*, & -(L-1) \leq u \leq -1; \\
0, & |u| \geq L.
\end{cases}
\]

If \( A = B \), \( C_{A,B}(\tau) \) reduces to the aperiodic auto-correlation of A and will be written as \( C_A(\tau) \) for simplicity.

**Definition 1:** For complex-valued sequence pair \( (A, B) \), define

\[
A \ast B := \sum_{h=-L}^{L} \left| C_A(u) + C_B(u) \right| = \|A\|_F^2 + \|B\|_F^2 + 2 \sum_{u=1}^{L-1} \left| C_A(u) + C_B(u) \right|,
\]

where \( \|A\|_F \) denotes the Frobenius norm of A, i.e., \( \|A\|_F^2 = \sum_{i=0}^{L-1} |A_i|^2 \) and \( \|B\|_F^2 = \sum_{i=0}^{L-1} |B_i|^2 \). It can be readily shown that

\[
A \ast B \leq L \cdot \left[ \|A\|_F^2 + \|B\|_F^2 \right].
\]

**(A, B)** is said to be a

1. GCP if \( A \ast B = \|A\|_F^2 + \|B\|_F^2 \), i.e., \( C_A(u) + C_B(u) = 0 \) for any \( u \neq 0 \);

2. or an NCP if \( \|A\|_F^2 + \|B\|_F^2 < A \ast B \leq L \cdot \left[ \|A\|_F^2 + \|B\|_F^2 \right] \). In particular, \((A, B)\) is said to be an ACP if the AASs \( i.e., C_A(u) + C_B(u) \), \( 1 \leq u \leq L - 1 \) are zero except at a certain time-shift over the set \( \{1, 2, \cdots, L - 1\} \).

**Remark 1:** We regard ACP as a special type of NCP. In this paper, we focus on ACPs over the 16-QAM constellation set.

#### B. Peak-to-Mean Power Control Problem in Code-keying OFDM System

Consider a code-keying OFDM system with \( L \) subcarriers, subcarrier spacing \( \triangle f \) and carrier frequency \( f_c \). For any length-\( L \) complex-valued codeword \( A \) in a codebook \( S \), the transmitted OFDM signal over a symbol period of \( T = \frac{1}{2\triangle f} \) is the real part of the signal below.

\[
T_A(t) = \sum_{i=0}^{L-1} A_i \xi(f_c + \triangle f) t^i, \quad 0 \leq t < T,
\]

where \( \xi = \exp \left( \sqrt{-2T} \pi \right) \). Moreover, let \( P_A(t) = |T_A(t)|^2 \) be the “instantaneous envelope power” of \( A \). Note that

\[
\frac{1}{T} \int_0^T P_A(t) dt = \frac{1}{T} \int_0^T T_A(t) dT = \frac{1}{L} \sum_{i=0}^{L-1} |A_i|^2.
\]

The “mean envelope power” when averaging over the entire codebook is defined as

\[
P_{av}(S) = \sum_{A \in S} p(A) \|A\|_F^2,
\]

where \( p(A) \) is the probability of transmitting the codeword \( A \). In particular, if each codeword in \( S \) has equal probability of occurrence, then \( P_{av}(S) \) reduces to

\[
P_{av}(S) = \frac{1}{|S|} \sum_{A \in S} \|A\|_F^2.
\]

Furthermore, let

\[
P_{EP}(A) = \sup_{t \in [0,T]} P_A(t)
\]

be the “peak envelope power (PEP)” of codeword \( A \). Define PMEPR of codebook \( S \) as

\[
\text{PMEPR}(S) := \max_{A \in S} \frac{P_{EP}(A)}{P_{av}(S)}.
\]

In addition, define the “instantaneous-to-mean envelope power ratio (IMEPR)” of \( A \) as

\[
\text{IMEPR}(A, t) := \frac{P_A(t)}{P_{av}(S)}.
\]
Clearly,
\[ \text{PMEPR}(S) = \max_{A \in S} \sup_{t \in [0,T]} \text{IMEPR}(A, t). \]  
(10)

By [20, Theorem 2], we have

**Lemma 1:** For complex-valued sequence pair \((A, B)\), the PEP of \(A\) (or \(B\)) is upper bounded by \(A * B\), i.e., \(\text{PEP}(A) \leq A * B\). Thus,
\[ \text{PMEPR}(A) \leq A * B / \|A\|_F^2. \]  
(11)

**Proof:** The proof of (11) follows immediately if we can prove \(\text{PEP}(A) \leq A * B\). To this end, we first note that
\[ P_A(t) = |T_A(t)|^2 = \|A\|_F^2 + 2 \sum_{u=1}^{L-1} \text{Re} \{ C_A(u) \xi^{|u|} \}. \]

Therefore, we have
\[ P_A(t) + P_B(t) = \|A\|_F^2 + \|B\|_F^2 + 2 \sum_{u=1}^{L-1} \text{Re} \{ [C_A(u) + C_B(u)] \xi^{|u|} \} \leq \|A\|_F^2 + \|B\|_F^2 + 2 \sum_{u=1}^{L-1} |C_A(u) + C_B(u)|. \]

Finally, by taking into account that \(P_B(t) \geq 0\), we obtain
\[ \text{PEP}(A) = \sup_{t \in (0,T)} P_A(t) \leq A * B. \]

The following lemma is straightforward by the above P-MEPR definition.

**Lemma 2:** Suppose codebook \(S\) is comprised of codewords of two or more which all have identical energy \(E\), i.e., \(|S| \geq 2\) and \(\|A\|_F^2 = E\) for any \(A \in S\). By (7), it follows that \(P_m(S) = E\). Furthermore, suppose for any \(A \in S\) there exists a distinct pairing sequence \(B \in S\) with \(A * B = F\), where \(F\) is a constant for all possible \((A, B)\) pairs. Then, the P-MEPR of \(S\) is upper bounded by \(F/E\), i.e.,
\[ \text{PMEPR}(S) \leq F/E = A * B / \|A\|_F^2. \]  
(12)

**Remark 2:** Note that PMEPR is defined with respect to a codebook. Although the PMEPR upper bound in (11) applies to the codebook consisting of sequence \(A\) only, it is useful if \(A\) is included in a larger codebook \(S\) satisfying the conditions specified in Lemma 2. Specifically, the upper bound in (11), i.e., \(A * B / \|A\|_F^2\), can also be used to upper-bound the P-MEPR of \(S\), as shown in (12). Applications of Lemma 1 and Lemma 2 will be shown in Section III and Section IV.

**C. Generalized Boolean Functions**

Denote by \(Z_q = \{0, 1, \cdots, q - 1\}\) the set of integers modulo \(q\), where \(q\) is a positive integer. For \(x = [x_0, x_1, \cdots, x_{m-1}] \in Z_q^m\), a generalized Boolean function \(f(x)\) (or \(f(x_0, x_1, \cdots, x_{m-1})\)) is defined as a mapping \(f : \{0, 1\}^m \to Z_q\). Let \((i_0, i_1, \cdots, i_{m-1})\) be the binary representation of the integer \(i = \sum_{k=0}^{m-1} i_k 2^{k-1}\), with \(i_{m-1}\) denoting the most significant bit. Given \(f(x)\) [or \(f(x_0, x_1, \cdots, x_{m-1})\)], define \(f_t := f(i_0, i_1, \cdots, i_{m-1})\), and \(f := (f(0, 0, \cdots, 0), f(1, 0, \cdots, 0), \cdots, f(1, 1, \cdots, 1))\).

For instance, if \(m = 3\) and \(q = 4\), then
\[
\begin{align*}
1 &= (1, 1, 1, 1, 1, 1, 1) \\
\chi_0 &= (0, 1, 0, 1, 0, 1, 0) \\
\chi_2 &= (0, 0, 0, 2, 2, 2, 2) \\
\chi_0 + 2\chi_2 + 1 &= (1, 2, 1, 2, 3, 0, 3, 0).
\end{align*}
\]

For \(\xi = \exp(\sqrt{-12}/\pi)\), define the complex-valued sequence associated with \(f\) as follows.
\[ \psi(f) := (\xi f_0, \xi f_1, \cdots, \xi f_{2^m-1}). \]

From now on, whenever the context is clear, we use \(C_{f,g}(\tau)\) and \(C_{f}(\tau)\) to represent \(C_{\psi(f),\psi(g)}(\tau)\) and \(C_{\psi(f)}(\tau)\), respectively, where \(f, g\) are two generalized Boolean functions in binary variables \(x_0, x_1, \cdots, x_{m-1}\).

**D. Existing Constructions of 16-QAM Golay- and Near-Complementary Pairs**

Rößing and Tarokh showed that 16-QAM GCPs can be constructed by the weighted sum of two (properly selected) quaternary GCPs [12]. The basic idea behind their construction is that any element drawn from the 16-QAM constellation set can be uniquely determined by two QPSK symbols. Later, Chong, Venkataramani and Tarokh extended the approach in [12] by algebraically constructing 16-QAM GCPs from generalized Boolean functions [13]. We now present their construction in the following lemma.

**Lemma 3:** Let
\[
A(x) = 2 \sum_{l=0}^{m-2} x_{\pi(l)} x_{\pi(l+1)} + \sum_{l=0}^{m-1} c_l x_{\pi(l)} + c \cdot 1, \]
(13a)
\[
\alpha(x) = A(x) + s(x), \]
(13b)
\[
B(x) = A(x) + 2x_{\pi(m-1)}, \]
(13c)
\[
\beta(x) = \alpha(x) + 2x_{\pi(m-1)} \]
(13d)
where \(x \in Z_4^m, c_l \in Z_4, c \in Z_4, \pi\) is a permutation of the symbols \(\{0, 1, \cdots, m - 1\}\), and \(s(x)\) is the offset. Note that all the multiplications and additions in (13) are carried out over \(Z_4\). For \(r_1 = \frac{1}{4}, r_2 = \frac{1}{4}\), \(\gamma = \exp(\sqrt{-1}/\pi/4), \xi = \exp(\sqrt{-1}/\pi/2)\), let
\[
C = (C_0, C_1, \cdots, C_{2m-1}), \quad D = (D_0, D_1, \cdots, D_{2m-1}),
\]
(14)
where \(C_i = \gamma(r_1 \xi^{A_i} + r_2 \xi^{A_i})\) and \(D_i = \gamma(r_1 \xi^{B_i} + r_2 \xi^{B_i})\). Then, \((C, D)\) is a 16-QAM GCP for the following offsets.
\[
s(x) = \begin{cases} 
0 + d_1 x_{\pi(0)}, & 1 \leq w \leq m - 1 \\
0 + d_1 x_{\pi(w)} + d_2 x_{\pi(w+1)}, & 2d_0 + d_1 + d_2 = 0
\end{cases}
\]
(15)
where \(d_0, d_1, d_2 \in Z_4\).

The construction of 16-QAM NCPs by Lee and Golomb resembles Lemma 3 but uses a quadratic offset [22].
III. PROPOSED CONSTRUCTIONS OF NOVEL 16-QAM ALMOST-COMPLEMENTARY PAIRS

In this section, we present our proposed constructions of 16-QAM ACPs in the theorem below. Our key findings are three novel classes of quadratic offsets, each giving rise to 16-QAM ACP with only one non-zero AAS over all positive (or negative) time-shifts. Inspired by several non-linear offset examples from computer search, we generalize to all possible cases and provide analytical proof for each ACP type.

Theorem 1: In the context of Lemma 3, we give below three novel classes of quadratic offsets and show that each of them gives rise to a 16-QAM ACP (C, D).

1) The first offset class is in the form of
\[ s(x) = dx_{x(0)x(1)} + e_0x_{x(0)} + e_1x_{x(1)} + f, \quad (16) \]
where \((d, e_0, e_1, f)\) is given in Table I.

TABLE I: Coefficients of The First Offset Class

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<th>No.</th>
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With this offset class, we obtain Type-I 16-QAM ACP (C, D). Specifically, for \(d = 2, u \geq 0\), we have
\[
\left| C_C(u) + C_D(u) \right| = \begin{cases} 
2^{m+1}, & \text{for } u = 0; \\
\left\lfloor \frac{2^{m+1}}{5} \right\rfloor, & \text{for } u = 2^{r(0)}; \\
0, & \text{otherwise.}
\end{cases}
\quad (17)
\]
For \(d \in \{1, 3\}, u \geq 0\), we have
\[
\left| C_C(u) + C_D(u) \right| = \begin{cases} 
\frac{4}{5} \cdot 2^{m+1}, & \text{for } u = 0; \\
\sqrt{\frac{2}{10}} \cdot 2^m, & \text{for } u = 2^{r(0)}; \\
0, & \text{otherwise.}
\end{cases}
\quad (18)
\]

2) The second offset class is in the form of
\[
s(x) = dx_{x(0)x(1)} + d_1x_{x(1)}x_{x(2)} + e_0x_{x(0)} + e_1x_{x(1)} + e_2x_{x(2)} + f,
\quad (19)
\]
where \((d_0, d_1, e_0, e_1, e_2, f)\) is given in Table II.

TABLE II: Coefficients of The Second Offset Class

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With this offset class, we obtain Type-II 16-QAM ACP (C, D). Specifically, for \(d_0 = d_1\) and \(u \geq 0\), we have
\[
\left| C_C(u) + C_D(u) \right| = \begin{cases} 
\frac{2}{5} \cdot 2^{m+1}, & \text{for } u = 0; \\
\frac{1}{10} \cdot 2^m, & \text{for } u = 2^{r(0)} + 2^{r(1)}; \\
0, & \text{otherwise.}
\end{cases}
\quad (20)
\]
Also, for \(d_0 \neq d_1\) and \(u \geq 0\), we have
\[
\left| C_C(u) + C_D(u) \right| = \begin{cases} 
\frac{6}{5} \cdot 2^{m+1}, & \text{for } u = 0; \\
\frac{1}{10} \cdot 2^m, & \text{for } u = 2^{r(0)} - 2^{r(1)}; \\
0, & \text{otherwise.}
\end{cases}
\quad (21)
\]

3) The third offset class is in the form of
\[
s(x) = 2x_{x(0)x(1)} + 2x_{x(1)x(2)} + e_0x_{x(0)} + e_1x_{x(1)} + e_2x_{x(2)} + f,
\quad (22)
\]
where \((e_0, e_1, e_2, f)\) is given in Table III.

TABLE III: Coefficients of The Third Offset Class

<table>
<thead>
<tr>
<th>No.</th>
<th>((e_0, e_1, e_2, f))</th>
<th>No.</th>
<th>((e_0, e_1, e_2, f))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1, 0, 3, 1)</td>
<td>5</td>
<td>(3, 0, 1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>(1, 0, 3, 3)</td>
<td>6</td>
<td>(3, 0, 1, 3)</td>
</tr>
<tr>
<td>3</td>
<td>(1, 2, 3, 0)</td>
<td>7</td>
<td>(3, 2, 1, 0)</td>
</tr>
<tr>
<td>4</td>
<td>(1, 2, 3, 2)</td>
<td>8</td>
<td>(3, 2, 1, 2)</td>
</tr>
</tbody>
</table>

With this offset class, we obtain Type-III 16-QAM ACP (C, D). Specifically, for \(u \geq 0\), we have
\[
\left| C_C(u) + C_D(u) \right| = \begin{cases} 
2^{m+1}, & \text{for } u = 0; \\
\frac{2^{m+1}}{5}, & \text{for } u = 2^{r(1)}; \\
0, & \text{otherwise.}
\end{cases}
\quad (23)
\]

A. Proof of ACPs from the First Class Offset

Denote by \(A, B, \alpha, \beta\) the sequences generated by generalized Boolean functions \(A, B, \alpha, \beta\), respectively. Then, we have
\[
\begin{align*}
C_C(u) &= r_1^2C_A(u) + r_1^2C_B(u) + r_1r_2[C_A,A(u) + C_A,A(u)], \\
C_D(u) &= r_1^2C_B(u) + r_1^2C_B(u) + r_1r_2[C_B,B(u) + C_B,B(u)].
\end{align*}
\quad (24)
\]
By [3, Theorem 3], one can see that A and B form a GCP, i.e.,
\[
C_A(u) + C_B(u) = 0, \quad \forall u \neq 0.
\quad (25)
\]
Therefore, we have
\[
C_C(u) + C_D(u) = r_1^2[C_A(u) + C_B(u)] + r_1r_2[C_A,A(u) + C_A,A(u) + C_B,B(u) + C_B,B(u)],
\quad (26)
\]
for any $u > 0$ and
\[
C_C(0) + C_D(0) = r_1^2 [C_A(0) + C_B(0)] + r_2^2 [C_A(0) + C_B(0)]
\]
\[+ r_1 r_2 [C_{A,A}(0) + C_{A,B}(0) + C_{B,B}(0)] = 2^{m+1} + r_1 r_2 [C_{A,A}(0) + C_{A,B}(0) + C_{B,B}(0)].
\]
(27)

By (26) and (27), we just need to calculate $C_A(u) + C_B(u)$ and $C_{A,A}(u) + C_{A,B}(u) + C_{B,B}(u)$. Note that
\[
C_A(u) + C_B(u) = \sum_{i=0}^{2^m-1} [\xi^{s_i} + \xi^{s_j}] \cdot [1 + (-1)^{s_{i\pi(m-1)} + s_{j\pi(m-1)}].
\]
(28)

Similar to [13, (23)], the cross-term in (26) can be expressed as
\[
C_{A,A}(u) + C_{A,B}(u) + C_{B,B}(u) + C_{A,B}(u) = \sum_{i=0}^{2^m-1} [\xi^{s_i} + \xi^{s_j}] \cdot [1 + (-1)^{s_{i\pi(m-1)} + s_{j\pi(m-1)}].
\]
(29)

Next, let $(i_0, i_1, \cdots, i_m)$ and $(j_0, j_1, \cdots, j_m)$ be the binary representations of $i$ and $j$, respectively, where $j = i + u$ and $1 \leq u \leq 2^m - 1$. Given the permutation $\pi$, a binary vector $(i\pi(0), i\pi(1), \cdots, i\pi(m-1))$ and $(j\pi(0), j\pi(1), \cdots, j\pi(m-1))$ can be obtained. For ease of presentation, we call them the binary permutations of $i$ and $j$, respectively. Suppose that $\nu$ is the largest index for which $i\pi(\nu) \neq j\pi(\nu)$, i.e.,
\[
(j\pi(0), \cdots, j\pi(\nu-1), j\pi(\nu), j\pi(\nu+1), \cdots, j\pi(m-1)) = (j\pi(0), \cdots, j\pi(\nu-1), 1 - i\pi(\nu), i\pi(\nu+1), \cdots, i\pi(m-1))
\]
(30)

It is interesting to note that we only need to consider $\nu \leq m - 2$ because in the case of $\nu = m - 1$, $[1 + (-1)^{s_{i\pi(m-1)} + s_{j\pi(m-1)}]}$ in (28) and (29) reduces to zero. With this, $C_A(u) + C_B(u)$ and $C_{A,A}(u) + C_{A,B}(u) + C_{B,B}(u)$ can be simplified to
\[
C_A(u) + C_B(u) = 2 \sum_{i=0}^{2^m-1} [\xi^{s_i} + \xi^{s_j}] \cdot [1 + (-1)^{s_{i\pi(m-1)} + s_{j\pi(m-1)}}].
\]
(31)

and
\[
C_{A,A}(u) + C_{A,B}(u) + C_{B,B}(u) + C_{A,B}(u) = 2 \sum_{i=0}^{2^m-1} [\xi^{s_i} + \xi^{s_j}] \cdot [1 + (-1)^{s_{i\pi(m-1)} + s_{j\pi(m-1)}}].
\]
(32)

Now consider another pair of integers $i'$ and $j'$, whose binary permutations are
\[
(i\pi(0), \cdots, i\pi(\nu-1), i\pi(\nu), \cdots, i\pi(m-1))
\]
\[= (j\pi(0), \cdots, j\pi(\nu-1), j\pi(\nu), 1 - i\pi(\nu), \cdots, i\pi(m-1)),
\]
\[= (j\pi(0), \cdots, j\pi(\nu-1), j\pi(\nu), 1 - j\pi(\nu+1), \cdots, 1 - j\pi(m-1))
\]
\[= (i\pi(0), \cdots, i\pi(\nu-1), i\pi(\nu), 1 - j\pi(\nu+1), \cdots, 1 - i\pi(m-1)).
\]
(33)

respectively. Note that the binary permutations of $i'$ and $j'$ take the complements of that of $i$ and $j$ at positions $\{\pi(q) : q \geq \nu + 1\}$, respectively. Clearly, $i' \neq i$, $j' \neq j$, $i' - j' = j - i = u$. Let
\[
I_u = \{i : 0 \leq i \leq 2^m - u - 1, i\pi(\nu-1) = j\pi(\nu-1)\}
\]
\[I_u' = \{i' : 0 \leq i' \leq 2^m - u - 1, i'\pi(\nu-1) = j'\pi(\nu-1)\}.
\]
(34)
The following two identities are given in [15, p. 2483] and [13, (28)], respectively.
\[
I_u' = I_u,
\]
\[\xi^{A_i - A_j} = -\xi^{A_i - A_j}'.
\]
(36)

To proceed, our key idea is to partition the set $\{i, j : i\pi(m-1) = j\pi(m-1)\}$ into $Q_1$ and $Q_2$ which enable simplification. Specifically, define
\[
Q_1 := \{(i, j) : i\pi(m-1) = j\pi(m-1), i\pi(2) \neq j\pi(2)\}
\]
and
\[
Q_2 := Q_{21} \cup Q_{22},
\]
\[
Q_{21} := \{(i, j) : i\pi(m-1) = j\pi(m-1), i\pi(0) = j\pi(0), i\pi(2) = j\pi(2)\}
\]
\[
Q_{22} := \{(i, j) : i\pi(m-1) = j\pi(m-1), i\pi(0) \neq j\pi(0), i\pi(2) = j\pi(2)\}
\]
(38)

Consider the $(i', j')$ pair defined in (33). It is clear that $\xi^{s_{i'} - s_j} = \xi^{\nu - s_{j'}}$ for $\nu \geq 2$. Thus, by (35) and (36), we have
\[
\sum_{(i, j) \in Q_1} \xi^{s_{i'} - s_j} \cdot \xi^{A_i - A_j} = 0.
\]
(39)

Similarly, we have
\[
\sum_{(i, j) \in Q_1} \xi^{s_{i'} - s_j} \cdot \xi^{A_i - A_j} = 0.
\]
(40)

Hence, the sums in (31) and (32) only depend on the following six cases.

Case 1: $(i, j) \in Q_1, \nu \in \{0, 1\}$;

Case 2: $(i, j) \in Q_{22}, \nu = 1$;

Case 3: $(i, j) \in Q_{22}, \nu = 1$;

Case 4: $(i, j) \in Q_{22}, \nu = 1$;

Case 5: $(i, j) \in Q_{21}, \nu = 0$;
Case 6: \((i, j) \in \mathbb{Q}_{22}, \nu = 0\).
In fact, one just needs to deal with Cases 3, 4, 6 because Cases 1, 2, 5 are impossible as the value of \(\nu\) contradicts with the range of \((i, j)\).

For Case 3, the binary permutations of \(i\) and \(j\) are
\[
(\{0\}, 0, i_0(2), \ldots, i_0(m-1)),
(\{1\}, 0, i_0(2), \ldots, i_0(m-1)),
\]
respectively, for \(j > i\). Hence,
\[
\sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} \xi^{i_0 - s_j} \cdot \xi^{A_1 - A_j} = \sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} \xi^{di_{\pi}(0) + e_1 + 2i_{\pi}(2) - c_1} = 0.
\]

For Case 4, the binary permutations of \(i\) and \(j\) are
\[
(\{0\}, 0, i_0(1), i_0(2), \ldots, i_0(m-1)),
(1 - i_0(0), 1 - i_0(1), i_0(2), \ldots, i_0(m-1)),
\]
respectively, for \(j > i\). Hence,
\[
\sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} \xi^{i_0 - s_j} \cdot \xi^{A_1 - A_j} = \sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} \xi^{(d+2)i_{\pi}(0) + e_1 + 2i_{\pi}(2) - c_1} = 0.
\]

where the last step is obtained by splitting the sum for \(i_{\pi}(2) = 0\) and \(i_{\pi}(2) = 1\), similar to the technique used in (42).

For Case 6, the binary permutations of \(i\) and \(j\) are
\[
(0, i_0(1), i_0(2), \ldots, i_0(m-1)),
(1, i_0(1), i_0(2), \ldots, i_0(m-1)),
\]
respectively, for \(j > i\). Hence,
\[
\sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 0} \xi^{i_0 - s_j} \cdot \xi^{A_1 - A_j} = \sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 0} \xi^{(d+2)i_{\pi}(0) + e_0 + 2i_{\pi}(2) - c_0} = 0.
\]

Next, we move to the calculate of \(C_{A,\alpha}(u) + C_{A,u}(u) + C_{B,\beta}(u) + C_{B,B}(u)\). It is readily to show that for \(u \in \{1, 2, \ldots, 2^m - 1\}\)
\[
\sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} \xi^{i_0 - s_j} \cdot \xi^{A_1 - A_j} = \sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} \xi^{s_j - s_i} \cdot \xi^{A_1 - A_j} = 0.
\]

By (32) and (48), \(C_{A,\alpha}(u) + C_{A,u}(u) + C_{B,\beta}(u) + C_{B,B}(u)\) equals to 0 for \(u \in \{1, 2, \ldots, 2^m - 1\} \setminus \{2^0(0)\}\)

for \(u = 2^\sigma(0)\).

In the case of \(u = 0\), by (32), it follows that
\[
C_{A,\alpha}(0) + C_{A,u}(0) + C_{B,\beta}(0) + C_{B,B}(0) = 2 \sum_{i_{\pi}(m-1) = J_{\pi}(m-1)} \xi^{s_i - s_j} \cdot \xi^{A_1 - A_j}.
\]

By substituting \(s_i = di_{\pi}(0) + e_0i_{\pi}(0) + e_1i_{\pi}(1) + f\) into (50), we have
\[
C_{A,\alpha}(0) + C_{A,u}(0) + C_{B,\beta}(0) + C_{B,B}(0) = 4 \sum_{i_{\pi}(m-1) = J_{\pi}(m-1)} \xi^{s_i} \cdot \xi^{A_1 - A_j} = 0.
\]

Thus,
\[
C_{A,\alpha}(u) + C_{A,u}(u) + C_{B,\beta}(u) + C_{B,B}(u)
\]
\[
= 2^{m-1} \cdot \xi^{s_0} \cdot \xi^{A_1 - A_j}.
\]

Substituting (47) and (52) into (26) or (27) (according to \(u > 0\) or \(u = 0\)), we obtain
\[
|C_C(u) + C_B(u)|
\]
\[
= \begin{cases} 0, & \text{for } u = 0; \\ 2^{m-1} + r_1 r_2 2^m |\xi^{s_0} + \xi^{s_1 + f} + \xi^{e_0 + e_1 + f} + \xi^{d + e_0 + e_1 + f}|, & \text{for } u \geq 2^0; \\ 2^{m-1} r_2 |\xi^{s_0} + \xi^{s_1 + f} + \xi^{e_0 - e_1 - f} + 2 \xi^{e_0 + e_1 + f} + 2 \xi^{d + e_0 + e_1 + f}|, & \text{for } u = 2^0; \\ 0, & \text{otherwise}. \end{cases}
\]
By the coefficients \( (d, e_0, e_1, f) \) in Table I, one can show that
\[
\text{Re} \left\{ \xi^f + \xi^{e_0+f} + \xi^{e_1+f} + \xi^{d+e_0+e_1+f} \right\} = \begin{cases} 
0, & \text{for } d = 2; \\
-1, & \text{for } d \in \{1, 3\}, 
\end{cases}
\]
(54)
and
\[
\left| \xi^{-e_0} + \xi^{2-d-e_0} + 2\xi^f + 2\xi^{-e_0-f} + 2\xi^{2+e_1+f} + 2\xi^{2-d-e_0-e_1-f} \right| = \begin{cases} 
2, & \text{for } d = 2; \\
\sqrt{2}, & \text{for } d \in \{1, 3\}. 
\end{cases}
\]
(55)
By (53)–(55), the proof of (17) and (18) follows.

\[ \text{B. Proof of ACPs from the Second Class Offset} \]

We closely follow the proof for the first offset class. To prove \((20)\) for \(d_0 = d_1\), we just need to show the following two equations.
\[
C_{\alpha}(u) + C_{\beta}(u) = \begin{cases} 
2^{m+1}, & \text{for } u = 0; \\
2^{m-1} \xi^{2-e_0-e_1-e_0-e_1}, & \text{for } u = 2^{(\pi(0)+\pi(1))}; \\
0, & \text{for } u \in \{1, 2, \cdots, 2^m - 1\} \setminus \{2^{(\pi(0)+\pi(1))}\}. 
\end{cases}
\]
(56)
\[
C_{\alpha,\alpha}(u) + C_{\alpha,\Lambda}(u) + C_{\beta,\beta}(u) + C_{\beta,B}(u) = \begin{cases} 
2^m, & \text{for } u = 0; \\
2^{m-1} \xi^{(c_1-c_0+e_1+f)} \cdot (1 + \xi^{(2(\pi(0)+\pi(1)))}, & \text{for } u = 2^{(\pi(0)-\pi(1))}; \\
0, & \text{for } u \in \{1, 2, \cdots, 2^m - 1\} \setminus \{2^{(\pi(0)-\pi(1))}\}. 
\end{cases}
\]
(62)
Substituting (61) and (62) into (26) or (27) (according to \(u > 0\) or \(u = 0\)), we obtain
\[
\left| C_C(u) + C_D(u) \right| = \begin{cases} 
2^{m+1} + r_1^2 2^m, & \text{for } u = 0; \\
2^{m-1} r_2^2 \xi^{e_0+e_1+e_0} + 2\xi^f + 2\xi^{2+e_1+f}, & \text{for } u = 2^{(\pi(0)+\pi(1))}; \\
0, & \text{otherwise}. 
\end{cases}
\]
(63)
By coefficients \((d_0, d_1, e_0, e_1, e_2, f)\) in Table II, one can show that
\[
\left| \xi^{-e_0-e_1} + 2\xi^f + 2\xi^{2+f+e_2} \right| = \sqrt{5}, \quad \text{for } d_0 \neq d_1. 
\]
(64)
By (31) and (65), \(C_{\alpha}(u) + C_{\beta}(u)\) reduces to
\[
2 \sum_{(i,j) \in \mathbb{Q}_{22}, \nu=1} \xi^{s_i} \cdot \xi^{\eta_A} 
= 2 \sum_{(i,j) \in \mathbb{Q}_{22}, \nu=1} \xi^{(2e_0+e_1+e_0)\xi_e + (2e_0+e_1+e_0)\xi_{e_1} + (2e_0+e_1+e_0)\xi_{e_2}} 
= \begin{cases} 
2^{m-1} \xi^{2-e_0-e_1-e_0}, & \text{for } i(\pi(0) + i(\pi(1)) = 1; \\
0, & \text{for } i(\pi(0) = i(\pi(1)). 
\end{cases}
\]
(66)
where the non-zero sum occurs at \(u = 2^{(\pi(0)+\pi(1))}\). Thus, the proof of (56) follows.

Next, we deal with the cross-term in (57). For \(u \in \{1, 2, \cdots, 2^m - 1\}\), it is shown in Appendix A that
\[
\sum_{(i,j) \in \mathbb{Q}_{22}, \nu=1} \left[ \xi^{s_i} + \xi^{-s_j} \right] \cdot \xi_A \cdot \xi_A = 0. 
\]
(67)
In the case of \((i, j) \in \mathbb{Q}_{22}, \nu = 1\), one can show that
\[
\xi^{s_i} + \xi^{-s_j} = \xi^{d_0(\pi(0)+\pi(1))\eta_e} + \xi^{c_0(\pi(0)+\pi(1))c_0} + \xi^{e_0(\pi(0)+\pi(1))e_0} 
\]
(68)
where the last step is obtained by the fact that $d_0 + e_2 = e_0 + e_1 + 2f = 0$, as shown in Table II for $d_0 = d_1$.

By (32), (67) and (68), $C_{A,\alpha}(u) + C_{\alpha, A}(u) + C_{B,\beta}(u) + C_{\beta, B}(u)$ reduces to

$$2 \cdot \sum_{(i,j) \in \mathbb{Q}_{2^2, \nu = 1}^2} [\xi^{s_i} + \xi^{-s_i}] \cdot \xi^{A_i - A_j}$$

$$= 4 \cdot \sum_{(i,j) \in \mathbb{Q}_{2^2, \nu = 1}^2} \xi^{d(i_\pi(0) + i_\pi(1))(i_\pi(2)) + e_0 i_\pi(0) + e_1 i_\pi(1)}$$

$$\cdot \xi^{2i_\pi(2) + f} + \xi^{2 + 2c_0} \xi^{(0)} + (2 + 2c_1) i_\pi(1) + 2i_\pi(2) + 2 - c_0 - c_1$$

$$= 4 \cdot \sum_{(i,j) \in \mathbb{Q}_{2^2, \nu = 1}^2} \xi^{(d(1_i(0) + d(i_{\pi}(1) + e_{2} + 2i_\pi(2)) + c_0 - c_1)}$$

$$\cdot \xi^{(2 + 2c_0 + e_0) i_\pi(0) + (2 + 2c_1 + e_1) i_\pi(1) + 2 - c_0 - c_1 + f}.$$  

(69)

Again, note that $d_0 + e_2 = 0$. Thus, $(d_0 i_\pi(0) + d_0 i_\pi(1) + e_2 + 2)$ $i_\pi(2)$ in (69) is equal to $2i_\pi(2)$ if $i_\pi(0) + i_\pi(1) = 1$. As a result, $C_{A,\alpha}(u) + C_{\alpha, A}(u) + C_{B,\beta}(u) + C_{\beta, B}(u)$ simplifies to

$$2 \cdot \sum_{(i,j) \in \mathbb{Q}_{2^2, \nu = 1}^2} [\xi^{s_i} + \xi^{-s_i}] \cdot \xi^{A_i - A_j}$$

$$= \left\{ \begin{array}{ll}
2^{m-1} \cdot (2^{c_0 - c_1} - f) \cdot [1 + \xi^{2c_2}] & \text{for } i_\pi(0) + i_\pi(1) = 1; \\
0 & \text{for } i_\pi(0) = i_\pi(1) = 0,
\end{array} \right.$$  

(70)

where the non-zero cross-term value occurs at $u = 2\pi(0) + 2\pi(1)$. Thus, the proof of (57) for $u > 0$ follows.

In the end, it is shown in Appendix B that

$$C_{A,\alpha}(0) + C_{\alpha, A}(0) + C_{B,\beta}(0) + C_{\beta, B}(0) = 2^m.  \quad (71)$$

Therefore, we complete the proof of (57) for $u = 0$.

C. Proof of ACPs from the Third Class Offset

To prove (23), it is equivalent to show the following two equations.

$$C_{\alpha}(u) + C_{\beta}(u)$$

$$= \left\{ \begin{array}{ll}
2^{m+1} & \text{for } u = 0; \\
2^{m-1} \cdot (\xi^{-e_1 - c_1}) & \text{for } u = 2\pi(1); \\
0 & \text{for } u \in \{1, 2, \ldots , 2^m - 1\} \setminus \{2^\pi(1)\},
\end{array} \right.$$  

(72)

$$C_{A,\alpha}(u) + C_{\alpha, A}(u) + C_{B,\beta}(u) + C_{\beta, B}(u) = 0, \quad (73)$$

for any integer $u$. The proof of (72) and (73) can be obtained easily by following a similar reasoning as that of (17)–(21) and by taking advantage of the following identities

$$e_0 + e_2 = 0, \quad e_0 = e_2 + 2, \quad \text{and} \quad e_0 + e_1 + e_2 + 2f = 2.$$

Hence, it is omitted. Substituting (72) and (73) into (26) or (27) (according to $u > 0$ or $u = 0$), we arrive at (23).

D. Discussions and Example

Denote by “Type-I, Type-II, Type-III” the 16-QAM ACSs arising from the proposed three offset classes, respectively. Next, we define the following three codebooks.

$$S_1 := \{ C \in \text{Type-I with } d = 2 \} \cup \{ C \in \text{Type-III}\};$$

$$S_2 := \{ C \in \text{Type-I with } d = 1 \text{ or } 3 \};$$

$$S_3 := \{ C \in \text{Type-II}\}. \quad (74)$$

One can check that each codebook in (74) satisfies the conditions in Lemma 2 and hence, its PMEPR can be upper bounded by the PMEPR upper bound of any codeword in this codebook. Specifically, by the correlation distributions of these 16-QAM ACPs, we have

**TABLE IV: PMEPRs and Enumerations for Codebooks in (74)**

<table>
<thead>
<tr>
<th>Codebook</th>
<th>PMEPR upper bound</th>
<th>Total Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_1$</td>
<td>2.4</td>
<td>16(m/2)^{4m+1}</td>
</tr>
<tr>
<td>$S_2$</td>
<td>2 + $\sqrt{2}/4 \approx 2.3636$</td>
<td>16(m/2)^{4m+1}</td>
</tr>
<tr>
<td>$S_3$</td>
<td>2 + $\sqrt{5}/6 \approx 2.3727$</td>
<td>8(m/2)^{4m+1}</td>
</tr>
</tbody>
</table>

We note that our proposed 16-QAM ACPs also belong to the family of ZCP, where ZCP is formally defined as follows.

**Definition 2:** C and D form a ZCP with zero-correlation zone (ZCZ) width of Z if and only if

$$C_C(u) + C_D(u) = 0, \quad \text{for any } 1 \leq u \leq Z - 1. \quad (75)$$

By the AASs shown in (17)-(18), (20)-(21), and (23), we give the following remark.

**Remark 3:** The three types of 16-QAM ACPs constructed in Sections III-A–C are ZCPs with maximum ZCZ width of $2^{m-1} + 2^{m-2}$, where $2^m$ is the sequence length.

We give a 16-QAM ACP with maximum ZCZ by the following example.

**Example 1:** Let $m = 4, \pi = (3, 2, 0, 1), (c_0, c_1, c_2, c_3, c) = (2, 1, 2, 0, 0)$. Consider a 16-QAM ACP from the second offset class with $(d_0, d_1, e_0, e_1, e_2, f) = (1, 1, 2, 2, 3, 2)$. Then, we have

$$A_i = 2i_\pi(0)i_\pi(1) + 2i_\pi(1)i_\pi(2) + 2i_\pi(2)i_\pi(3) + 2i_\pi(0) + i_\pi(1) + 2i_\pi(2);$$

$$\alpha_i = 2i_\pi(0)i_\pi(1) + 3i_\pi(1)i_\pi(2) + 2i_\pi(2)i_\pi(3) + i_\pi(0) + i_\pi(2) + 2;$$

$$B_i = A_i + 2i_\pi(3), \quad \beta_i = \alpha_i + 2i_\pi(3).$$  

(76)

where $i \in \{0, 1, \ldots , 15\}$. Here, $(i_0, i_1, i_2, i_3)$ stands for the binary representation of $i$, whereas $(i_\pi(0), i_\pi(1), i_\pi(2), i_\pi(3))$ denotes the binary permutation of $i$ with $\pi$ being the permutation vector. In this example, $(i_\pi(0), i_\pi(1), i_\pi(2), i_\pi(3)) = (i_3, i_2, i_0, i_1)$. We note that there exists an efficient hardware generator for $A_i, \alpha_i, B_i, \beta_i$ using the property of generalized Boolean functions over $\mathbb{Z}_4$. We have shown such hardware generator for $A_i, \alpha_i$ in Fig. 1. Furthermore, substituting $A_i, \alpha_i, B_i, \beta_i$ into

$$C_i = \gamma(r_1 \xi^{A_i} + r_2 \xi^{\alpha_i}, \quad D_i = \gamma(r_1 \xi^{B_i} + r_2 \xi^{\beta_i}), \quad$$

where

$$\gamma$$

is the correlation function.
\[ C = \frac{\exp\left(\frac{-\sqrt{2m}}{\sqrt{5}}\right)}{\sqrt{5}} \left(1, -2 - \xi, 1, 2 + \xi, 3\xi, 3\xi, -3\xi, -3, -3, -3, -3, -3, -3, -3, -3, -3, -3, 1 + 2\xi, \xi, -1 + 2\xi\right), \]
\[ D = \frac{\exp\left(\frac{-\sqrt{2m}}{\sqrt{5}}\right)}{\sqrt{5}} \left(1, -2 - \xi, -1, -2 - \xi, 3\xi, 3\xi, -3\xi, -3, 3, 3, 3, 3, 3, 1 + 2\xi, -\xi, 1 + 2\xi\right). \]  

we obtain \((C, D)\) shown in (77). It can be shown that
\[
\frac{1}{5} \left(192, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -16 - 8\xi, 0, 0, 0\right),
\]
where the non-zero out-of-phase AAS occurs at \(u = 2\pi(0) + 2\pi(1) = 12\). See Fig. 2.a for the plot of AASs of \((C, D)\).

By Lemma 1, we have
\[
PMEPR(C) \leq \frac{C \ast D}{2^m + r_1 r_2 2^{m-1}} = \frac{192 + 2 \times 8 \times \sqrt{5}}{6 \cdot 16} = 2 + \sqrt{5}/6 \approx 2.3727.
\]
From Fig. 2.b, one can see that \(PMEPR(C) = 2.2273\), \(PMEPR(D) = 2.3449\) which are well upper bounded by \(2 + \sqrt{5}/6\).

Finally, we compared our proposed codebook \((i.e., S_1 \cup S_2 \cup S_3)\) in Table V with the codebooks in [12], [13] and [22]. Note that the code rate of a code-keying OFDM is defined as \(R(C) \coloneqq \frac{\log_2 |C|}{L}\), where \(|C|\) denotes the set size of codebook \(C\), \(L\) the number of subcarriers. Compared to the codebook in [22] which has the second lowest PMEPR value in Table V, our proposed codebook has a larger set size, hence higher code rate, but the lowest PMEPR value in Table V. Moreover, we have compared in Table VI the code-rate of our proposed codebook with that of the codebooks in [13], (31)] and [22] for \(L \in \{4, 8, 16, 32, 64, 128, 256, 512, 1024\}\). A common point of these three codebooks is that they all have PMEPRs not greater than 3. One can see that our proposed codebook achieves a code-rate comparable to that in [13], (31)]. Although the code-rates drop quickly for larger \(L\), they may be suitable for OFDM systems with smaller number of subcarriers or for pilot design as mentioned in [13].

IV. CONCLUSIONS

This paper is devoted to the constructions of 16-QAM ACPs, each having zero out-of-phase AASs except at time-shifts of \(\pm \tau\), where \(\tau\) is a certain positive time-shift. Such ACPs may serve as an alternative to GCPs in many applications. By three novel classes of quadratic offsets, we have constructed three types of 16-QAM ACPs with maximum ZCZ width of \(2^{m-1} + 2^{m-2}\), giving rise to 16-QAM ACSs with maximum PMEPR of 2.4. By properly arranging the obtained 16-QAM sequences into three codebooks, \(i.e., S_1, S_2, S_3\), we have shown in Table IV that they possess PMEPR upper bounds of \(2.4, 2 + \sqrt{2}/4 \approx 2.3636, 2 + \sqrt{5}/6 \approx 2.3727\), respectively.

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APPENDIX A

PROOF OF (67)

1) \((i, j) \in 2\nu, \nu = 1\). In this case, one can show that
\[
\xi^{s_i} + \xi^{-s_j} = \xi^{d_0 i_\nu(0) + d_1 i_\nu(2)} + \xi^{d_0 i_\nu(0) + e_0 i_\nu(0) + 2 e_2 i_\nu(2)} + f
\]
\[+ \xi^{d_0 i_\nu(0) + d_1 i_\nu(2) - e_0 i_\nu(0) - e_2 i_\nu(2) - e_1 - f} \]

By expanding \(\sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{s_i} + \xi^{-s_j}] \cdot \xi^{A_i - A_j}\) for \((i, j) \in 2\nu, \nu = 1\), we have
\[
\sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{s_i} + \xi^{-s_j}] \cdot \xi^{A_i - A_j}
\]
\[= \sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{s_i} + \xi^{-s_j}] \cdot \xi^{c_1 + 2 i_\nu(0) + 2 i_\nu(2)}
\]
\[= \sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{e_2 i_\nu(2) + f} + \xi^{-d_1 i_\nu(2) - e_2 i_\nu(2) - e_1 - f}]
\]
\[= \xi^{c_1}, \sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{f} + \xi^{-e_2 - f} + \xi^{e_2 + f} + \xi^{-d_1 - e_2 - e_1 - f}]
\]
\[= \xi^{c_1}, \sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{f} + \xi^{e_2 + f} + \xi^{-e_2 - e_1 - f}]
\]
\[= \xi^{c_1}, \sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{e_2 + f} + \xi^{-e_2 - e_1 - f}]
\]
\[= \xi^{c_1}, \sum_{(i, j) \in 2\nu, \nu = 1} [\xi^{e_2 + f} + \xi^{-e_2 - e_1 - f}]
\]
\[= 0.\]
Also, since \(e_0 + e_1 + 2f + d_0 + d_1 + 2e_2 = 0\), we have

\[
\begin{align*}
\xi_0^{d_0+e_0+e_2+f} + \xi^{-d_0-d_1-e_0-e_1-e_2-f} \\
- \xi^{d_0+e_0+e_2+f} - \xi^{-d_0-d_1-e_0-e_1-e_2-f} \\
= \xi^{-d_0-d_1-e_0-e_1-e_2-f} - \xi^{e_2+f} \\
+ \xi^{d_0+e_0+e_2+f} - \xi^{-d_1-e_2-e_1-f} & \quad \text{(81)}
\end{align*}
\]

\(= 0\).

Substituting (80) and (81) into (79), we arrive at

\[
\sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 1} [\xi^{s_i} + \xi^{-s_j}] \cdot \xi^{A_i-A_j} = 0.
\]

2) \((i, j) \in \mathbb{Q}_{22}, \nu = 0\). In this case, one has

\[
\begin{align*}
\xi^{s_i} + \xi^{-s_j} = & \xi^{d_1i_{\pi(1)}i_{\pi(2)} + e_1i_{\pi(1)} + e_2i_{\pi(2)} + f} \\
& + \xi^{-d_1i_{\pi(1)}i_{\pi(2)} - e_1i_{\pi(1)} - (d_0 + e_2)i_{\pi(2)} - e_0 - f} & \quad \text{(82)}
\end{align*}
\]

By expanding \(\sum_{(i,j) \in \mathbb{Q}_{22}, \nu = 0} [\xi^{s_i} + \xi^{-s_j}] \cdot \xi^{A_i-A_j}\) for \((i_{\pi(1)}, i_{\pi(2)}) \in \mathbb{Z}_2^2\), we obtain (83) which completes the proof.
where the first and the second Re \{ \cdot \} are obtained for \( i_{\pi(2)} = 0 \) and \( i_{\pi(2)} = 1 \), respectively. We now proceed with the discussions in the following cases.

1) \( e_0 = 2 \). In this case, according to Table II, we have
\[ e_0 = e_1 = f = 2, \quad d_0 + d_1 = 2, \quad d_0 + e_2 = d_1 + e_2 = 0. \]

Therefore,
\[
\text{Re} \left\{ \xi f + \xi e_0 f + \xi e_1 f + \xi e_0 + e_1 f \right\} = 0,
\]

and
\[
\text{Re} \left\{ \xi e_2 f + \xi d_0 + e_0 + e_2 + f \right\} = 2.
\]

By (84)–(86), the proof of (71) follows.

2) \( e_0 \in \{1, 3\} \). In this case, according to Table II, we have
\[ e_0 + d_0 = e_0 + e_1 = 2, \quad e_0 + f = e_1 + f = 0. \]
Therefore,
\[
\Re \left\{ e^{2\pi f t} + e^{2\pi d_0 t + e_0 t + e_2 t + \frac{f^2}{2}} + e^{2\pi d_1 t + e_1 t + e_2 t + \frac{f^2}{2}} \right\} = 0
\]
and
\[
\Re \left\{ e^{2\pi f t} + e^{2\pi e_0 t + e_1 t + e_2 t + \frac{f^2}{2}} + e^{2\pi e_0 t + e_1 t + e_2 t + \frac{f^2}{2}} \right\} = 2.
\]

By (84) and (87)–(88), the proof of (71) follows.

REFERENCES