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CEV Asymptotics of American options

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Abstract

The constant elasticity of variance (CEV) model is a practical approach to option pricing by fitting to the implied volatility smile. Its application to American-style derivatives, however, poses analytical and numerical challenges. By taking the Laplace-Carson transform (LCT) to the free-boundary value problem characterizing the option value function and the early exercise boundary, the analytical result involves confluent hyper-geometric functions. Thus, the numerical computation could be unstable and inefficient for certain set of parameter values. We solve this problem by an asymptotic approach to the American option pricing problem under the CEV model. We demonstrate the use of the proposed approach using perpetual and finite-time American puts.

Key words: CEV model, American options, Partial differential equation, Perturbation technique

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1. Introduction

American options are options that can be exercised at any time prior to maturity. The early exercise feature makes American options more valuable and

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attractive in the financial market. However, the optimal exercise boundary (strategy) also makes the valuation of American options a highly challenging problem in finance because the corresponding valuation is related to an optimal stopping problem in probability or a free-boundary value problem in partial differential equations (PDEs). Apart from American calls and puts, many financial products essentially belong to the class of American options. For instance, a stock loan can be transformed into a perpetual American call option [1]. Russian option or no-regret option is the nickname of the perpetual American lookback option [2]. Therefore, the valuation of American options is the central problem in this paper.

Empirical studies suggest that the Black-Scholes (BS) model is inadequate to explain the volatility smile observed in the financial market. One popular alternative is the constant elasticity of variance (CEV) model, introduced by Cox [3], which captures the volatility smile through a power of the underlying asset price as the local volatility of the asset price process. The CEV model has stimulated many interesting studies since [3]. For instance, Emanuel and MacBeth [4] derive a closed-form solution for European options under the CEV model. Davydov and Linetsky [5] apply the Laplace transform to obtain analytical formulas for the prices of several path-dependent options under the CEV model. Wong and Zhao [6] construct an artificial boundary finite difference method to compute American options under the CEV model. By means of homotopy analysis method, Zhao and Wong [7] derive a closed-form solution for American option prices under general diffusion which nests the CEV model. However, the implementation of the homotopy solution requires a complicated iterative integration. Wong and Zhao [8] derive closed-form solutions for American put under the CEV model by taking the Laplace-Carson transform (LCT).

Although there are several analytical results on American option pricing under CEV in the literature, their numerical use is far from being satisfactory. Specifically, the pricing formula in [8] requires the computation of confluent hypergeometric functions followed by a Laplace inversion. The numerical implementation is hardly made in a stable and efficient manner. For instance, we experience that their analytical formula fails to produce a numerical output when the elasticity of variance is negative and close to zero. This motivates us to apply the perturbation technique to the free-boundary value problems of American options with respect to the elasticity of variance.

Our approach is based on Park and Kim [9], who propose an asymptotic PDE approach to stabilize the numerical computation of path-dependent options under CEV. An asymptotic analysis of a similar model but in a different limit is also addressed in [10]. Similar asymptotic PDE approach also appears in the literature of stochastic volatility model such as [11] and reference therein. However, these studies only consider European-style contract without taking into account the optimal exercise decision.

Recently, Wong and Wong [12] construct an asymptotic approach for stock loan under the fast mean-reverting stochastic volatility model. As the valuation of stock loan involves the optimal exercise boundary, their asymptotic analysis contains two asymptotic expansions for the value function and the optimal exercise boundary, respectively. These two expansion formulas are simultaneously solved from the free-boundary value problem governing the stock loan. Inspired by them, this paper combines the approaches of [9] and [12] to American option pricing problems under CEV. We investigate both perpetual and finite-time American options.

For finite maturity American options, we first apply the LCT to the option value function and the early exercise boundary as proposed by [13] and [8]. The problem is then transformed to the one related to perpetual American options. Under the assumption of a small value of elasticity of variance, we derive the analytical recursive formulas for each order of approximation in the asymptotic expansion. We show numerically that the asymptotic formulas are well behaved and offer reasonable numerical values when traditional analytical techniques fail to do so.

The remainder of this paper is organized as follows. Section 2 introduces the CEV model and the free-boundary value problem for American options. Section 3 presents the asymptotic solutions of American option price and its optimal exercise boundary. Section 4 provides numerical examples to illustrate the asymptotic solutions. Section 5 concludes this paper and suggests possible future works.

2. Problem formulation

2.1. The CEV model

The CEV model is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the probability measure $\mathbb{P}$ is the market-implied risk-neutral probability, and the filtration $\mathcal{F}$ is the σ-field generated by the risk-neutral process $\{S_s\}_{0\leq s\leq t}$ that satisfies the stochastic differential equation (SDE): $\frac{dS_t}{S_t} = (r-q)\,dt+\delta S_t^{\beta}\,dW_t$, where r is risk-free interest rate, q is the dividend yield, and W_t is the Wiener process. The parameter β is known as the elasticity of variance ($\frac{S}{\sigma}\frac{d\sigma}{dS} = \beta$) while

δ is the scale parameter fixing the initial instantaneous volatility ($\sigma_0 = \delta S_0^{\beta}$), where the local volatility σ is the diffusion coefficient.

As stated in [5], the CEV model nests several financial models as its special cases, such as the BS model and the square-root volatility model. The value of β controls the slope of the volatility smile implied by European options. Empirical studies suggest that β is usually negative and close to zero for the fact that implied volatility smiles are usually downward sloping. When $\beta = 0$, the CEV model reduces to the BS model. Hence, our asymptotic expansion is perturbed around the BS solution.

The put-call symmetry of American options enables the American call pricing formula to be inferred by its put counterpart. Thus, we focus on American put in this paper. However, the proposed approach is generally useful for a wide range of American options on single asset.

2.2. *Perpetual American put*

To simplify matters, we begin with the perpetual American put. The no-arbitrage price of this option is given as follows.

$$V(S) = \operatorname{ess\,sup}_{\theta \in \mathcal{T}_\infty} \mathbb{E}[e^{-r\theta}(K - S_\theta)^+ | S_t = S] \tag{1}$$

where $(\bullet)^+ = \max(\bullet, 0)$, and $\mathcal{T}_\infty$ is the set of stopping times. Specifically, $\mathcal{T}_\infty = \inf\{\xi > t | V(S_\xi) \leq K - S_\xi\}$, which can be shown to be equivalent to $\mathcal{T}_\infty = \inf\{\xi > t | S_\xi \leq S^*\}$, where the constant, S^*, is the optimal exercise boundary that maximizes the value function V. Note that all expectations in this paper are taken under the the risk-neutral probability $\mathbb{P}$.

It is well known that the problem in (1) can be formulated as a free-boundary value problem [14] as follows.

$$\begin{aligned}
\frac{1}{2}\delta^2 S^{2\beta+2}\frac{d^2V}{dS^2} + (r-q)S\frac{dV}{dS} - rV &= 0, \quad S \geq S^*, & (2)\\
V(S^*) &= K - S^*, & (3)\\
\frac{dV(S^*)}{dS} &= -1, & (4)\\
\lim_{S\to\infty} V(S) &= 0, & (5)
\end{aligned}$$

where (3), (4) and (5) are the value-matching condition, smooth-pasting condition and far-field boundary condition, respectively. In the region $(S^*, +\infty)$, $V(S) > (K-S)^+$ so that the option should be held rather than exercised and hence it is called the continuation region. Thus, the exercise boundary, S^*, determines the strategy for investors to exercise their options optimally.

2.3. Finite-time American put

When the American put has a maturity date, it becomes the finite-time American put. The price representation of the finite-time American put is given by

$$P(S,t) = \operatorname{ess\,sup}_{\theta\in\mathcal{T}_T} \mathbb{E}[e^{-r\theta}(K - S_\theta)^+ | S_t = S], \tag{6}$$

where $\mathcal{T}_T = \inf\{t \leq \xi \leq T | P(S(\xi), \xi) \leq K - S_\xi\}$. The set of stopping times of the American put can be shown to be $\mathcal{T}_T = \inf\{t \leq \xi \leq T | S_\xi \leq S^*(\xi)\}$, where the deterministic function $S^*(t)$ is the optimal exercise boundary, varying with time. The optimal stopping problem (6) can be transformed into a free-boundary value problem. Let $\tau = T - t$. Then, $(P(S,\tau), S^*(\tau))$ is the solution pair to the free-boundary value PDE:

$$\begin{aligned}
\frac{1}{2}\delta^2 S^{2\beta+2}\frac{\partial^2 P}{\partial S^2}+(r-q)S\frac{\partial P}{\partial S}-rP &= \frac{\partial P}{\partial \tau},\;\; S\geq S^*(\tau), && (7)\\
P(S,0) &= (K-S)^+, && (8)\\
P(S^*(\tau),\tau) &= K-S^*(\tau), && (9)\\
\frac{\partial P(S^*(\tau),\tau)}{\partial S} &= -1, && (10)\\
\lim_{S\to\infty} P(S,\tau) &= 0, && (11)
\end{aligned}$$

where (8) is the initial condition, (9) is the value-matching condition, (10) is the smooth-pasting condition and (11) is the far-field boundary condition for the American put option. This PDE problem resembles the ODE problem (2)-(5) except that a partial derivative with respect to τ appears in (2) and the initial condition is added. It is shown in [7] that $S^*(0^+)=\min(rK/q,K)$.

By taking the Laplace-Carson transform, an analytical solution pair to (7)- (11) is obtained in [8]. However, their analytical option pricing function consists of an inverse Laplace transform on a linear combination of confluent hyper-geometric functions. The numerical computation of the value function and the optimal exercise boundary could be unstable and time consuming for some sets of parameters. This motivates us to investigate asymptotic solutions to the free-boundary problems associated with perpetual and finite-time American options.

3. Asymptotic expansion of American put

3.1. Perpetual American put

Consider the problem of (2)-(5) and the asymptotic expansion for $V(S)$:

$$V(S)=V_0(S)+\beta V_1(S)+\beta^2 V_2(S)+\cdots, \tag{12}$$

where $V_i(S), i = 0, 1, 2, \cdots$, are solved successively until a certain accuracy is attained. We also consider the expansion of the optimal exercise boundary as

$$\ln S^* = X^* = x_0 + \beta x_1 + \beta^2 x_2 + \cdots . \tag{13}$$

We derive finite sums (layers) to approximate $V(S)$ and $\ln S^*$.

For a fixed $N = 0, 1, \ldots$, define the $N-$th boundary layer as

$$\ln S_N^* = X_N^* = x_0 + \beta x_1 + \cdots + \beta^N x_N, \tag{14}$$

and the $N-$th price layer as

$$\bar{V}_N(S) = V_{0,N}(S) + \beta V_{1,N}(S) + \cdots + \beta^N V_{N,N}(S), \tag{15}$$

where the correction terms $V_{n,N}(S)$ are functions based on the same N-th boundary layer $\ln S_N^*$. Thus, the second subscript indicates the corresponding order of the boundary layer. To simplify notation, we often write $V_{n,N} = V_{n,N}(S)$ so that the dependence of S is suppressed.

Our analytical correction terms are derived based on the following lemma.

Lemma 3.1. Suppose that $f(\cdot)$ is an infinitely differentiable function and c_n are constant values for all $n = 0, 1, \cdots, N$ and a given N. Then, we have the following Taylor expansion.

$$f\left(\sum_{i=0}^{N} c_i \beta^i\right) = \sum_{k=0}^{N} a_k \beta^k + \mathcal{O}\left(\beta^{N+1}\right), \ a_k = \frac{1}{k!}\frac{\partial^k}{\partial \beta^k}\left[f\left(\sum_{i=0}^{k} c_i \beta^i\right)\right]_{\beta=0}.$$

Proof. See appendix A. □

Lemma 3.1 implies that the value of $V_{n,N}$ in our later analysis is independent of $V_{n+1,N}, \cdots, V_{N,N}$. Hence, $V_{n,N}$ can be analytically calculated through a recursion starting from the initial function $V_{0,N}$, which is taken to be the BS solution for any fixed integer N.

Proposition 3.1. Suppose that the solution pair $(V(S), S^*)$ to the ODE problem (2)-(5) has a pair of asymptotic expansions (15) and (14), respectively. Then, for any fixed $N = 0, 1, \ldots,$ $(V_{0,N}, x_0)$ is the solution pair to the perpetual American put under the BS model: $V_{0,N} = c_0 S^{\gamma_-}$ for $S \in [S_N^*, +\infty)$, where

$$c_0 = -\frac{1}{\gamma_-}\left(1 - \frac{1}{\gamma_-}\right)^{\gamma_- - 1} K^{-(\gamma_- - 1)},\ x_0 = \ln\left[\left(1 - \frac{1}{\gamma_-}\right)^{-1} K\right],$$

$$\gamma_\pm = \frac{\delta^2 - 2(r-q) \pm \sqrt{(2r - 2q - \delta^2)^2 + 8r\delta^2}}{2\delta^2}.$$

In addition, the n-th order correction pairs $(V_{n,N}, x_n)$ for $n = 1, \cdots, N$, are recursively obtained as $V_{n,N}(S) = c_n S^{\gamma_-} + u_{n,N}(S)$ for $S \in [S_N^*, +\infty)$, where

$$\begin{bmatrix} x_n \\ c_n \end{bmatrix} = -M^{-1} L_n = -M^{-1} \begin{bmatrix} L_n^1 \\ L_n^2 \end{bmatrix},$$

$$\begin{aligned}
u_{n,N}(S) &= \frac{1}{\gamma_+ - \gamma_-}\left[S^{\gamma_-}\int S^{1-\gamma_-} f_{n,N}(S)dS - S^{\gamma_+}\int S^{1-\gamma_+} f_{n,N}(S)dS\right], \\
f_{n,N}(S) &= \sum_{k=0}^{n-1} \frac{(2\ln S)^{n-k}}{(n-k)!}\frac{\partial^2 V_{k,N}}{\partial S^2},\ M = \begin{bmatrix} c_0\gamma_- e^{\gamma_- x_0} + e^{x_0} & e^{\gamma_- x_0} \\ c_0\gamma_-(\gamma_- - 1)e^{(\gamma_- - 1)x_0} & \gamma_- e^{(\gamma_- - 1)x_0} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
L_n^1 &= \sum_{k=1}^{n-1}[c_k a_{n-k}(\gamma_-) + b_{n-k,k}] + c_0 a_n(\gamma_-; x_n = 0) + u_{n,N}(e^{x_0}) + a_n(1; x_n = 0), \\
L_n^2 &= \sum_{k=1}^{n-1}[c_k\gamma_- a_{n-k}(\gamma_- - 1) + d_{n-k,k}] + c_0\gamma_- a_n(\gamma_- - 1; x_n = 0) + u'_{n,N}(e^{x_0}), \\
a_k(c) &= \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}\left[e^{c\sum_{i=0}^k x_i\beta^i}\right]_{\beta=0},\ a_k(c; x_k = 0) = \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}\left[e^{c\sum_{i=0}^k x_i\beta^i}\right]_{\beta=0, x_k=0}, \\
b_{k,n} &= \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}\left[u_{n,N}(e^{\sum_{i=0}^k x_i\beta^i})\right]_{\beta=0},\ d_{k,n} = \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}\left[u'_{n,N}(e^{\sum_{i=0}^k x_i\beta^i})\right]_{\beta=0}.
\end{aligned}$$

Proof. Denote the ordinary differential operator $\mathcal{L}_0 = \frac{1}{2}\delta^2 S^2 \frac{d^2}{dS^2} + (r-q)S\frac{d}{dS} - r\bullet$. Substituting (15) into (2) and applying the Taylor expansion on $S^{2\beta}$ $(= e^{2\beta \ln S})$ with respect to β, we decompose the governing ODE (2) into the following ODEs.

$$\mathcal{O}(1): \qquad \mathcal{L}_0 V_{0,N} = 0, \tag{16}$$

$$\mathcal{O}(\beta^n): \qquad \mathcal{L}_0 V_{n,N} = -\frac{1}{2}\delta^2 S^2 f_{n,N}(S), \quad \text{for } n = 1, \cdots, N, \tag{17}$$

where $f_{n,N}(S)$ is defined in the proposition.

All of the ODEs in (16) and (17) share the same pair of fundamental solutions: S^{γ_+} and S^{γ_-}. Using these two fundamental solutions, the Wronskian method is applied to obtain the particular solution to the inhomogeneous ODE. The particular solution is exactly the $u_{n,N}(S)$ in the proposition. Thus, for a given value of S_N^*, the solutions to (16) and (17) take the form: $V_{n,N} = D_n S^{\gamma_+} + c_n S^{\gamma_-} + u_{n,N}(S), \quad S \in [S_N^*, +\infty)$, where c_n and D_n respectively absorb the coefficients of S^{γ_-} and S^{γ_+} generated by the indefinite integral in $u_{n,N}(S)$. The boundary condition (5) implies that $D_n = 0$ for all $n = 0, 1, \cdots, N$.

Enforcing (15) to satisfy the value-matching condition (3) up to $\mathcal{O}(\beta^N)$, we have

$$\sum_{n=0}^{N} \beta^n [c_n {S_N}^{*\gamma_-} + u_{n,N}(S_N^*)] = K - S_N^*. \tag{18}$$

To apply the perturbation technique, we attempt to expand $S_N^* (= e^{X_N^*})$ and $u_{n,N}(S_N^*)$ by Lemma 3.1. To do this, we have to ensure the infinite differentiability of $u_{n,N}(S_N^*)$ in advance. It is trivial because $u_{n,N}(S_N^*)$ is simply an integration of elementary functions which are infinitely differentiable. Denote

$$S_N^{*c} = e^{c\sum_{i=0}^{N} x_i \beta^i} := \sum_{k=0}^{N} a_k(c)\beta^k + \mathcal{O}(\beta^{N+1}),$$

$$u_{n,N}(S_N^*) = u_{n,N}(e^{\sum_{i=0}^N x_i\beta^i}) := \sum_{k=0}^N b_{k,n}\beta^k + \mathcal{O}(\beta^{N+1}),$$

where $a_k(l)$ and $b_{k,n}$ are displayed in this proposition. Substituting these two expansions into (18) results in

$$\sum_{n=0}^N \beta^n [c_n \sum_{k=0}^N a_k(\gamma_-)\beta^k + \sum_{k=0}^N b_{k,n}\beta^k] + \mathcal{O}(\beta^{N+1}) = K - \sum_{n=0}^N a_n(1)\beta^n.$$

Matching orders in the value-matching condition (3) yields,

$$\mathcal{O}(1): \qquad c_0 e^{\gamma_- x_0} = K - e^{x_0}, \tag{19}$$

$$\mathcal{O}(\beta^n): \qquad \sum_{k=0}^n [c_k a_{n-k}(\gamma_-) + b_{n-k,k}] = -a_n(1), \quad \text{for } n = 1, \cdots, N. \tag{20}$$

Similarly, the smooth-pasting condition should be satisfied up to $\mathcal{O}(\beta^N)$. Hence,

$$\sum_{n=0}^N \beta^n [c_n\gamma_- \sum_{k=0}^N a_k(\gamma_- - 1)\beta^k + \sum_{k=0}^N d_{k,n}\beta^k] + \mathcal{O}(\beta^{N+1}) = -1.$$

Matching orders in the smooth-pasting condition (4) becomes,

$$\mathcal{O}(1): \qquad c_0\gamma_- e^{(\gamma_- - 1)x_0} = -1, \tag{21}$$

$$\mathcal{O}(\beta^n): \qquad \sum_{k=0}^n [c_k\gamma_- a_{n-k}(\gamma_- - 1) + d_{n-k,k}] = 0, \quad \text{for } n = 1, \cdots, N. \tag{22}$$

Finally, we impose the far-field boundary condition (5) to hold for each order in the expansion. Therefore, $\lim_{S\to\infty} V_{n,N}(S) = 0$ for all integers $n \le N$.

The ODE (16), boundary conditions (19), (21) and the far-field boundary condition ensure that $V_{0,N}$ is the BS pricing formula of the perpetual American put, and e^{x_0} is the corresponding optimal exercise boundary. Their formulas are presented in the proposition.

For $n \geq 1$, the coefficients of the n-th order correction term $V_{n,N}$ and the correction to the exercise boundary x_n are solutions to the algebraic equations (20) and (22), given the functions of $V_{0,N}, \cdots, V_{n-1,N}$.

Note that Lemma 3.1 implies that x_n is implicitly contained in $a_n(c)$ for each n. Specifically, we prove in Appendix B that $a_n(c) = ce^{cx_0}x_n + a_n(c; x_n = 0)$, where $a_n(c; x_n = 0)$ denotes the remaining part of $a_n(c)$ containing no x_n. By rewriting (20) and (22) in matrix form, we have $M(x_n, c_n)' + L_n = 0$, where M and L_n are defined in the proposition. It is easy to show that M is invertible and the result follows. $\square$

Remark: As $u_{n,N}$ in Proposition 3.1 is an indefinite integral, one expects two unknown coefficients of S^{γ_+} and S^{γ_-} to be generated from it. However, our proof show that the coefficient of S^{γ_+} is zero and the coefficient of S^{γ_-} is absorbed in c_n defined in the proposition. This remark aims at reducing confusion.

Although Proposition 3.1 offers the asymptotic expansions to arbitrary order, our numerical examples only use the first and second order approximations. Thus, the following two corollaries summarize explicit solutions to the first and second order approximations.

Corollary 3.1. The first-order asymptotic solution pair $\left(\bar{V}_1(S), S_1^*\right)$ is given by

$$\bar{V}_1(S) = V_{0,1} + \beta V_{1,1} \text{ for } S \in [S_1^*, +\infty),\ S_1^* = \exp\left(x_0 + \beta x_1\right),$$

where $V_{0,1}(S)$ is the BS perpetual American put with optimal exercise boundary S_1^*, and M, x_0 and $\gamma_\pm$ are stated in Proposition 3.1. In addition,

$$V_{1,1}(S) = c_1 S^{\gamma_-} + u_{1,1}(S),\ S \in [S_1^*, +\infty), \quad \begin{bmatrix} x_1 \\ c_1 \end{bmatrix} = -M^{-1} \begin{bmatrix} u_{1,1}(e^{x_0}) \\ u'_{1,1}(e^{x_0}) \end{bmatrix},$$

$$u_{1,1}(S) = \frac{2\gamma_-(\gamma_- - 1)c_0}{\gamma_+ - \gamma_-} S^{\gamma_-} \left[\frac{1}{2}(\ln S)^2 + \frac{1}{\gamma_+ - \gamma_-} \ln S + \frac{1}{(\gamma_+ - \gamma_-)^2} \right].$$

As $u_{1,1}$ is useful in the finite-time American put, let $u_1^*(S; c_0, \gamma_+, \gamma_-) = u_{1,1}(S)$.

Corollary 3.2. The second-order asymptotic solution pair $\left(\bar{V}_2(S), S_2^*\right)$ is given by

$$\bar{V}_2(S) = V_{0,2} + \beta V_{1,2} + \beta^2 V_{2,2} \text{ for } S \in [S_2^*, +\infty), \;\; S_2^* = e^{x_0 + \beta x_1 + \beta^2 x_2},$$

where $V_{0,2}(S)$ is the BS perpetual American put price with optimal exercise boundary S_2^*, $V_{1,2}(S)$ is identical to $V_{1,1}(S)$ in Corollary 3.1 with S_1^* replaced by S_2^*, and M, x_0, x_1 and $\gamma_\pm$ are stated in Proposition 3.1 and Corollary 3.1. In addition,

$$V_{2,2}(S) = c_2 S^{\gamma_-} + u_{2,2}(S) \text{ for } S \in [S_2^*, +\infty), \;\; (x_2, c_2)' = -M^{-1} L_2,$$

$$\begin{aligned}
u_{2,2}(S) &= \frac{1}{\gamma_+ - \gamma_-} \left[S^{\gamma_-} \int S^{1-\gamma_-} f_{2,2}(S) dS - S^{\gamma_+} \int S^{1-\gamma_+} f_{2,2}(S) dS \right], \\
f_{2,2}(S) &= 2C_0 \gamma_-(\gamma_- 1) S^{\gamma_- - 2} (\ln S)^2 + 2C_1 \gamma_-(\gamma_- 1) S^{\gamma_- - 2} \ln S + 2u''_{1,2}(S) \ln S,
\end{aligned}$$

$$L_2 = \left[\begin{array}{c} \frac{1}{2}\gamma_-^2 c_0 x_1^2 e^{\gamma_- x_0} + \gamma_- c_1 x_1 e^{\gamma_- x_0} + b_{1,1} + u_{2,2}(e^{x_0}) + \frac{x_1^2 e^{x_0}}{2} \\ \frac{1}{2}\gamma_-(\gamma_- - 1)^2 c_0 x_1^2 e^{(\gamma_- - 1)x_0} + \gamma_-(\gamma_- - 1) c_1 x_1 e^{(\gamma_- - 1)x_0} + d_{1,1} + u'_{2,2}(e^{x_0}) \end{array} \right],$$

$$\begin{aligned}
b_{1,1} &= \frac{2\gamma_-(\gamma_- - 1) c_0 x_1 e^{\gamma_- x_0}}{\gamma_+ - \gamma_-} \left(\frac{\gamma_-}{2} x_0^2 + \frac{\gamma_+ x_0}{\gamma_+ - \gamma_-} + \frac{\gamma_+}{(\gamma_+ - \gamma_-)^2} \right), \\
d_{1,1} &= \frac{2\gamma_-(\gamma_- - 1) c_0 x_1 e^{(\gamma_- - 1)x_0}}{\gamma_+ - \gamma_-} \left(\gamma_- x_0 + \frac{\gamma_+}{\gamma_+ - \gamma_-} \right) + (\gamma_- - 1) e^{-x_0} b_{1,1}.
\end{aligned}$$

Let $b_{1,1}^*(c_0, x_0, x_1, \gamma_+, \gamma_-) = b_{1,1}$ and $d_{1,1}^*(c_0, x_0, x_1, \gamma_+, \gamma_-) = d_{1,1}$ as they are useful for the finite-time American put.

Remark: Although $u_{2,2}(S)$ in Corollary 3.2 contains indefinite integrals, it can be explicitly obtained via integration by parts because the integrations in it are of the form $\int S^m(\ln S)^n dS$. Symbolic software can efficiently produce the result but the entire expression is rather long. The explicit formula can be provided upon request.

Proposition 3.1 renders an efficient asymptotic solution pair to the ODE free-boundary conditions (2)-(5). It gains computational efficiency by satisfying the value-matching and smooth-pasting conditions approximately (up to $\mathcal{O}(\beta^N)$). Therefore, the early exercise boundary is obtained in an explicit form through x_n, $n = 0, 1, \cdots, N$. This pair of expansions is expected to perform well for a large integer N.

However, the asymptotic approach in Proposition 3.1 is not without limitation. Our numerical experiment in a later section reveals that the error generated from the approximate boundary conditions deteriorates the accuracy of the solutions for a relatively lower order of expansion, such as $N = 1$ or 2. To attain certain accuracy for a wide region of S, a high-order approximation is required but computational time would be much longer. Notice that the level of the value function is controlled by the value-matching condition. Satisfying it approximately would cause the significant overall error. Therefore, we propose to adjust the value function so that it satisfies the value-matching condition exactly.

Corollary 3.3. The adjusted asymptotic solution of perpetual American put price,

$$\widetilde{V}_N(S) = \bar{V}_N(S) - \bar{V}_N(S_N^*) + K - S_N^*, \;\; S \in [S_N^*, +\infty), \tag{23}$$

where S_N^* is obtained via Proposition 3.1, satisfies the value-matching condition (3) and far-field boundary condition (5) exactly, but the governing ODE (2) and smooth-pasting condition (4) approximately up to $\mathcal{O}(\beta^N)$.

It is clear that the adjusted option value function satisfies the value matching condition, while other conditions are satisfied according to Proposition 3.1. An advantage of this adjustment is that the asymptotic early exercise boundary is still obtained in explicit form as in Proposition 3.1. Our numerical experiment shows that this adjustment greatly enhances the accuracy with similar degree of computational efficiency.

3.2. Finite-time American put

The free-boundary value PDE (7)-(11) of the finite-time American put contains an additional partial derivatives with respect to τ, the time to maturity. To reduce the dimension of the problem, we adopt the approach of Wong and Zhao [8] to take Laplace-Carson transforms (LCTs) to the value function and the optimal exercise boundary with respect to τ. For $\lambda > 0$, the LCTs of the American put price $P(S,\tau)$ and its optimal exercise boundary $S^*(\tau)$ are defined as

$$\begin{aligned}\widehat{P}(S,\lambda) &= \mathcal{LC}[P(S,\tau)](\lambda) = \int_0^\infty P(S,\tau)\lambda e^{-\lambda\tau}\,d\tau.\\ \widehat{S}^*(\lambda) &= \mathcal{LC}[S^*(\tau)](\lambda) = \int_0^\infty S^*(\tau)\lambda e^{-\lambda\tau}\,d\tau.\end{aligned}$$

There is no essential difference between the LCT and the Laplace transform (LT) by recognizing the relationship between them: $\mathcal{LC}[P(S,\tau)](\lambda) = \lambda\mathcal{L}[P(S,\tau)](\lambda)$. Therefore, the finite-time American put price and its optimal exercise boundary can be numerically obtained by inverting the Laplace transforms. The use of LCT instead of LT solely aims to simplify the ODE after the entire transformation.

After taking the LCT, the PDE problem of (7)-(11) is transformed into a free-boundary value ODE for $\widehat{P}(S,\lambda)$ as follows

$$
\begin{aligned}
\frac{\delta^2 S^{2\beta+2}}{2}\frac{\partial^2 \widehat{P}}{\partial S^2} + (r-q)S\frac{\partial \widehat{P}}{\partial S} - (\lambda + r)\widehat{P} &= -\lambda(K-S)^+,\ S \geq \widehat{S}^*(\lambda), &(24)\\
\widehat{P}(\widehat{S}^*(\lambda),\lambda) &= K - \widehat{S}^*(\lambda), &(25)\\
\frac{\partial \widehat{P}(\widehat{S}^*(\lambda),\lambda)}{\partial S} &= -1, &(26)\\
\lim_{S\to\infty} \widehat{P}(S,\lambda) &= 0, &(27)
\end{aligned}
$$

where λ is regarded as a constant when solving this ODE problem. The ODE of (24)-(27) resembles the free-boundary value ODE (2)-(5) for the perpetual American put except that there is an inhomogeneous term $-\lambda(K-S)^+$ in the governing equation. Therefore, it is natural for us to consider the asymptotic expansions for $\widehat{P}(S,\lambda)$ and $\widehat{S}^*(\lambda)$ as

$$\widehat{P}(S,\lambda) = P_0(S,\lambda) + \beta P_1(S,\lambda) + \beta^2 P_2(S,\lambda) + \cdots, \tag{28}$$

$$\ln \widehat{S}^*(\lambda) = \widehat{X}^*(\lambda) = X_0(\lambda) + \beta X_1(\lambda) + \beta^2 X_2(\lambda) + \cdots. \tag{29}$$

For each integer N, let

$$\ln \widehat{S}_N^*(\lambda) = \widehat{X}_N^*(\lambda) = X_0(\lambda) + \beta X_1(\lambda) + \cdots + \beta^N X_N(\lambda) \tag{30}$$

be a series of boundary layers. We also define the series of price layers as

$$\widehat{P}_N(S,\lambda) = P_{0,N}(S,\lambda) + \beta P_{1,N}(S,\lambda) + \cdots + \beta^N P_{N,N}(S,\lambda), \tag{31}$$

where the correction terms, $P_{i,j}(S,\lambda), j \geq i$, share the j-th boundary layer $\widehat{S}_j^*(\lambda)$.

Proposition 3.2. If the solution pair $(\widehat{P}(S,\lambda), \widehat{S}^*(\lambda))$ to the free-boundary value ODE problem (24)-(27) has the pair of asymptotic expansions (31) and (30), respectively, then $(P_{0,N}(S,\lambda), X_0(\lambda))$ is the solution pair of the form,

$$P_{0,N}(S,\lambda) = \begin{cases} A_0 S^{\xi_-} + \overline{u}_{0,N}(S) & , \ S \in [K, +\infty) \\ B_0 S^{\xi_+} + C_0 S^{\xi_-} + \underline{u}_{0,N}(S) & , \ S \in [\widehat{S}_N^*(\lambda), K) \end{cases},$$

$$\overline{u}_{0,N}(S) = 0,\ \underline{u}_{0,N}(S) = \frac{\lambda}{\lambda+r}K - \frac{\lambda}{\lambda+q}S,$$

$$B_0 = \frac{\lambda^2 + \lambda r - \xi_-(\lambda r - q\lambda)}{(\xi_+ - \xi_-)(\lambda+q)(\lambda+r)} K^{1-\xi_+},\ \ C_0 = \frac{-\frac{q}{\lambda+q} - B_0 \xi_+ e^{(\xi_+ - 1)X_0}}{\xi_- e^{(\xi_- - 1)X_0}},$$

$$A_0 = B_0 K^{\xi_+ - \xi_-} + C_0 - \frac{\lambda(r-q)}{(\lambda+q)(\lambda+r)} K^{1-\xi_-},$$

where $X_0(\lambda)$ is the solution to $\left(1 - \frac{\xi_+}{\xi_-}\right) B_0 e^{\xi_+ X_0} - \frac{q(1-\xi_-)}{(\lambda+q)\xi_-} e^{X_0} - \frac{r}{\lambda+r}K = 0$ and equals to $\ln\left[K\left(\frac{r\xi_-}{r\xi_- - \lambda - r}\right)^{\frac{1}{\xi_+}}\right]$ if $q = 0$. Furthermore, the correction terms $P_{n,N}(S,\lambda)$ and $X_n(\lambda)$, $n = 1, 2, \cdots, N$, are recursively obtained as

$$P_{n,N}(S,\lambda) = \begin{cases} A_n S^{\xi_-} + \overline{u}_{n,N}(S) & , \ S \in [K, +\infty) \\ B_n S^{\xi_+} + C_n S^{\xi_-} + \underline{u}_{n,N}(S) & , \ S \in [\widehat{S}_N^*(\lambda), K) \end{cases},$$

$$\begin{bmatrix} X_n \\ C_n \end{bmatrix} = -\widehat{M}^{-1}\widehat{L}_n = -\widehat{M}^{-1}\begin{bmatrix} \widehat{L}_n^1 \\ \widehat{L}_n^2 \end{bmatrix},$$

where $\widehat{M} \in \mathbb{R}^{2\times 2}$, $\xi_\pm = \frac{\delta^2 - 2(r-q) \pm \sqrt{4(r-q)^2 + 4(2\lambda + r + q)\delta^2 + \delta^4}}{2\delta^2}$,

$$\widehat{f}_{n,N}(S) = \sum_{k=0}^{n-1} \frac{(2\ln S)^{n-k}}{(n-k)!} \frac{\partial^2 P_{k,N}}{\partial S^2},$$

$$\widehat{u}_{n,N}(S) = \frac{1}{\xi_+ - \xi_-}\left[S^{\xi_-}\int S^{1-\xi_-}\widehat{f}_{n,N}(S)dS - S^{\xi_+}\int S^{1-\xi_+}\widehat{f}_{n,N}(S)dS\right],$$

$$\overline{u}_{n,N}(S) = \widehat{u}_{n,N}(S)\mathbb{1}_{\{S\geq K\}},\ \ \underline{u}_{n,N}(S) = \widehat{u}_{n,N}(S)\mathbb{1}_{\{S<K\}},$$

$$B_n = \frac{K[\overline{u}'_{n,N}(K) - \underline{u}'_{n,N}(K)] - \xi_-[\overline{u}_{n,N}(K) - \underline{u}_{n,N}(K)]}{(\xi_+ - \xi_-)K^{\xi_+}},$$

$$A_n = B_n K^{\xi_+ - \xi_-} + C_n + [\underline{u}_{n,N}(K) - \overline{u}_{n,N}(K)]K^{-\xi_-},$$

$$\widehat{M} = \left[\begin{array}{cc} B_0\xi_+ e^{\xi_+ X_0} + C_0\xi_- e^{\xi_- X_0} + \frac{q}{\lambda+q} e^{X_0} & e^{\xi_- X_0} \\ B_0\xi_+(\xi_+ - 1)e^{(\xi_+ -1)X_0} + C_0\xi_-(\xi_- - 1)e^{(\xi_- -1)X_0} & \xi_- e^{(\xi_- -1)X_0} \end{array} \right],$$

$$\begin{aligned}
\widehat{L}_n^1 &= \sum_{k=1}^{n-1}[B_k\widehat{a}_{n-k}(\xi_+) + C_k\widehat{a}_{n-k}(\xi_-) + \widehat{b}_{n-k,k}] + B_0\widehat{a}_n(\xi_+; X_n = 0) \\
&\quad + C_0\widehat{a}_n(\xi_-; X_n = 0) + B_n e^{\xi_+ X_0} + \underline{u}_{n,N}(e^{X_0}) + \frac{q}{\lambda+q}\widehat{a}_n(1; X_n = 0), \\
\widehat{L}_n^2 &= \sum_{k=1}^{n-1}[B_k\xi_+\widehat{a}_{n-k}(\xi_+ - 1) + C_k\xi_-\widehat{a}_{n-k}(\xi_- - 1) + \widehat{d}_{n-k,k}] \\
&\quad + B_0\xi_+\widehat{a}_n(\xi_+ - 1; X_n = 0) + C_0\xi_-\widehat{a}_n(\xi_- - 1; X_n = 0) \\
&\quad + B_n\xi_+ e^{(\xi_+ -1)X_0} + \underline{u}'_{n,N}(e^{X_0}),
\end{aligned}$$

$$\widehat{a}_k(c) = \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}[e^{c\sum_{i=0}^k X_i\beta^i}]_{\beta=0},\ \widehat{a}_k(c; X_k = 0) = \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}[e^{c\sum_{i=0}^k X_i\beta^i}]_{\beta=0, X_k=0},$$

$$\widehat{b}_{k,n} = \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}[\underline{u}_{n,N}(e^{\sum_{i=0}^k X_i\beta^i})]_{\beta=0},\ \widehat{d}_{k,n} = \frac{1}{k!}\frac{\partial^k}{\partial\beta^k}[\underline{u}'_{n,N}(e^{\sum_{i=0}^k X_i\beta^i})]_{\beta=0}.$$

Proof. Let $\mathcal{L}_1 = \frac{1}{2}\delta^2 S^2 \frac{d^2}{dS^2} + (r-q)S\frac{d}{dS} - (\lambda + r)\bullet$. Substituting (31) into (24) and applying the Taylor expansion to $S^{2\beta}(= e^{2\beta \ln S})$ with respect to β, we obtain the following order of ODEs.

$$\mathcal{O}(1): \qquad \mathcal{L}_1 P_{0,N} = -\lambda(K - S)^+, \tag{32}$$

$$\mathcal{O}(\beta^n): \qquad \mathcal{L}_1 P_{n,N} = -\frac{1}{2}\delta^2 S^2 \widehat{f}_{n,N}(S),\ \text{for } n = 1, \cdots, N, \tag{33}$$

where $\widehat{f}_{n,N}(S)$ is given in the proposition. Unlike the case of perpetual American option, the zeroth order ODE (32) contains an inhomogeneous term. The fundamental solutions still take the form: S^{ξ_+} and S^{ξ_-}, but the $\xi_\pm$ have to be revised to incorporate λ as shown in the proposition. By the Wronskian method, particular solutions to the inhomogeneous ODEs (32) and (33) are obtained and denoted as

$\widehat{u}_{n,N}(S)$. As the inhomogeneous term in (32) involves the maximum operator, we have to separate the stock price domain into two intervals so that $P_{n,N}$ and $\widehat{u}_{n,N}$ are possibly step functions at this stage. For a fixed $\widehat{S}_N^*(\lambda)$, the solutions to (32) and (33) are of the form, for $n = 0, 1, \cdots, N$,

$$P_{n,N}(S,\lambda) = \begin{cases} D_n S^{\xi_+} + A_n S^{\xi_-} + \overline{u}_{n,N}(S) & , \quad S \in [K, +\infty) \\ B_n S^{\xi_+} + C_n S^{\xi_-} + \underline{u}_{n,N}(S) & , \quad S \in [\widehat{S}_N^*(\lambda), K) \end{cases},$$

where coefficients of S^{ξ_-} and S^{ξ_+} generated by the indefinite integrals of $\overline{u}_{n,N}(S)$ and $\underline{u}_{n,N}(S)$ are absorbed into the constants A_n, B_n, C_n and D_n. The far-field boundary condition (27) implies that $D_n = 0$, for $n = 0, 1, \cdots, N$.

As the desired solution should be differentiable, we impose path-wise continuity on each correction term at $S = K$, for each $n = 0, 1, \cdots, N$. The continuity leads to the following system of equations.

$$\begin{cases} A_n K^{\xi_-} + \overline{u}_{n,N}(K) &= B_n K^{\xi_+} + C_n K^{\xi_-} + \underline{u}_{n,N}(K), \\ \xi_- A_n K^{\xi_- - 1} + \overline{u}'_{n,N}(K) &= \xi_+ B_n K^{\xi_+ - 1} + \xi_- C_n K^{\xi_- - 1} + \underline{u}'_{n,N}(K). \end{cases}$$

Solving it produces the values of A_n and B_n as shown in this proposition.

However, the values of A_n and B_n depend on C_n. In addition, X_n is still unknown. The remaining task makes use of the value-matching and smooth-pasting conditions to form a system of equations for C_n and X_n. The formulation and the resulting equation are similar to the one presented in the proof of Proposition 3.1. Hence, we omit the detail. Results are displayed in the present proposition. □

To better illustrate the results in Proposition 3.2, the explicit formulas for the second-order boundary and price layers are summarized in the following corollary.

Corollary 3.4. The second-order asymptotic solution pair to (24)-(27) is given by

$$
\begin{aligned}
\widehat{P}_2(S,\lambda) &= P_{0,2}(S,\lambda) + \beta P_{1,2}(S,\lambda) + \beta^2 P_{2,2}(S,\lambda), \quad S \in [\widehat{S}_2^*(\lambda), +\infty), \\
\widehat{S}_2^*(\lambda) &= \exp\left(X_0(\lambda) + \beta X_1(\lambda) + \beta^2 X_2(\lambda)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
P_{n,2}(S,\lambda) &= \begin{cases} A_n S^{\xi_-} + \overline{u}_{n,2}(S) & , \ S \in [K, +\infty) \\ B_n S^{\xi_+} + C_n S^{\xi_-} + \underline{u}_{n,2}(S) & , \ S \in [\widehat{S}_2^*(\lambda), K) \end{cases}, \ n = 1,2 \\
\begin{bmatrix} X_n \\ C_n \end{bmatrix} &= -\widehat{M}^{-1} \begin{bmatrix} \widehat{L}_n^1 \\ \widehat{L}_n^2 \end{bmatrix}, \ n = 1,2
\end{aligned}
$$

$$
\begin{aligned}
\overline{u}_{1,2}(S) &= u_1^*(S; A_0, \xi_+, \xi_-), \ \ \underline{u}_{1,2}(S) = u_1^*(S; C_0, \xi_+, \xi_-) + u_1^*(S; B_0, \xi_-, \xi_+), \\
\widehat{f}_{2,2}(S) &= 2(\ln S)^2 \frac{\partial^2 P_{0,2}}{\partial S^2} + 2\ln S \frac{\partial^2 P_{1,2}}{\partial S^2}, \\
\widehat{u}_{2,2}(S) &= \frac{1}{\xi_+ - \xi_-}\left[S^{\xi_-}\int S^{1-\xi_-}\widehat{f}_{2,2}(S)dS - S^{\xi_+}\int S^{1-\xi_+}\widehat{f}_{2,2}(S)dS\right], \\
\overline{u}_{2,2}(S) &= \widehat{u}_{2,2}(S)\mathbb{1}_{\{S \geq K\}}, \ \ \underline{u}_{2,2}(S) \ = \ \widehat{u}_{2,2}(S)\mathbb{1}_{\{S < K\}},
\end{aligned}
$$

$$
\begin{aligned}
\widehat{b}_{1,1} &= b_{1,1}^*(C_0, X_0, X_1, \xi_+, \xi_-) + b_{1,1}^*(B_0, X_0, X_1, \xi_-, \xi_+), \\
\widehat{d}_{1,1} &= d_{1,1}^*(C_0, X_0, X_1, \xi_+, \xi_-) + d_{1,1}^*(B_0, X_0, X_1, \xi_-, \xi_+), \\
\widehat{L}_1^1 &= B_1 e^{\gamma_+ X_0} + \underline{u}_{1,2}'(e^{X_0}), \ \ \widehat{L}_1^2 = B_1\gamma_+ e^{(\gamma_+ - 1)X_0} + \underline{u}_{1,2}(e^{X_0}), \\
\widehat{L}_2^1 &= B_1\xi_+ X_1 e^{\xi_+ X_0} + C_1 \xi_- X_1 e^{\xi_- X_0} + \widehat{b}_{1,1} + B_0 \frac{(\xi_+ X_1)^2}{2} e^{\xi_+ X_0} \\
&\quad + C_0 \frac{(\xi_- X_1)^2}{2} e^{\xi_- X_0} + B_2 e^{\xi_+ X_0} + \underline{u}_{2,2}(e^{X_0}), \\
\widehat{L}_2^2 &= B_1 \xi_+ (\xi_+ - 1) X_1 e^{(\xi_+ - 1)X_0} + C_1 \xi_- (\xi_- - 1) X_1 e^{(\xi_- - 1)X_0} + \widehat{d}_{1,1} \\
&\quad + B_0 \xi_+ \frac{((\xi_+ - 1)X_1)^2}{2} e^{(\xi_+ - 1)X_0} + C_0 \xi_- \frac{((\xi_- - 1)X_1)^2}{2} e^{(\xi_- - 1)X_0} \\
&\quad + B_2 \xi_+ e^{(\xi_+ - 1)X_0} + \underline{u}_{2,2}'(e^{X_0}),
\end{aligned}
$$

and $P_{0,2}$, $\widehat{M}$, X_0, $\xi_\pm$ and $A_i, B_i, i = 1, 2$ are stated in Proposition 3.2. In addition, u_1^* is defined in Corollary 3.1 while $b_{1,1}^*$ and $d_{1,1}^*$ are defined in Corollary 3.2.

Although the explicit formula for $\widehat{u}_{2,2}(S)$ can be easily deduced using integration by parts, the entire formula is omitted due to its very long expression but is available upon request.

Similar to the case of perpetual American put, we adjust asymptotic value function in Proposition 3.2 to fit the value-matching condition exactly.

Corollary 3.5. The following adjusted asymptotic expansion to the LCT of the American put price

$$\widetilde{P}_N(S, \lambda) = \widehat{P}_N(S, \lambda) - \widehat{P}_N(\widehat{S}_N^*(\lambda), \lambda) + K - \widehat{S}_N^*(\lambda), \quad S \geq \widehat{S}_N^*(\lambda), \tag{34}$$

satisfies (25) and (27) exactly but (24) and (26) approximately up to $\mathcal{O}(\beta^N)$.

Proof. The proof is obvious as it is a direct consequence of Proposition 3.2. $\square$

Once we obtain the asymptotic LCT value of the American put, we can convert it to the true price by Laplace inversion. In fact, we advocate the approach of adjusting the asymptotic result using Corollary 3.5 prior to taking the Laplace inversion. Therefore, $P(S, \tau) \simeq \mathcal{L}^{-1}\left\{\widetilde{P}_N(\lambda)/\lambda\right\}$. The optimal exercise policy is computed via the Laplace inversion as well. Specifically, $S^*(\tau) \simeq \mathcal{L}^{-1}\left\{\widehat{S}_N^*(\lambda)/\lambda\right\}$.

All of our numerical examples on finite-time American puts are implemented with the adjustment specified in Corollary 3.5.

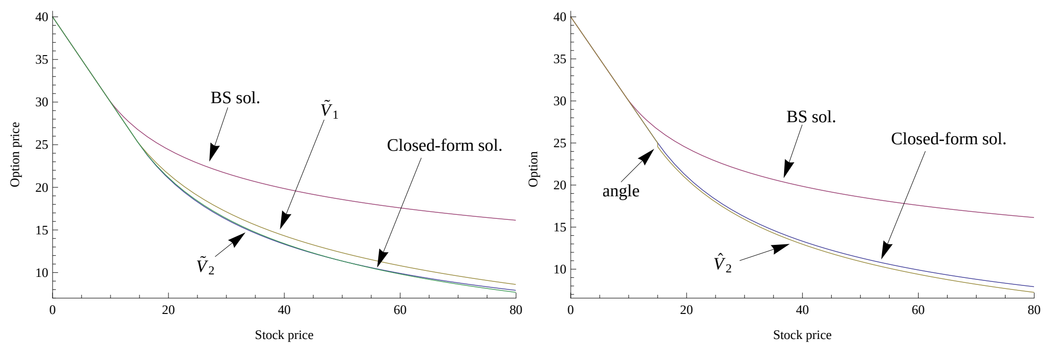
4. Numerical examples

This section conducts numerical experiments to verify the accuracy and efficiency of our asymptotic approach. We examine both the perpetual and finite-time American options, including their prices and optimal exercise boundaries. For the case of perpetual American option, an advantage of our expansion solution is that the optimal exercise boundary is obtained explicitly while the traditional analytical method solves it from a non-linear algebraic equation, which invokes a numerical root-finding procedure. For the case of finite-time American option, we contrast our asymptotic pricing formula and optimal exercise boundary to those in [8]. Appendix C exhibits the required formulas. As the Laplace inversion is involved, we adopt the Gaussian quadrature scheme proposed by Piessens [15]. Our numerical experiments use the following parameters:

$$r = 0.05, q = 0, \sigma_0 = 0.4, \beta = -0.1, K = 40, S_0 = 40 \text{ and } \delta = \sigma_0 \mathrm{S}_0^{-\beta} = 0.5785$$

4.1. Perpetual American put

Figure 1a plots the CEV perpetual American put value against the underlying asset price. The optimal exercise boundaries are 9.20, 14.58, 14.99 and 14.71 for the BS model, first-order approximation, second-order approximation and closed-form solution, respectively. Therefore, the BS model underestimates the optimal exercise boundary quite significantly. Under the CEV model, option holders are more likely to exercise. The asymptotic approximations reported in here refer to the adjusted option value in Corollary 3.3. It can be seen from Figure 1a that the asymptotic expansion converges to the closed-form solution quite rapidly. By



(a) With adjustment (b) Without adjustment

Figure 1: Perpetual American put values

inspection, the second-order approximation is already very close to the closed-form solution. In fact, the performance of the second-order approximation can be quantified by the root mean squared error (RMSE) between the approximation and the closed-form solution over the interval $[0, 80]$. The RMSE is only 0.104. This error is insignificant in practice. Certainly, the error can be further reduced by including more correction terms.

We stress that the adjustment in Corollary 3.3 is indispensable. Figure 1b illustrates the problem of using Proposition 3.1 without the adjustment. It can be seen from Figure 1b that there is an angle near the exercise boundary from the asymptotic solutions. The angle is caused by approximating the value-matching condition. The error associated with the approximation propagates from the point near the exercise boundary to the entire range of the underlying asset price. Consequently, the RMSE (0.416) is larger than that of the adjusted option value (0.104). More importantly, the situation gets even worse for the case of finite-time American option if the adjustment does not apply. The reason is that the finite-time American option requires an additional Laplace inversion, which becomes unstable near the exercise region if the adjustment is absent.

4.2. Finite-time American put

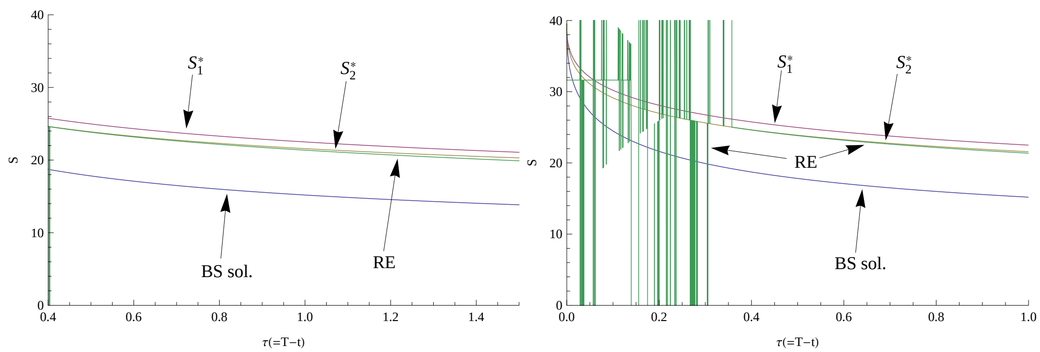


Figure 2: Optimal exercise boundaries

In this subsection, we use the same parameters to study the finite-time case. Figure 2 plots the optimal exercise boundaries of the finite-time American put against the time to maturity. We experience that the closed-form solution often fails to offer a numerical value when $|\beta| < 0.5$. Here the closed-form solution to American put price means the analytical solution with Laplace inversion in [8]. When the β is negative but close to zero, the functional equation for the optimal exercise boundary in [8] is insensitive to the change in $\widehat{S}^*(\lambda)$ so that a root-finding numerical procedure is unable to output a numerical value over a long running time. However, the asymptotic boundary value is always available. Therefore, no exercise boundary corresponding to the closed-form solution in [8] is provided in Figure 2. To estimate the accuracy of the first- and second-order approximations, we use the Richardson extrapolation (RE) to project the exercise boundary of the closed-form solution. The extrapolation is based on the closed-form optimal exercise boundaries corresponding to $\beta = -0.6, -0.7, -0.8, -0.9$ and -1. Figure 2a (left) only plots for the case of $\tau \geq 0.4$ because the projection from RE is still unable when $\tau < 0.4$. Figure 2b (right) shows the situation when $\tau < 0.4$. It can be seen that the closed-form solution oscillates quite sharply when τ gets close to zero. The instability of the closed-form solution is more pronounced with a less negative β and short time-to-maturity. However, our asymptotic solution is stable and accurate. The accuracy of our solution can be partially reflected by the fact that $S^*(0^+) = K$ is satisfied.

Table 1: Asymptotic results of American puts ($\beta = -0.1$ and $\tau = 1$)

S	BS sol.	1st-order	2nd-order	RE
20	19.7979	20	20	20
30	12.6119	11.1815	10.7256	10.7299
40	7.98526	5.77404	5.36116	5.33151
50	5.20209	2.84158	2.58372	2.54957
60	3.38609	1.23602	1.18842	1.13693

Table 1 shows numerical values of the American put with different approaches, where the maturity of the option is set to one year. As the RE works quite well for a long maturity option, we regard it as the benchmark for the American put with maturity one year. It can be seen that our approximation is very close to the benchmark. Our second-order approximation has relative error of only 1% compared with the benchmark. Although the estimate from the RE method may not accurate enough and our solution could be even more accurate, we are comfortable to conclude that our solution is practically accurate and efficient.

5. Conclusion

The valuation of American options under the CEV model has been a challenge in financial mathematics. The classical analytic solution method using the Laplace transform requires the computation of confluent hyper-geometric functions, making the Laplace inversion inevitably unstable under certain circumstances. Our asymptotic solution provides an efficient and accurate pricing alternative to deal with some problematic cases. The perturbation technique proposed in this paper is also applicable to other types of American options such as stock loans and Russian options, but the calculation should be modified accordingly. Investigating the convergence of our approximation is certainly an interesting future research.

A. Proof of Lemma 3.1

From the Taylor expansion of f, it is clear that

$$a_k = \frac{1}{k!}\sum_{l=0}^{N} f^{(l)}(0)\left[\frac{\partial^k}{\partial\beta^k}(\sum_{i=0}^{N} c_i\beta^i)^l\right]_{\beta=0}.$$

Our goal is to prove that, for any integers k and l, the following equality holds.

$$\left[\frac{\partial^k}{\partial\beta^k}\left(\sum_{i=0}^{N} c_i\beta^i\right)^l\right]_{\beta=0} = \left[\frac{\partial^k}{\partial\beta^k}\left(\sum_{i=0}^{k} c_i\beta^i\right)^l\right]_{\beta=0}. \tag{35}$$

Let $z = \sum_{i=0}^{N} c_i\beta^i$, $w_k = \sum_{i=0}^{k} c_i\beta^i$ and $v_k = \sum_{i=k+1}^{N} c_i\beta^i$. We have

$$\frac{\partial^k}{\partial\beta^k} z^l = \frac{\partial^k}{\partial\beta^k}(w_k + v_k)^l = \frac{\partial^k}{\partial\beta^k} w_k^l + \frac{\partial^k}{\partial\beta^k}\left[\sum_{r=0}^{l-1}\begin{pmatrix} l \\ r \end{pmatrix} w_k^r v_k^{l-r}\right].$$

Therefore, the statement in (35) is equivalent to that, for $0 \le r \le l-1$,

$$\left[\frac{\partial^k}{\partial\beta^k}(w_k^r v_k^{l-r})\right]_{\beta=0} = 0 \Longleftrightarrow \left[\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\frac{\partial^{k-j}}{\partial\beta^{k-j}} w_k^r \frac{\partial^j}{\partial\beta^j} v_k^{l-r}\right]_{\beta=0} = 0,$$

Let $v_k = \beta^{k+1}\sum_{i=0}^{N} c_{k+1+i}\beta^i := \beta^{k+1} g(\beta)$. Then, it is clear that

$$\frac{\partial^j}{\partial\beta^j} v_k^{l-r} = \sum_{h=0}^{j}\begin{pmatrix} j \\ h \end{pmatrix}\frac{\partial^{j-h}}{\partial\beta^{j-h}} g^{l-r}(\beta)\frac{\partial^h}{\partial\beta^h}\beta^{(k+1)(l-r)}.$$

For all $l-r \ge 1$, $0 \le h \le j$ and $0 \le j \le k$, a simple calculation shows that

$$\frac{\partial^h \beta^{(k+1)(l-r)}}{\partial\beta^h} = (k{+}1)(l{-}r)[(k{+}1)(l{-}r){-}1]\cdots[(k{+}1)(l{-}r){-}h{+}1]\beta^{(k+1)(l-r)-h}$$

equals to zero when $\beta = 0$. The proof is completed.

B. Property of a_k

We claim that, $a_k(c) = ce^{cx_0}x_k + a_k(c; x_k = 0)$ for $k = 0, 1, \cdots, N$ and the function $a_n(c; x_n = 0)$ independent of x_n.

We prove this claim by induction. When $k = 1$, we have $a_1(c) = ce^{cx_0}x_1$. For $k \ge 2$, we write $a_k(c)$ as follows.

$$a_k(c) = \frac{1}{k!}\left[\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}\frac{\partial^j}{\partial\beta^j}(e^{cx_k\beta^k})\frac{\partial^{k-j}}{\partial\beta^{k-j}}(e^{c\sum_{i=0}^{k-1}x_i\beta^i})\right]_{\beta=0}.$$

First, we need to prove that,

$$\left[\frac{\partial^j}{\partial\beta^j}(e^{cx_k\beta^k})\right]_{\beta=0} = 0 \text{ for } 1 \le j \le k-1 \text{ and } k \ge 2. \tag{36}$$

By Taylor expansion of e^x, we have

$$\frac{\partial^j(e^{cx_k\beta^k})}{\partial\beta^j} = \sum_{i=1}^{\infty}\frac{1}{i!}\frac{\partial^j}{\partial\beta^j}[(cx_k\beta^k)^i] = \sum_{i=1}^{\infty}\frac{1}{i!}[(cx_k)^i ki(ki-1)\cdots(ki-j+1)\beta^{ki-j}].$$

Since $ki - j \ge 1$ in this case, the above formula vanishes at $\beta = 0$ and (36) holds. Then, we recognize that

$$\frac{\partial^k}{\partial\beta^k}(e^{cx_k\beta^k}) = k!cx_k + \sum_{i=2}^{\infty}\frac{1}{i!}(cx_k)^i ki(ki-1)\cdots(ki-k+1)\beta^{ki-k},$$

where the second term vanishes at $\beta = 0$. Altogether, we establish that the only term containing x_k in the expression of $a_k(c)$ is $ce^{cx_0}x_k$ for $k = 1, 2, \cdots, N$.

C. Closed-form solutions

Proposition C.1. For the case of $\beta < 0, r \neq q$, the closed-form solution to the free boundary value problem (2) - (5) is given by $V(S) = D\phi(S)$ for $S \ge S^*$, where $\phi(S) = S^{\beta+\frac{1}{2}}e^{-\frac{x}{2}}W_{k,m}(x)$, $D = -1/\phi'(S^*)$,

$$x = -\frac{|r-q|}{\delta^2\beta}S^{-2\beta},\ \epsilon = sign(q-r),\ k = \frac{r}{2\beta|r-q|} + \epsilon(\frac{1}{2} + \frac{1}{4\beta}),\ m = -\frac{1}{4\beta},$$

in which $W_{k,m}(x)$ is the Whittaker function. The optimal exercise boundary (S^*) satisfies $\phi(S^*) + \phi'(S^*)(K - S^*) = 0$.

Proposition C.2. For the case of $\beta < 0, r \neq q$, the LCT of the American put option price under the CEV model is given by

$$\widehat{P}_A(S,\lambda) = \begin{cases} C_{11}\phi(S), & S \in [K,\infty), \\ C_{21}\phi(S) + C_{22}\psi(S) + u(S), & S \in [\widehat{S}^*(\lambda), K), \end{cases}$$

where $\psi(S) = S^{\beta+\frac{1}{2}} e^{\epsilon\frac{x}{2}} M_{k,m}(x)$, $\phi(S) = S^{\beta+\frac{1}{2}} e^{\epsilon\frac{x}{2}} W_{k,m}(x)$,

$$u(S) = -\frac{\lambda}{\lambda+q}S + \frac{\lambda K}{\lambda+r},$$

$$C_{11} = \frac{a_5(a_2b_2 - a_4b_1) + a_6(a_3b_1 - a_1b_2)}{(a_2a_3 - a_1a_4)a_5} + \frac{b_3}{a_5},$$

$$C_{21} = \frac{a_6(a_3b_1 - a_1b_2)}{(a_2a_3 - a_1a_4)a_5} + \frac{b_3}{a_5},\ C_{22} = \frac{a_1b_2 - a_3b_1}{a_2a_3 - a_1a_4}$$

$$a_1 = \phi(K),\ a_2 = \psi(K),\ a_3 = \left.\frac{d\phi(S)}{dS}\right|_{S=K},\ a_4 = \left.\frac{d\psi(S)}{dS}\right|_{S=K},\ a_5 = \left.\frac{d\phi(S)}{dS}\right|_{S=\widehat{S}^*(\lambda)},$$

$$a_6 = \left.\frac{d\psi(S)}{dS}\right|_{S=\widehat{S}^*(\lambda)},\ b_1 = u(K),\ b_2 = -\frac{\lambda}{\lambda+q},\ b_3 = -\frac{q}{\lambda+q},$$

$$x = -\frac{|r-q|}{\delta^2\beta}S^{-2\beta},\ \epsilon = sign(q-r),\ k = \frac{\lambda+r}{2\beta|r-q|} + \epsilon(\frac{1}{2} + \frac{1}{4\beta}),\ m = -\frac{1}{4\beta}.$$

$M_{k,m}(x)$ and $W_{k,m}(x)$ are the Whittaker functions. The LCT of the optimal exercise boundary of the American put option under the CEV model satistfies

$$A_1\Lambda(\widehat{S}^*(\lambda)) = A_2\phi(\widehat{S}^*(\lambda)) + A_3\widehat{S}^*(\lambda)\left.\frac{d\phi(S)}{dS}\right|_{S=\widehat{S}^*(\lambda)} + A_4\left.\frac{d\phi(S)}{dS}\right|_{S=\widehat{S}^*(\lambda)},$$

where

$$\Lambda(S) = \phi(S)\frac{d\psi(S)}{dS} - \psi(S)\frac{d\phi(S)}{dS} = \xi(S)\omega,$$

$$\xi(S) = \exp\left(\frac{r-q}{\delta^2\beta}S^{-2\beta}\right), \omega = \frac{2|r-q|\Gamma(2m+1)}{\delta^2\Gamma(m-k+1/2)},$$

$$A_1 = \left(\frac{\lambda K}{\lambda+r} - \frac{\lambda}{\lambda+q}K\right)\left.\frac{d\phi(S)}{dS}\right|_{S=K} + \frac{\lambda\phi(K)}{\lambda+q},$$

$$A_2 = -\frac{q\Lambda(K)}{\lambda+q},\ A_3 = \frac{q\Lambda(K)}{\lambda+q},\ A_4 = -\frac{rK\Lambda(K)}{\lambda+r}.$$

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