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Towards a feasible implementation of quantum neural networks using quantum dots

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Towards a feasible implementation of quantum neural networks using quantum dots

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We propose an implementation of quantum neural networks using an array of quantum dots with dipole-dipole interactions. We demonstrate that this implementation is both feasible and versatile by studying it within the framework of GaAs based quantum dot qubits coupled to a reservoir of acoustic phonons. Using numerically exact Feynman integral calculations, we have found that the quantum coherence in our neural networks survive for over a hundred ps even at liquid nitrogen temperatures (77 K), which is three orders of magnitude higher than current implementations, which are based on SQUID-based systems operating at temperatures in the mK range. © 2016 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4943622>]

Quantum neural networks (QNNs) have earned tremendous attention recently since Google and NASA's Quantum Artificial Intelligence Lab announced their use of D-Wave processors for machine supervised learning and big data classification.^{1–3} However, these machines use SQUIDs, which require temperatures in the mK range in order to keep quantum coherence for sufficiently long.⁴ This limitation would be lifted by orders of magnitude when the neural network is implemented with quantum dots (QD), which we show have coherence survival times on the ns scale even at 77 K.

QD are small conductive regions of semiconductor heterostructure that contain a precisely controlled number of excess electrons, and have been the subject of several excellent reviews, such as Refs. 5–7. The electrons are locked to a small region by external electric and magnetic fields, which define the shape and size of the dot, typically a few nanometers to a few hundred nanometers in size. Controllable QDs are often made on the basis of a 2D electron gas of GaAs heterostructures. The energy levels of QDs are precisely controlled by the size of the dot and the strength of external electric and magnetic fields.^{8–15} By arranging the quantum dots in a regular array on a layer of semiconductor heterostructure one can form a matrix for a quantum register, composed of either charge-based or spin-based qubits, to be used for quantum computations.¹⁶ The fact that QDs can easily be controlled¹⁷ makes arrays of quantum dots particularly attractive for quantum neural networks, where the coherence

requirements are not as strict as in the circuit-based quantum computing; instead, the system just needs to find the minimum of energy state, which can be found quicker with the aid of quantum tunneling rather than solely classical hopping.⁴

Using an array of GaAs QDs for quantum neural networks was first proposed by Behrman *et al.*¹⁸ Their original idea assumed the use of quantum dot molecules interacting with each other only by means of their shared phonon bath. Within this framework, it would be nearly impossible to control the training of the QNN since manipulating a phonon bath is an arduous task.¹⁹ In this letter, we present a more achievable quantum dot based QNN architecture, where the QDs interact with each other via dipole-dipole coupling. We present realistic physical parameters for all couplings, and use a numerically exact Feynman integral calculation to study the time evolution of the phonon-damped coherence in a pair of one-electron QDs in such a network.

A careful series of experiments on decoherence rates in SQUIDs was reported in Ref. 20, which showed that at 80 mK decoherence rates can be as high as 110 ps⁻¹, but it is well known that superconducting devices cannot operate at much higher temperatures. We will present numerically exact calculations to show that the quantum coherence in our quantum dot based architecture survives for durations on the same time scale, but at much higher temperatures.

Hypothetical QNN built of an array of QDs implements the idea of a Hopfield network as the SQUID based processors do. To ensure potential advantages of QD based architecture one must prove that the three or more QDs interaction terms do not change dramatically the form of the Hamiltonian, and hence, it is valid in its standard Ising form $H = A_{ij}s_i s_j + B_i s_i$. Another required assumption is that the common bath of

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acoustic phonons propagating in the semiconductor heterostructure is stable enough and is not significantly affected by the linear interaction with QDs (4). Under these assumptions, it is sufficient to show that if a pairwise interaction between two QDs separated by a given distance between neighbouring QDs allows for the survival of quantum superposition of states at a given temperature (of $\sim 10^2$ K order), we can expect the whole QD array to exhibit quantum coherence at high temperatures of the same order.

We consider small QDs of $d = 3.3$ nm diameter in a GaAs based substrate,²¹ where the excitons interact with their bath of acoustic phonons.^{21–23} The Hamiltonian of a pair of QD excitons embedded in a semiconductor heterostructure can be written in the form

$$H = H_{\text{Ex}} + H_{\text{Ph}} + V \equiv H_0 + V, \quad (1)$$

where

$$H_{\text{Ex}} = \sum_{i=1}^2 \frac{\Delta_i}{2} \sigma_z^{(i)} + \frac{K_i}{2} \cos(\omega_L t) \sigma_x^{(i)} + \sum_{i \neq j} J_{ij} \sigma_+^{(i)} \sigma_-^{(j)} \quad (2)$$

describes the energy of the excitons, with Δ_i being the energy gap between the ground state and the first excited state of the i -th exciton; K_i is a coupling due to an external driving field with Rabi frequency ω_L ; and J_{ij} is the dipole-dipole coupling between the dots, constructed in analogy to the dipole-dipole interaction of atoms.²⁴ The pseudo-spin operators of the i th QD are

$$\sigma_z^{(i)} = |X_i\rangle\langle X_i| - |0_i\rangle\langle 0_i|, \quad \sigma_x^{(i)} = |0_i\rangle\langle X_i| + |X_i\rangle\langle 0_i|, \\ \sigma_+^{(i)} = |X_i\rangle\langle 0_i|, \quad \sigma_-^{(i)} = |0_i\rangle\langle X_i|.$$

The free phonon Hamiltonian is

$$H_{\text{Ph}} = \sum_{\alpha} \frac{p_{\alpha}^2}{2m_{\alpha}} + \frac{m_{\alpha} \omega_{\alpha}^2 x_{\alpha}^2}{2}, \quad (3)$$

and the interaction between the phonons and the QDs is given by

$$V = \sum_{\alpha, i} g_{\alpha} x_{\alpha} |X_i\rangle\langle X_i|. \quad (4)$$

We consider a pair of identical QDs ($\Delta_1 = \Delta_2 = \Delta$, $J_{12} = J_{21} = J$, $K_1 = K_2 = K$) in which we can see that in the limit of vanishing driving field ($K \rightarrow 0$) the eigenstates of H_{Ex} are

$$\frac{|X0\rangle - |0X\rangle}{\sqrt{2}}, \quad \frac{|X0\rangle + |0X\rangle}{\sqrt{2}}, \quad |00\rangle, \quad |XX\rangle \quad (5)$$

corresponding to the eigenvalues $(-J, J, -\Delta, \Delta)$. The first two states of Eq. (5) have zero eigenvalue with respect to the interaction with phonons, given by Eq. (4), and thus survive in coherent superposition even in the presence of a bath of acoustic phonons.

After making a unitary transformation of the Hamiltonian into a frame rotating around the z -axis with the frequency ω_L , and making a Rotating Wave Approximation, we arrive at a Hamiltonian in a matrix form

$$H_{\text{Ex}} = \begin{pmatrix} 0 & K & K & 0 \\ K & 0 & J & K \\ K & J & 0 & K \\ 0 & K & K & 0 \end{pmatrix},$$

where we assume $K = 0.24$ ps⁻¹ is the driving field parameter, which corresponds to low intensity fields of about 0.95 kV/cm. For the dipole frequency, we use the estimation:^{17,24} $J = \frac{V_{\text{dip}}}{\hbar} = \frac{\mu^2}{\epsilon L^3} \approx 1.4$ ps⁻¹, where $\epsilon \approx 10$ is the dielectric permittivity of GaAs, $L = 7.5$ nm is a typical inter-dot distance, and $\mu = \langle X0|er_x|00\rangle \approx 79$ Debye is the transition dipole moment of the QDs.

The system of QDs interacting with phonons can be described in terms of the von Neumann equation for the reduced density matrix

$$\dot{\rho} = \text{tr}_{\text{Ph}} \left(-\frac{i}{\hbar} [H, \rho_{\text{tot}}] \right). \quad (6)$$

Eq. (6) can be solved numerically exactly using the quasi-adiabatic propagator path integral technique²⁵ using the free open source MATLAB code FeynDyn.²⁶ We use the initial condition

$$\rho_{\text{tot}}(0) = \rho(0) \otimes \frac{e^{-\beta H_{\text{Ph}}}}{\text{tr}(e^{-\beta H_{\text{Ph}}})},$$

where

$$\rho(0) = |\psi(0)\rangle\langle\psi(0)|, \quad |\psi(0)\rangle = \frac{1}{\sqrt{2}}(|0X\rangle + |X0\rangle), \quad (7)$$

and we treat the phonon bath as continuous, by defining the spectral density

$$J(\omega) = \frac{\pi}{2} \sum_{\alpha} \frac{g_{\alpha}}{m_{\alpha} \omega_{\alpha}} \delta(\omega - \omega_{\alpha}). \quad (8)$$

The spectral density $J(\omega)$ for acoustic phonons in GaAs QDs is extremely well characterized. The form for $J(\omega)$ has been derived from first principles, and the agreement with experiments is within the error bars of the experiment⁸

$$J(\omega) = \alpha \omega^3 e^{-(\omega/\omega_c)^2} \quad (9)$$

with $\alpha = 0.027\pi$ ps² and $\omega_c = 2.2$ ps⁻¹.⁸ This form for $J(\omega)$ along with these specific parameters has been used consistently in a plethora of studies, which is in excellent agreement between experiment and theory.^{8,23,26–28}

In the Feynman integral representation of Eq. (6), the spectral density and temperature determine the bath correlation function²⁸

$$R(t) = \int_0^{\infty} \frac{d\omega}{\pi} J(\omega) \left[\cos(\omega t) \coth\left(\frac{\omega}{2k_B T}\right) - i \sin(\omega t) \right],$$

which in turn determines the time scale of the QD's memory. For the specific spectral density form and parameters used here, it has been shown that the memory lasts for about 2.5 ps.^{27,28} Therefore, having now defined all physical parameters used in the Hamiltonian, we determined through numerical experiments that a time step of $\Delta t = 1$ ps was

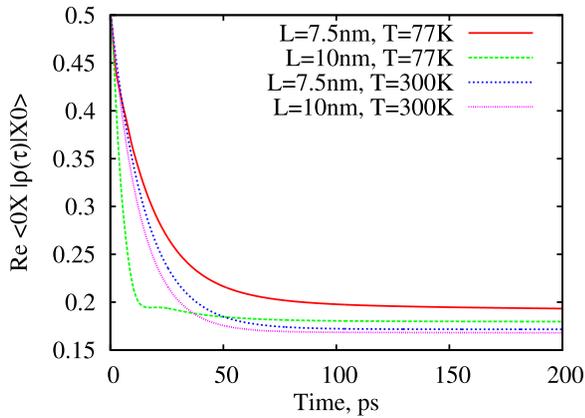


FIG. 1. Real part of an off-diagonal density matrix element with respect to time, demonstrating that the coherence survives for over 100 ps even at 77 K. The graphs are shown for two regimes: a strong coupling $L = 7.5$ nm and a moderate coupling $L = 10$ nm.

sufficient for numerical convergence of the Feynman integral, and to ensure that the full memory length of 2.5 ps was captured, we set the memory length in the Feynman integral to 3 time steps.

Converged Feynman integral calculations for all elements of the QD density matrix $\rho(t)$ were obtained at temperatures of $T = 77$ K and 300 K, and in Fig. 1, it is shown that the real part of the coherence $\text{Re}[\langle X0|\rho(t)|X0\rangle]$ lasts for over 100ps even at $T = 77$ K.

While the dynamics of a coherence is expected to begin with a zero slope, the decoherence due to the bath causes the coherence to rapidly decay, in a manner closely resembling a decaying exponential: $\exp(-\gamma t)$. To study the dependence of the decoherence rate with respect to temperature, we calculated the dynamics at 30 different temperatures between 40 K and 300 K, and for each case, we fitted $\text{Re}[\langle 0X|\rho(t)|X0\rangle]$ to a decaying exponential to determine a decoherence rate γ .

It is shown in Fig. 2 that decoherence rates of 1 ns^{-1} order are still maintained even at 77 K, which is accessible by liquid nitrogen. At the same time, the coherence dynamics essentially depends on the relative strength of the coupling constant (J) with respect to the driving field (K). Changing the interdot distance, or varying the steepness of the QD confining potential with an external magnetic field, we can achieve different

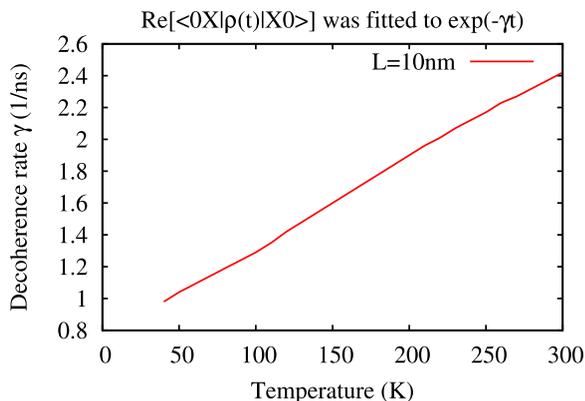


FIG. 2. The decoherence rate in a pair of interacting QDs calculated as a linear fit of $\text{Re}[\langle 0X|\rho(t)|X0\rangle] \sim \exp(-\gamma t)$. The typical decoherence rate remains at 1 ns^{-1} scale. Calculated for $d = 3.3$ nm and $L = 10$ nm using 200 ps overaging window.

regimes of coherence oscillations or suppression. The $L = 10$ nm, $T = 77$ K curve in Fig. 1 shows the remnant of quantum oscillations, which is suppressed for smaller L (7.5 nm) or higher temperature (300 K). The decoherence rates can be controlled in the same way.

The idea of a quantum neural network²⁹ is to connect a set of quantum elements (in our case, the QDs qubits) by tunable weights, so that a certain quadratic optimization problem can be solved. The optimization then becomes a physical problem of evolving a quantum system at the non-zero temperature toward the minimum energy state. The dissipative bath plays an integral role in this quantum annealing process.^{30,31} This is different from a QNN implemented as a circuit-based quantum computer,³² where the interaction with the environment poses the main obstacle for creation of stable superpositions of quantum states. In a quantum annealer, the interaction of the system with the environment, i.e., the noise, in contrast can increase the effective barrier transparency between the local minima and the desired ground state, therefore enhancing the efficiency of the computation.^{33,34}

Solid state quantum annealing computers, produced by D-wave Systems, Inc.,^{4,35} are based on SQUID qubits with programmable weights implemented as inductive couplings. Such systems operate at the temperatures below 1 K, requiring power on the kW range for cooling the system. In an array of dipole-dipole coupled QDs with a low driving frequency, the coupling weights J_{ij} can be tuned by either external fields or by changing the material properties in the area between the dots, and we have shown using a numerically exact approach that such devices can maintain coherence at 77 K.

The difference between our design described by the Hamiltonian (2) and the classical Hopfield neural network with the $J_{ij}s_i^z s_j^z$ interactions, as well as quantum annealers on SQUIDs, is that the interaction $J_{ij}\sigma_i^+ \sigma_j^-$ flips the states of two interacting qubits dynamically, in the presence of a fluctuating environment. In this sense, our model is closer to the biological settings of the original Hopfield work³⁶ than the spin glass type energy minimizing models. The Hamiltonians considered in this letter can be used both for networks with self-organization and feed-forward networks.³⁷

The possible application of QDs to the construction of quantum neural networks has already been discussed in the literature.^{7,18} These considerations however involve only the problem of charge control of the qubit state and requires manipulation of the phonon bath in order to work. In the present letter, we have introduced a model element of a quantum dot neural network architecture. The QDs are assumed to be well separated from each other, so that wave function overlap can be neglected. For such array, the selection rules for the interlevel transitions are the same as for isolated QDs.³⁸ It is possible to control the QDs by the near-field of plasmon modes propagating in the substrate beneath the dots. The QDs can also be controlled by external magnetic field, which changes the confining potential steepness. The authors understand that the use of QDs with $n \geq 2$ electrons might be more insightful from the standpoint of using collective excitations of the whole QD array,^{38,39} but such complicated models can be considered only as a future perspective for the quantum neural networks.⁴⁰ We strongly advocate for dipole-dipole

coupled QDs as an architecture for the construction of quantum neural networks that are robust and feasible at higher temperatures than current SQUID-based architectures.

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