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Non-zero-sum Reinsurance Games subject to Ambiguous Correlations

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Abstract

This paper studies the economic implications of ambiguous correlation in a non-zero-sum game between two insurers. We establish the general framework of Nash equilibrium for the coupled optimization problems. For the constant absolute risk aversion (CARA) insurers, we show that the equilibrium reinsurance strategies admit closed-form solutions. Our results indicate that the ambiguous correlation leads to an increase in the equilibrium demand of reinsurance protection for both insurers. Numerical studies examine the effect on the quality of the correlation estimations.

Keywords: Reinsurance, Non-zero-sum Stochastic Differential Game, Relative Performance Concerns, Ambiguous Correlation, G -Brownian motion, Hamiltonian-Jacobi-Bellman-Isaacs Equation, Nash Equilibrium, Externalities

1. Introduction

The optimal reinsurance and investment (IR) problems under different stochastic environments have been extensively studied in the fields of insurance and control theory. Representative works include but not limited to [1, 3, 10, 12, 13]. However, the aforementioned studies do not take into account the effect of interactions among the insurance companies. In fact, economical and sociological studies have pointed out that human beings or firms tend to compare themselves to

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their peers, and that such relative performance concerns have significant impacts on one's decision-making. For example, [4] shows that the concept of relative performance concerns is relevant to financial bubbles and excess volatility. [6] establishes the unique existence of the Nash equilibrium for the optimal investment problems subject to the relative performance concerns in a N-agent economy under the Brownian motion framework. Subsequently, [2] extends the tractability of the non-zero-sum game framework to the IR problems with two insurers under the mixed regime-switching framework. [14] introduces model uncertainty into the associated IR games in [2], but does not effectively address the sensitivity of the correlation to the equilibrium strategies of the insurers.

In this paper, we study the robust reinsurance games between two insurers. Our present work differs from [2] in two key aspects. First, we treat the correlation coefficient (ρ) between two insurers' surplus processes as an ambiguous parameter which could be stochastic, whereas the correlation coefficient is a constant in [2]. Secondly, we allow the insurers to be either cooperative or competitive to highlight the impact of the ambiguous correlation in the non-zero-sum games between two competitive as well as two cooperative insurers, whereas only the case of competitive insurers is considered in [2]. Each insurer has her own confidence interval for ρ , where the bounds could be different (different constraints sets), and she maximizes her expected utility of her relative terminal surplus with respect to that of her counterparty by choosing her proportional reinsurance protection under the worst-case scenario of ρ . We show that the associated reinsurance game with ambiguous correlation fits naturally into the two-dimensional G -Brownian motion framework that is first introduced in [11] and has been subsequently applied to other stochastic control problems, as shown in [5, 7].

Using the dynamic programming principle, we provide the Nash equilibrium of the robust non-zero-sum stochastic differential reinsurance game as the solution of a system of coupled Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations, for general utility functions. More importantly, we show that the Nash equilibrium reinsurance strategies and value functions of the insurers admit explicit solutions for the case of constant absolute risk aversion (CARA) insurers. Our results indicate that the ambiguity in correlation leads to an increase in the demand of the reinsurance protections for both insurers, whether they are cooperative or competitive. Furthermore, our welfare analysis

shows that the nature of externalities in the game between competitive insurers is different from that in the game between two cooperative insurers. To the best of our knowledge, the ambiguous correlation risks in the non-zero-sum reinsurance game has not been studied in the existing literature.

The rest of this paper is organized as follows. Section 2 formulates optimization problems of our interest with the surplus processes defined using G -Brownian motions. In Section 3, we apply the dynamic programming principle to the optimization problems and provide the sufficient conditions that the Nash equilibrium for the coupled problems exists. We also provide an explicit solution for the case of exponential utilities. Section 4 provides the numerical examples for the case of the CARA insurers, together with economic interpretations. Section 5 concludes the paper and discusses the possible extensions.

2. Problem Formulation

We formulate the non-zero-sum game problem between two insurers using two-dimensional G -Brownian motion, which is introduced in [11]. That is, the associated game problem is studied in a complete space generated by the corresponding G -expectation. However, to best motivate the necessity of the G -Brownian motion framework, we shall begin with the standard insurance models under a physical measure \mathbb{P} generated by the standard Brownian motion, and point out the ill-defined components under the assumption of the ambiguous correlation. We then apply the G -framework to reformulate our original game problem such that the mentioned ill-defined components become well-defined under the G -framework.

2.1. The Model

We begin with the model of the surplus process of each insurer. Following [2], we adopt the standard Cramér-Lundberg diffusion approximation to model the surplus process of the insurer $k \in \{1, 2\}$, denoted by $\{X_k(t)\}_{t \geq 0}$. See [9] for the treatise on diffusion approximation in insurance models. Specifically, $X_k(t)$ satisfies the stochastic differential equation (SDE)

$$dX_k(t) = (p_k - \lambda_k \mathbb{E}[\eta_k])dt + \sqrt{\lambda_k \mathbb{E}[\eta_k^2]} d\mathcal{W}_k(t), \quad (1)$$

where $p_k > 0$ is the premium rate, $\lambda_k > 0$ is the arrival rate of the claims, $\eta_k \neq 0$ is a random variable representing the size of the claims with $\mathbb{E}[\eta_k^2] < \infty$ and $\{\mathcal{W}_k(t)\}_{t \geq 0}$ is a standard \mathbb{P} -Brownian motion for $k = 1, 2$. The dependence between two insurers is reflected by the correlation between $\{\mathcal{W}_1(t)\}_{t \geq 0}$ and $\{\mathcal{W}_2(t)\}_{t \geq 0}$, i.e. $d\mathcal{W}_1(t)d\mathcal{W}_2(t) = \rho(t)dt$, but $\rho(t)$ is uncertain and possibly stochastic, in the sense that the insurer k only knows $\rho(t) \in [\underline{\rho}_k, \bar{\rho}_k]$. In what follows, we shall assume that the parameters of the model dynamics are constants and independent of time. We do so to explicitly capture the consequence of the ambiguous correlation on the Nash equilibrium in the associated non-zero-sum game. The extension to the time-varying parameters is rather immediate but yields no additional economic insights.

Suppose that there is a reinsurance company, then the insurer $k \in \{1, 2\}$ can manage her insurance risks through purchasing proportional reinsurance protection at the premium rate $\theta_k > p_k > 0$. Let $1 - q_k(t)$ be the reinsurance proportion of the insurer $k \in \{1, 2\}$ at time t . Then the reinsurance company will cover $(1 - q_k(t))100\%$ of the claims while the insurer k will cover the remaining. The reinsurance strategy of the insurer $k \in \{1, 2\}$ is characterized by $\{q_k(t)\}_{t \geq 0}$, which is a \mathcal{F}_t -progressively measurable process valued in $[0, 1]$ and $\mathcal{F}_t = \sigma(\{(\mathcal{W}_1(s), \mathcal{W}_2(s))\}_{s=0}^t)$. We denote $Q_k = \{q_k(t) \in \mathcal{F}_t | q_k(t) \in [0, 1]\}$ the set of convex reinsurance strategies of insurer k . With reinsurance, the surplus process $\{X_k^{q_k}(t)\}_{t \geq 0}$ of the insurer $k \in \{1, 2\}$ becomes

$$\begin{aligned} dX_k^{q_k}(t) &= [p_k - \theta_k(1 - q_k(t)) - \lambda_k \mathbb{E}[\eta_k] q_k(t)] dt + \sqrt{\lambda_k \mathbb{E}[\eta_k^2] q_k(t)} d\mathcal{W}_k(t), \\ &=: [\delta_k + \mu_k q_k(t)] dt + \sigma_k q_k(t) d\mathcal{W}_k(t), \end{aligned} \quad (2)$$

where $\delta_k = p_k - \theta_k < 0$ is the premium difference, $\mu_k = \theta_k - \lambda_k \mathbb{E}[\eta_k]$ is the relative safety loading and $\sigma_k = \sqrt{\lambda_k \mathbb{E}[\eta_k^2]}$ is the volatility of the claims process. We assume the initial reserve of the insurer $k \in \{1, 2\}$ is $X_k^{q_k}(0) = x_k > 0$.

2.2. Objectives of the Insurers

Suppose that the insurer $k \in \{1, 2\}$ has a utility function, denoted by U_k , which is increasing and strictly concave function valued in \mathbb{R} and satisfies Inada conditions:

$$\left. \frac{\partial U_k}{\partial x} \right|_{x \rightarrow -\infty} = +\infty, \quad \left. \frac{\partial U_k}{\partial x} \right|_{x \rightarrow +\infty} = 0.$$

To incorporate the ambiguity of the correlation and the interaction between two insurers, we assume the objective of each insurer is to maximize the expected utility of a linear combination of both insurers' surpluses at terminal time $T > 0$ under the worst-case scenario of the correlation. Mathematically, we consider the following optimization problem for the insurer $k \in \{1, 2\}$:

$$\sup_{q_k \in Q_k} \inf_{\rho \in [\underline{\rho}_k, \bar{\rho}_k]} \mathbb{E} [U_k(X_k^{q_k}(T) - \kappa_k X_m^{q_m}(T))] \quad (3)$$

for $m \neq k \in \{1, 2\}$, where $\kappa_k \in [-1, 1]$ reflects the level of relative performance concern of the insurer k . Indeed, when $\kappa_k = 0$, for $k = 1, 2$, we return to the single-agent optimal reinsurance problem, in which ambiguous correlation would play no role in the insurer's optimization problem. In light of this, we shall hereafter assume that $\kappa_k \neq 0$. When $\kappa_k \in (0, 1]$ (resp. $\kappa_k \in [-1, 0)$), for $k = 1, 2$, insurer k treats insurer m , for $k \neq m \in \{1, 2\}$ as competitor (resp. cooperator), as her optimization problem in (3) would indicate that she would optimally purchase reinsurance protection to maximize the difference between her terminal surplus against (resp. the sum of her terminal surplus with) that her competitor under the correlation estimate that yields the worse expected payoff. Although the Nash equilibrium in Section 3 also includes the case when insurer k is competitive ($\kappa_k > 0$) and insurer m is cooperative ($\kappa_m < 0$), for $k \neq m \in \{1, 2\}$, we choose not to study this case for there is no clear economic rationale on the establishment of a game between one competitive and one cooperative insurers. See [2, 6, 14] for the optimization under the relative performance concerns when $\kappa_k \in [0, 1]$; and [7] for the maximin formulation of the robust portfolio optimization with ambiguous correlation.

Major technical hurdle arising from our problem formulation is that the underlying measure of the expectation and the admissible set of the reinsurance strategies in (3) are not clear. More specifically, we denote $\bar{X}_k^{q_k, q_m}(t) = X_k^{q_k}(t) - \kappa_k X_m^{q_m}(t)$ the relative surplus (performance) process of the insurer k for $k \neq m \in \{1, 2\}$. Then the dynamics of $\bar{X}_k^{q_k, q_m}(t)$ is given by

$$\begin{aligned} d\bar{X}_k^{q_k, q_m}(t) &= [\delta_k - \kappa_k \delta_m + \mu_k q_k(t) - \kappa_k \mu_m q_m(t)] dt + \sigma_k q_k(t) d\mathcal{W}_k(t) - \kappa_k \sigma_m q_m(t) d\mathcal{W}_m(t) \\ &= [\bar{\delta}_k + \mu_k q_k(t) - \kappa_k \mu_m q_m(t)] dt + (\sigma_k q_k(t), -\kappa_k \sigma_m q_m(t)) d\vec{\mathcal{W}}^{(k)}(t) \end{aligned} \quad (4)$$

with $\bar{X}_k^{q_k, q_m}(0) = x_k - \kappa_k x_m =: \bar{x}_k$, where $\bar{\delta}_k = \delta_k - \kappa_k \delta_m$ and $\vec{\mathcal{W}}^{(k)}(t) = (\mathcal{W}_k(t), \mathcal{W}_m(t))'$. The dynamics of $\{\vec{\mathcal{W}}^{(k)}(t)\}_{t \geq 0}$ is uncertain and thus the admissibility of $q_k(t)$ is not well-defined.

Similar to [7], we characterize the problem of our interest through the G -framework of [11]. Because two insurers are heterogeneous (different confidence regions of ρ and different utility functions), we need to introduce two sets of G -expectation and G -Brownian motion. We then study the insurers' optimization problems respectively on two G -expectation spaces. To this end, we establish the probabilistic setup from the perspective of the insurer $k \neq m \in \{1, 2\}$. We first define by Θ_k the set of all insurer k 's feasible correlation ($\rho \in [\underline{\rho}_k, \bar{\rho}_k]$) choices that ensure a unique strong solution to the SDEs (1). Then the set of (non-equivalent) priors \mathcal{P}^{Θ_k} is defined as the set of probability measures \mathbb{P}_ρ on (Ω, \mathcal{F}_t) induced by \mathbb{P} : $\mathcal{P}^{\Theta_k} = \{\mathbb{P}_\rho : \rho \in \Theta_k\}$, where $\mathbb{P}_\rho(A) = \mathbb{P}(\{\omega : X \in A\})$, $A \in \mathcal{F}_T$ and $X = (X_1, X_2)'$ is the unique strong solution to SDEs (1) given ρ .

We denote by $\text{lip}(\mathbb{R}^2)$ the space of all bounded and Lipschitz functions on \mathbb{R}^2 . Then following [11], we define a G_k -normal distribution $P_t^{G_k}(\phi) : \text{lip}(\mathbb{R}^2) \rightarrow \mathbb{R}$ as

$$P_t^{G_k}(\phi(\vec{x})) = v_k(t, \vec{x}),$$

where $v_k(t, \vec{x})$ is a bounded continuous function on $[0, T] \times \mathbb{R}^2$ which is the viscosity solution of the nonlinear partial differential equation (PDE):

$$\frac{\partial v_k}{\partial t} - G_k(D^2 v_k) = 0, \quad v_k(0, \vec{x}) = \phi(\vec{x}), \quad \text{where } G_k(A) = \frac{1}{2} \sup_{\gamma \in \Gamma_k} \text{tr}[\gamma \gamma^T A]$$

for $A \in \mathbb{S}_2$, in which $D^2 v$ is the Hessian matrix of v , i.e. $D^2 v = \left(\frac{\partial^2 v}{\partial x_i \partial x_j} \right)$ and

$$\Gamma_k := \left\{ \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix} : \rho \in \Theta_k \right\} \subset \mathbb{R}^{2 \times 2},$$

which is bounded and closed. A *two-dimensional G_k -Brownian motion* $\vec{B}^{(k)}(t) = (B_k^{(k)}(t), B_m^{(k)}(t))'$ under a G_k -expectation $\widehat{\mathbb{E}}_k$ is then defined as follows:

1. For each $s, t \geq 0$ and $\phi \in \text{lip}(\mathbb{R}^2)$, $\vec{B}^{(k)}(t)$ and $\vec{B}^{(k)}(t+s) - \vec{B}^{(k)}(s)$ are identically distributed: $\widehat{\mathbb{E}}_k[\phi(\vec{B}^{(k)}(t+s) - \vec{B}^{(k)}(s))] = \widehat{\mathbb{E}}_k[\phi(\vec{B}^{(k)}(t))] := P_t^{G_k}(\phi)$.
2. For $0 \leq t_1 < \dots < t_m < \infty$, $m = 1, 2, \dots$, the increment $\vec{B}^{(k)}(t_m) - \vec{B}^{(k)}(t_{m-1})$ is “backwardly” independent from $\vec{B}^{(k)}(t_1), \dots, \vec{B}^{(k)}(t_{m-1})$: for each $\phi \in \text{lip}(\mathbb{R}^{2 \times m})$, $\widehat{\mathbb{E}}_k[\phi(\vec{B}^{(k)}(t_1), \dots, \vec{B}^{(k)}(t_{m-1}), \vec{B}^{(k)}(t_m))] = \widehat{\mathbb{E}}_k[\phi_1(\vec{B}^{(k)}(t_1), \dots, \vec{B}^{(k)}(t_{m-1}))]$, where $\phi_1(x^1, \dots, x^{m-1}) := \widehat{\mathbb{E}}_k[\phi(x^1, \dots, x^{m-1}, \vec{B}^{(k)}(t_m) - \vec{B}^{(k)}(t_{m-1}) + x^{m-1})]$, $x^1, \dots, x^{m-1} \in \mathbb{R}^2$.

The G_k -expectation $\widehat{\mathbb{E}}_k[\cdot]$ can be viewed as a proxy of $\sup_{\mathbb{P}_\rho \in \mathcal{P}^{\Theta_k}} \mathbb{E}^{\mathbb{P}_\rho}[\cdot]$ with our choice of G_k , where $\mathbb{E}^{\mathbb{P}_\rho}[\cdot]$ is the expectation operator under \mathbb{P}_ρ . Loosely speaking, denoting $\stackrel{d}{=}$ to be equality in distribution, we have $\vec{\mathcal{W}}^{(k)}(t) \stackrel{d}{=} \vec{B}^{(k)}(t)$ marginally because $\widehat{\mathbb{E}}_k[\phi(B_j^{(k)}(t))] = \mathbb{E}[\phi(\mathcal{W}_j(t))]$ for $j = k, m$ and $\phi \in \text{lip}(\mathbb{R})$. However, it is important to note that $\vec{\mathcal{W}}^{(k)}(t)$ and $\vec{B}^{(k)}(t)$ are not equivalent.

Replacing the original Brownian motion $\vec{\mathcal{W}}^{(k)}(t)$ with G -Brownian motion $\vec{B}^{(k)}(t)$ in (4), the dynamics of $\bar{X}_k^{q_k, q_m}(t)$ is given by

$$d\bar{X}_k^{q_k, q_m}(t) = [\bar{\delta}_k + \mu_k q_k(t) - \kappa_k \mu_m q_m(t)]dt + (\sigma_k q_k(t), -\kappa_k \sigma_m q_m(t))d\vec{B}^{(k)}(t) \quad (5)$$

with $\bar{X}_k^{q_k, q_m}(0) = \bar{x}_k$. As in [5, 7], the worst-case utility function is defined as

$$\bar{U}_k^{t, \bar{x}_k, q_k} := -\widehat{\mathbb{E}}_k[-U_k(\bar{X}_k^{q_k, q_m}(T)) | \mathcal{F}_t^{(k)}] = \inf_{\mathbb{P}_\rho \in \mathcal{P}^{\Theta_k}} \mathbb{E}^{\mathbb{P}_\rho}[U_k(\bar{X}_k^{q_k, q_m}(T)) | \mathcal{F}_t^{(k)}],$$

where $\mathcal{F}_t^{(k)} = \sigma(\{\vec{B}^{(k)}(s)\}_{s=0}^t)$. The admissible set of the insurer k 's reinsurance strategies is $Q_k = \{q_k(t) \in \mathcal{F}_t^{(k)} \mid q_k(t) \in [0, 1]\}$. Now the optimization problem (3) is well-defined under the G -framework. Hereafter, we replace the set " $\mathbb{P}_\rho \in \mathcal{P}^{\Theta_k}$ " by the notation " $\rho \in [\underline{\rho}_k, \bar{\rho}_k]$ " that reminds us the bounds of the correlation coefficient.

The coupled optimization problem (3) for two insurers forms a non-zero-sum game. A Nash equilibrium for two insurers is a 2-tuple $(q_1^*, q_2^*) \in Q_1 \times Q_2$ that satisfies the inequalities:

$$\begin{aligned} \textbf{Problem 1} \quad \inf_{\rho \in [\underline{\rho}_1, \bar{\rho}_1]} \mathbb{E}^{\mathbb{P}_\rho}[U_1(\bar{X}_1^{q_1^*, q_2^*}(T))] &\leq \inf_{\rho \in [\underline{\rho}_1, \bar{\rho}_1]} \mathbb{E}^{\mathbb{P}_\rho}[U_1(\bar{X}_1^{q_1^*, q_2^*}(T))], \\ \inf_{\rho \in [\underline{\rho}_2, \bar{\rho}_2]} \mathbb{E}^{\mathbb{P}_\rho}[U_2(\bar{X}_2^{q_2^*, q_1^*}(T))] &\leq \inf_{\rho \in [\underline{\rho}_2, \bar{\rho}_2]} \mathbb{E}^{\mathbb{P}_\rho}[U_2(\bar{X}_2^{q_2^*, q_1^*}(T))]. \end{aligned}$$

3. Nash Equilibrium

3.1. General case

In this section, we shall characterize the Nash equilibrium via the dynamic programming principle under the G -framework, as shown in [8]. Define the value function of the insurer $k \neq m \in \{1, 2\}$ as

$$V_k(t, \bar{x}_k) := \sup_{q_k \in Q_k} \bar{U}_k^{t, \bar{x}_k, q_k} = \sup_{q_k \in Q_k} \inf_{\rho \in [\underline{\rho}_k, \bar{\rho}_k]} \mathbb{E}^{\mathbb{P}_\rho} \left[U_k(\bar{X}_k^{q_k, q_m^*}(T)) \mid \bar{X}_k^{q_k, q_m^*}(t) = \bar{x}_k \right]. \quad (6)$$

Under the assumption that the utility function U_k satisfies the Inada condition, it can be checked that V_k is also increasing and strictly concave in $\bar{x}_k \in \mathbb{R}$, but we omit it here due to the page limit.

To simplify matter, we suppress the arguments of the functions. Denote $\mathcal{L}^{q_k, q_m; \rho_k}$ to be the infinitesimal generator for the relative surplus process $\bar{X}_k^{q_k, q_m}$. Then,

$$\mathcal{L}^{q_k, q_m; \rho_k} := (\bar{\delta}_k + \mu_k q_k - \kappa_k \mu_m q_m) \frac{\partial}{\partial \bar{x}_k} + \left(\frac{\sigma_k^2}{2} q_k^2 - \rho_k \kappa_k \sigma_k \sigma_m q_k q_m + \frac{\kappa_k^2 \sigma_m^2 (q_m)^2}{2} \right) \frac{\partial^2}{\partial \bar{x}_k^2}. \quad (7)$$

Analogous to [7, 8], we have the following verification theorem.

Theorem 3.1. The value function V_k in (6) is the unique deterministic continuous viscosity solution of the following Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation:

$$\frac{\partial V_k}{\partial t} + \sup_{q_k \in Q_k} \inf_{\rho_k \in [\underline{\rho}_k, \bar{\rho}_k]} \mathcal{L}^{q_k, q_m^*; \rho_k} V_k = 0 \quad (8)$$

with the terminal condition $V_k(T, \bar{x}_k) = U_k(\bar{x}_k)$.

Proof. The proof can be found in [8]. □

If $\frac{\partial^2 V_k}{\partial \bar{x}_k^2} < 0$, the infimum, we denoted by ρ_k^* , can be solved straightforwardly:

$$\rho_k^* := \arg \min_{\rho_k \in [\underline{\rho}_k, \bar{\rho}_k]} \mathcal{L}^{q_k, q_m^*; \rho_k} V_k(t, \bar{x}_k) = \underline{\rho}_k \mathbf{1}_{\{\kappa_k > 0\}} + \bar{\rho}_k \mathbf{1}_{\{\kappa_k < 0\}} = \frac{1}{\kappa_k} \min(\kappa_k \underline{\rho}_k, \kappa_k \bar{\rho}_k). \quad (9)$$

Then the HJBI equation (8) is reduced to a HJB equation:

$$\frac{\partial V_k}{\partial t} + \sup_{q_k \in Q_k} \mathcal{L}^{q_k, q_m^*; \rho_k^*} V_k = 0 \quad (10)$$

with the terminal condition $V_k(T, \bar{x}_k) = U_k(\bar{x}_k)$. By maximizing the quadratic form of q_k in (10), the expression of q_k^* is deduced:

$$\begin{aligned} q_k^* &:= \arg \max_{q_k \in Q_k} \left\{ \mu_k q_k \frac{\partial V_k(t, \bar{x}_k)}{\partial \bar{x}_k} + \left(\frac{\sigma_k^2}{2} q_k^2 - \rho_k^* \kappa_k \sigma_k \sigma_m q_k q_m^* \right) \frac{\partial^2 V_k(t, \bar{x}_k)}{\partial \bar{x}_k^2} \right\} \\ &= \left[\min \left(\rho_k^* \kappa_k \frac{\sigma_m}{\sigma_k} q_m^* - \frac{\mu_k}{\sigma_k^2} \left(\frac{\partial V_k(t, \bar{x}_k)}{\partial \bar{x}_k} \bigg/ \frac{\partial^2 V_k(t, \bar{x}_k)}{\partial \bar{x}_k^2} \right), 1 \right) \right]^+ \in Q_k, \end{aligned} \quad (11)$$

where $x^+ = \max(x, 0)$ and $-\frac{\partial V_k}{\partial \bar{x}_k} \bigg/ \frac{\partial^2 V_k}{\partial \bar{x}_k^2}$ represents the risk-tolerance level of the insurer k . Substituting q_k^* into the HJB equation (10) yields a nonlinear partial differential equation (PDE) for V_k : $\partial V_k / \partial t + \mathcal{L}^{q_k^*, q_m^*; \rho_k^*} V_k = 0$.

Now, we discuss how to verify the solution pair $(q_k^*, \bar{X}_k^{q_k^*, q_m^*})$, where $\bar{X}_k^{q_k^*, q_m^*}$ is the strong solution to (5) with q_k^* and q_m^* , is an optimal pair of our robust optimization problem. Suppose that we have found the value function $V_k \in C^{1,2}$ via solving the HJBI equation (8). Then, by G -Itô's lemma (on $-V_k$), we have

$$\begin{aligned} & V_k(t, \bar{x}_k) - V_k(T, \bar{X}_k^{q_k^*, q_m^*}(T)) \\ &= \int_t^T \left[-\frac{\partial V_k}{\partial t} - \inf_{\rho \in [\underline{\rho}_k, \bar{\rho}_k]} \mathcal{L}^{q_k^*, q_m^*; \rho_k} V_k \right] du - \int_t^T \frac{\partial V_k}{\partial \bar{x}_k} \bar{\sigma}^{(k)} d\bar{B}^{(k)}(u) \\ & \quad - \int_t^T \frac{1}{2} \frac{\partial^2 V_k}{\partial \bar{x}_k^2} \sum_{i,j=1}^2 \bar{\sigma}_i^{(k)} \bar{\sigma}_j^{(k)} d\langle B^{(k,i)}, B^{(k,j)} \rangle_u - \int_t^T 2G_k \left(-\frac{1}{2} \frac{\partial^2 V_k}{\partial \bar{x}_k^2} \bar{\Sigma}^{(k)} \right) du, \quad (12) \end{aligned}$$

where $\bar{\sigma}^{(k)} = (\bar{\sigma}_1^{(k)}, \bar{\sigma}_2^{(k)}) := (\sigma_k q_k^*(t), -\kappa_k \sigma_m q_m^*(t))$, $B^{(k)} := (B^{(k,1)}, B^{(k,2)})'$, $\langle B^{(k,i)}, B^{(k,j)} \rangle$ is mutual variation process of $B^{(k,i)}$ and $B^{(k,j)}$, and $\bar{\Sigma}^{(k)} = \bar{\sigma}^{(k)'} \bar{\sigma}^{(k)}$. Notice that

$$\arg \min_{\rho \in [\underline{\rho}_k, \bar{\rho}_k]} \mathcal{L}^{q_k^*, q_m^*; \rho_k} V_k = \rho_k^*.$$

Hence, $\partial V_k / \partial t + \inf_{\rho \in [\underline{\rho}_k, \bar{\rho}_k]} \mathcal{L}^{q_k^*, q_m^*; \rho_k} V_k = \partial V_k / \partial t + \mathcal{L}^{q_k^*, q_m^*; \rho_k^*} V_k = 0$. Moreover, as shown in [11],

$$\begin{aligned} \xi_T &:= - \int_t^T \frac{\partial V_k}{\partial \bar{x}_k} \bar{\sigma}^{(k)} d\bar{B}^{(k)}(u) - \int_t^T \frac{1}{2} \frac{\partial^2 V_k}{\partial \bar{x}_k^2} \sum_{i,j=1}^2 \bar{\sigma}_i^{(k)} \bar{\sigma}_j^{(k)} d\langle B^{(k,i)}, B^{(k,j)} \rangle_u \\ & \quad - \int_t^T 2G_k \left(-\frac{1}{2} \frac{\partial^2 V_k}{\partial \bar{x}_k^2} \bar{\Sigma}^{(k)} \right) du, \quad T \geq t, \end{aligned}$$

is a G_k -martingale. Hence, we have $\hat{\mathbb{E}}_k[\xi_T | \mathcal{F}_t^{(k)}] = \xi_t = 0$. Taking $\hat{\mathbb{E}}_k[\cdot | \mathcal{F}_t^{(k)}]$ on both sides of (12) yields

$$V_k(t, \bar{x}_k) = -\hat{\mathbb{E}}_k[-U(\bar{X}_k^{q_k^*, q_m^*}(T)) | \mathcal{F}_t^{(k)}] = \bar{U}_k^{t, \bar{x}_k, q_k^*}.$$

The left-hand side of the above equation is the value function of our robust optimization problem, while the right-hand side is the worst-case utility for the reinsurance strategy q_k^* (given q_m^*). This yields that q_k^* is an optimal strategy of our robust optimization problem.

Remark: For CARA insurers considered in Section 3.2, the value functions are sufficiently smooth for the verification. However, the verification for a non-smooth value function V_k poses an interesting question for future research but certainly beyond the scope of this paper.

Theorem 3.2 characterizes the Nash equilibrium of Problem 1 as the solution of the coupled partial differential equations. More specifically, we have

Theorem 3.2. Assume that $\frac{\partial^2 V_k(t, \bar{x}_k)}{\partial x_k^2} < 0$, for $k = 1, 2$, where V_k is the solution to the HJBI equation in (8). The Nash equilibrium reinsurance strategy pair for Problem 1 is the solution of the following coupled non-linear equations:

$$\begin{cases} q_1^*(t) = \left[\min \left(\rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} q_2^* - \frac{\mu_1}{\sigma_1^2} \left(\frac{\partial V_1(t, \bar{x}_1)}{\partial \bar{x}_1} \Big/ \frac{\partial^2 V_1(t, \bar{x}_1)}{\partial \bar{x}_1^2} \right), 1 \right) \right]^+, \\ q_2^*(t) = \left[\min \left(\rho_2^* \kappa_2 \frac{\sigma_1}{\sigma_2} q_1^* - \frac{\mu_2}{\sigma_2^2} \left(\frac{\partial V_2(t, \bar{x}_2)}{\partial \bar{x}_2} \Big/ \frac{\partial^2 V_2(t, \bar{x}_2)}{\partial \bar{x}_2^2} \right), 1 \right) \right]^+, \end{cases} \quad (13)$$

where ρ_k^* admits the form in (9), and the Nash equilibrium value functions are V_1 and V_2 , which then become the solutions of the following system of coupled PDEs:

$$\begin{cases} \frac{\partial V_1(t, \bar{x}_1)}{\partial t} + \mathcal{L}^{q_1^*, q_2^*; \rho_1^*} V_1(t, \bar{x}_2) = 0, \\ \frac{\partial V_2(t, \bar{x}_2)}{\partial t} + \mathcal{L}^{q_2^*, q_1^*; \rho_2^*} V_2(t, \bar{x}_2) = 0, \end{cases} \quad (14)$$

with the terminal conditions $V_1(T, \bar{x}_1) = U_1(\bar{x}_1)$ and $V_2(T, \bar{x}_2) = U_2(\bar{x}_2)$.

The Nash equilibrium reinsurance strategies (q_1^*, q_2^*) in (13) reproduces to that in [2] for the case of certain correlation. See also [13] for the associated Nash equilibrium for the non-zero-sum reinsurance game with model uncertainty. The key difference (q_1^*, q_2^*) in (13) from those in the existing literature is that $\rho_1^* \kappa_1$ and $\rho_2^* \kappa_2$ can have different signs when there is ambiguous correlation. It complicates the analysis of this system.

More importantly, Theorem 3.2 shows that the existence of Nash equilibrium is equivalent to the solvability of the coupled systems in (13), which in turn is equivalent to the solvability of the coupled PDEs in (14). As remarked in [2], the general existence of the solution to the coupled PDEs in (14) is very difficult to establish for any $T > 0$. For a sufficiently small time $T > 0$, however, the local existence and uniqueness of the solution to the coupled PDEs in (14) can be established via the Cauchy-Kowalevski Theorem. Interestingly enough, we show that for the representative case of CARA insurers, the corresponding coupled equations in (13) and coupled PDEs in (14) can be explicitly solved and that the Nash equilibrium is established for any $T > 0$.

3.2. CARA Insurers

In this section, we consider constant absolute risk aversion (CARA) insurer $k \in \{1, 2\}$ who has an exponential utility function

$$U_k(x) = -\frac{1}{\gamma_k} \exp(-\gamma_k x), \quad (15)$$

where $\gamma_k > 0$ is the risk aversion coefficient of the insurer k . The following theorem shows that the Nash equilibrium reinsurance strategies and value functions in Theorem 3.2 admit closed-forms for the case of two CARA insurers.

Theorem 3.3. Assume that $\kappa_1\kappa_2\rho_1^*\rho_2^* \neq 1$. If the insurer k , for $k = 1, 2$, has the exponential utility (15), the Nash equilibrium value function (6) of the insurer k admits a closed-form solution:

$$V_k(t, \bar{x}_k) = -\frac{1}{\gamma_k} \exp(-\gamma_k[\bar{x}_k + f_k(t)]),$$

where

$$f_k(t) = \left[\bar{\delta}_k + \mu_k q_k^* - \kappa_k \mu_m q_m^* - \gamma_k \left(\frac{\sigma_k^2 (q_k^*)^2}{2} - \rho_k^* \kappa_k \sigma_k \sigma_m q_k^* q_m^* + \frac{\kappa_k^2 \sigma_m^2 (q_m^*)^2}{2} \right) \right] (T - t) \quad (16)$$

with the optimal choice of ρ for the insurer $k \in \{1, 2\}$:

$$\rho_k^* = \underline{\rho}_k \mathbf{1}_{\{\kappa_k > 0\}} + \bar{\rho}_k \mathbf{1}_{\{\kappa_k < 0\}} = \frac{1}{\kappa_k} \min(\kappa_k \underline{\rho}_k, \kappa_k \bar{\rho}_k), \quad (17)$$

and the Nash equilibrium reinsurance strategies (q_1^*, q_2^*) are specified as follows. Define

$$\begin{aligned} \tilde{q}_1 &:= \frac{1}{1 - \rho_1^* \rho_2^* \kappa_1 \kappa_2} \left(\frac{\rho_1^* \kappa_1 \mu_2}{\gamma_2 \sigma_1 \sigma_2} + \frac{\mu_1}{\gamma_1 \sigma_1^2} \right), \quad \tilde{q}_2 := \frac{1}{1 - \rho_1^* \rho_2^* \kappa_1 \kappa_2} \left(\frac{\rho_2^* \kappa_2 \mu_1}{\gamma_1 \sigma_1 \sigma_2} + \frac{\mu_2}{\gamma_2 \sigma_2^2} \right), \quad (18) \\ h_{1h} &:= \rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} + \frac{\mu_1}{\gamma_1 \sigma_1^2}, \quad h_{2v} := \rho_2^* \kappa_2 \frac{\sigma_1}{\sigma_2} + \frac{\mu_2}{\gamma_2 \sigma_2^2}, \quad h_{1x} := \frac{\mu_1}{\gamma_1 \sigma_1^2}, \quad h_{2y} := \frac{\mu_2}{\gamma_2 \sigma_2^2}. \end{aligned}$$

Following cases are possible.

1. If $\tilde{q}_1 > 0, \tilde{q}_2 > 0$,

$$(q_1^*, q_2^*) = \begin{cases} (\min(\tilde{q}_1, h_{1h}, 1), \min(\tilde{q}_2, h_{2v}, 1)), & \text{if } \kappa_1 \rho_1^* \geq 0, \kappa_2 \rho_2^* \geq 0, \\ (\min(\tilde{q}_1, h_{1h}, 1), \min(\max(\tilde{q}_2, h_{2v}), 1)), & \text{if } \kappa_1 \rho_1^* \geq 0, \kappa_2 \rho_2^* < 0, \\ (\min(\max(\tilde{q}_1, h_{1h}), 1), \min(\tilde{q}_2, h_{2v}, 1)), & \text{if } \kappa_1 \rho_1^* < 0, \kappa_2 \rho_2^* \geq 0, \\ (\min(\max(\tilde{q}_1, h_{1h}), 1), \min(\max(\tilde{q}_2, h_{2v}), 1)), & \text{if } \kappa_1 \rho_1^* < 0, \kappa_2 \rho_2^* < 0; \end{cases}$$

2. If $\tilde{q}_1 > 0, \tilde{q}_2 \leq 0, (q_1^*, q_2^*) = (\min(h_{1x}, 1), [\min(h_{2v}, 1)]^+)$;

3. If $\tilde{q}_1 \leq 0, \tilde{q}_2 > 0, (q_1^*, q_2^*) = ([\min(h_{1h}, 1)]^+, \min(h_{2y}, 1))$.

Proof. To solve (13) and (14), we begin with the following *Ansatz* V_k , for $k = 1, 2$,

$$V_k(t, \bar{x}_k) = -\frac{1}{\gamma_k} \exp(-\gamma_k[\bar{x}_k + f_k(t)]), \quad (19)$$

where $f_k(t)$ is to be determined. Plugging the ansatz (19) into the system (13), we have

$$\begin{cases} q_1^* &= \left[\min \left(\rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} q_2^* + \frac{\mu_1}{\gamma_1 \sigma_1^2}, 1 \right) \right]^+, \\ q_2^* &= \left[\min \left(\rho_2^* \kappa_2 \frac{\sigma_1}{\sigma_2} q_1^* + \frac{\mu_2}{\gamma_2 \sigma_2^2}, 1 \right) \right]^+. \end{cases} \quad (20)$$

For the cases of $\kappa_1 \rho_1^*$ and $\kappa_2 \rho_2^*$ having the same sign, the system (20) is solved similarly to that of [2, 14]. Hence, we only discuss the case of $\kappa_1 \rho_1^* \geq 0$, $\kappa_2 \rho_2^* < 0$, while the case of $\kappa_1 \rho_1^* < 0$, $\kappa_2 \rho_2^* \geq 0$ is treated similarly as these two cases are symmetric.

The pair $(\tilde{q}_1, \tilde{q}_2)$ defined in (18) is the intersection point of the following lines on the (q_1, q_2) -plane:

$$\iota_1 : q_1 = \rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} q_2 + \frac{\mu_1}{\gamma_1 \sigma_1^2}, \quad \iota_2 : q_2 = \rho_2^* \kappa_2 \frac{\sigma_1}{\sigma_2} q_1 + \frac{\mu_2}{\gamma_2 \sigma_2^2}.$$

Denote the q_1 - and q_2 -intercepts of ι_1 and ι_2 by $(h_{1x}, 0)$, $(0, h_{1y})$ and $(h_{2x}, 0)$, $(0, h_{2y})$, respectively.

Then,

$$h_{1x} = \frac{\mu_1}{\gamma_1 \sigma_1^2}, \quad h_{1y} = -\frac{\mu_1}{\rho_1^* \kappa_1 \gamma_1 \sigma_1 \sigma_2}, \quad h_{2x} = -\frac{\mu_2}{\rho_2^* \kappa_2 \gamma_2 \sigma_1 \sigma_2}, \quad h_{2y} = \frac{\mu_2}{\gamma_2 \sigma_2^2}.$$

Let ℓ_h be the horizontal line $\ell_h : q_2 = 1$, ℓ_v be the vertical line $\ell_v : q_1 = 1$, h_{1h} (h_{2h}) be the q_1 -intersect of ι_1 (ι_2) and ℓ_h , and h_{1v} (h_{2v}) be the q_2 -intersect of ι_1 (ι_2) and ℓ_v . Then,

$$\begin{aligned} h_{1h} &= \rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} + \frac{\mu_1}{\gamma_1 \sigma_1^2}, \quad h_{2h} = \left(1 - \frac{\mu_2}{\gamma_2 \sigma_2^2} \right) / \left(\rho_2^* \kappa_2 \frac{\sigma_1}{\sigma_2} \right), \\ h_{1v} &= \left(1 - \frac{\mu_1}{\gamma_1 \sigma_1^2} \right) / \left(\rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} \right), \quad h_{2v} = \rho_2^* \kappa_2 \frac{\sigma_1}{\sigma_2} + \frac{\mu_2}{\gamma_2 \sigma_2^2}. \end{aligned}$$

Since $\kappa_1 \rho_1^* \geq 0$, $\kappa_2 \rho_2^* < 0$, we have

$$h_{1x} > 0, \quad h_{1y} < 0, \quad h_{2x} > 0, \quad h_{2y} > 0, \quad h_{1h} > 0, \quad \tilde{q}_1 > 0$$

and the following relationships hold: $h_{1x} \geq h_{2x} \Leftrightarrow \tilde{q}_2 \leq 0$;

$$h_{1h} > 0 \Leftrightarrow h_{1y} < 1; \quad h_{1v} > 0 \Leftrightarrow h_{1x} < 1; \quad h_{1v} > 1 \Leftrightarrow h_{1h} < 1;$$

$$h_{2v} < 0 \Leftrightarrow h_{2x} < 1; \quad h_{2h} < 0 \Leftrightarrow h_{2y} < 1; \quad h_{2v} < 1 \Leftrightarrow h_{2h} < 1.$$

(q_1^*, q_2^*) is the intersection point of the following lines:

$$\ell_1 : \begin{cases} q_1 = 0, & q_2 \leq h_{1y}, \\ \iota_1, & h_{1y} < q_2 < h_{1v}, \\ q_1 = 1, & q_2 \geq h_{1v}, \end{cases} \quad \ell_2 : \begin{cases} q_2 = 1, & q_1 \leq h_{2h}, \\ \iota_2, & h_{2h} < q_1 < h_{2x}, \\ q_2 = 0, & q_1 \geq h_{2x}. \end{cases}$$

It is clear that ℓ_1 is increasing and ℓ_2 is decreasing in q_1 and q_2 .

- If $\tilde{q}_2 \leq 0$, then $h_{1x} \geq h_{2x}$.
 - If $h_{2x} \leq 1$, then $h_{2v} \leq 0$ and ℓ_2 for $q_1 < h_{2x}$ ($\Rightarrow q_2 > 0$) does not intersect with ℓ_1 for $q_1 < \min(h_{1x}, 1)$ ($\Rightarrow q_2 < 0$). Thus, $(q_1^*, q_2^*) = (\min(h_{1x}, 1), 0)$.
 - If $h_{2x} > 1$, then $h_{1x} > 1$ and ℓ_1 for $q_1 < 1$ always below q_1 -axis. Hence, $q_1^* = 1$ and we know from (20) that $q_2^* = [\min(h_{2v}, 1)]^+$.

To sum up, in this case ($\tilde{q}_1 > 0$, $\tilde{q}_2 \leq 0$), we have

$$(q_1^*, q_2^*) = (\min(h_{1x}, 1), [\min(h_{2v}, 1)]^+).$$

- If $\tilde{q}_2 > 0$, then $h_{1x} < h_{2x}$. Notice that $\tilde{q}_1 < h_{1h} \Leftrightarrow \tilde{q}_2 < 1$ and $\tilde{q}_2 < h_{2v} \Leftrightarrow \tilde{q}_1 > 1$.
 - If $\tilde{q}_1 < 1$, $\tilde{q}_2 < 1$, then $\tilde{q}_1 < h_{1h}$, $h_{2v} < \tilde{q}_2$ and it is clear that $(q_1^*, q_2^*) = (\tilde{q}_1, \tilde{q}_2)$.
 - If $\tilde{q}_1 < 1$, $\tilde{q}_2 \geq 1$, then $h_{1h} \leq \tilde{q}_1 \leq h_{2h}$, $h_{2v} < \tilde{q}_2 < h_{1v}$ and ℓ_2 for $q_2 < 1$ ($\Rightarrow q_1 > h_{2h}$) does not intersect with ℓ_1 for $q_1 < h_{1h}$. Hence, $q_2^* = 1$ and $q_1^* = [\min(h_{1h}, 1)]^+ = h_{1h}$.
 - If $\tilde{q}_1 \geq 1$, $\tilde{q}_2 < 1$, then similar to the previous case, we have $h_{1h} > \tilde{q}_1 > h_{2h}$, $h_{2v} \geq \tilde{q}_2 \geq h_{1v}$ and $(q_1^*, q_2^*) = (1, [\min(h_{2v}, 1)]^+)$. Notice that $h_{2x} > \tilde{q}_1 \geq 1$. Hence $h_{2v} > 0$ and $(q_1^*, q_2^*) = (1, \min(h_{2v}, 1))$.
 - If $\tilde{q}_1 \geq 1$, $\tilde{q}_2 \geq 1$, then $h_{1h} \leq \tilde{q}_1 \leq h_{2h}$, $h_{1v} \leq \tilde{q}_2 \leq h_{2v}$, and ℓ_2 for $q_2 < 1$ ($\Rightarrow q_1 > h_{2h}$) does not intersect with ℓ_1 for $q_1 < h_{1h}$. Hence, $q_2^* = 1$ and $q_1^* = [\min(h_{1h}, 1)]^+ = \min(h_{1h}, 1)$.

To sum up, in this case ($\tilde{q}_1 > 0$, $\tilde{q}_2 > 0$), we have

$$(q_1^*, q_2^*) = (\min(\tilde{q}_1, h_{1h}, 1), \min(\max(\tilde{q}_2, h_{2v}), 1)).$$

Together with the results of [14], the equilibrium reinsurance strategies q_1^* and q_2^* are completely solved.

Using explicit solution of (q_1^*, q_2^*) and the ansatz (19), the PDE (14) for V_k becomes an ordinary differential equation (ODE) for f_k :

$$\frac{\partial f_k}{\partial t} + \bar{\delta}_k + \mu_k q_k^* - \kappa_k \mu_m q_m^* - \gamma_k \left(\frac{\sigma_k^2 (q_k^*)^2}{2} - \rho_k^* \kappa_k \sigma_k \sigma_m q_k^* q_m^* + \frac{\kappa_k^2 \sigma_m^2 (q_m^*)^2}{2} \right) = 0$$

with terminal condition $f_k(T) = 0$. The solution of $f_k(t)$ is presented in (16). □

Since $\kappa_k \in [-1, 1]$, $\rho_k \in [-1, 1]$, for $k = 1, 2$, and the fact that there is no clear economic rationale behind a game between a competitive ($\kappa_k > 0$) insurer and a cooperative ($\kappa_m < 0$) insurer, the assumption $\kappa_1 \kappa_2 \rho_1^* \rho_2^* \neq 1$ in Theorem 3.3 is relatively mild for it constitutes the following extreme scenarios:

Case (1a) $\kappa_k = 1$, $\underline{\rho}_k = \bar{\rho}_k = 1$, for $k = 1, 2$;

Case (1b) $\kappa_k = 1$, $\underline{\rho}_k = -1$, $\bar{\rho}_k \in (-1, 1]$, for $k = 1, 2$;

Case (2a) $\kappa_k = -1$, $\underline{\rho}_k = \bar{\rho}_k = -1$, for $k = 1, 2$;

Case (2b) $\kappa_k = -1$, $\underline{\rho}_k \in [-1, 1)$, $\bar{\rho}_k = 1$, for $k = 1, 2$;

Case (1a) is uninteresting as there is no ambiguity in correlation, which has already been studied in [2]. In Case (1b), both competing insurers have their lowest correlation estimates to be equal to -1 , i.e. $\underline{\rho}_k = -1$. From (17), it follows readily that $\rho_1^* = \rho_2^* = -1$. In face of $\rho_k^* = -1$, insurer k would increase demand of the reinsurance protection as her claim risk is perfectly negative-correlated with that her competitor. Yet, increase demand of the reinsurance protection implies an increase in expenditure, which in turn erodes her relative performance surplus. Hence, there does not exist a strategy for each insurer that can both hedge against claim risk via reinsurance purchase while at the same time maximize her relative terminal surplus. Cases (2a) and (2b) can be interpreted analogously. In this respect, Cases (1a)-(2b) constitute extreme scenarios that yield little economic insights into the strategic demand of reinsurance protection of the insurers, and we shall therefore focus on the case when $\kappa_1 \kappa_2 \rho_1^* \rho_2^* \neq 1$ in the numerical studies below.

Insurer 1							Insurer 2						
x_1	p_1	θ_1	λ_1	$\mathbb{E}[\eta_1]$	$\mathbb{E}[\eta_1^2]$	γ_1	x_2	p_2	θ_2	λ_2	$\mathbb{E}[\eta_2]$	$\mathbb{E}[\eta_2^2]$	γ_2
5	5	7	0.8	2.5	80	0.4	7	2	4	0.5	2	50	0.5

Table 1: Model parameters

4. Numerical Examples

In this section, we provide the numerical studies on the effect of the ambiguous correlation to the equilibrium reinsurance demands and the value functions of the CARA insurers in Section 3.2. Unless otherwise stated, the following numerical studies are performed based on the model parameters in Table 1. Due to the symmetric nature of the game, it suffices to consider exclusively on the equilibrium demand and value function of insurer 1. We shall denote $\rho_{min}^k := \underline{\rho}_k$ and $\rho_{max}^k := \bar{\rho}_k$, for $k = 1, 2$, in the following figures.

4.1. Effect of $[\underline{\rho}_k, \bar{\rho}_k]$ on the equilibrium reinsurance demand of insurer k , for $k = 1, 2$.

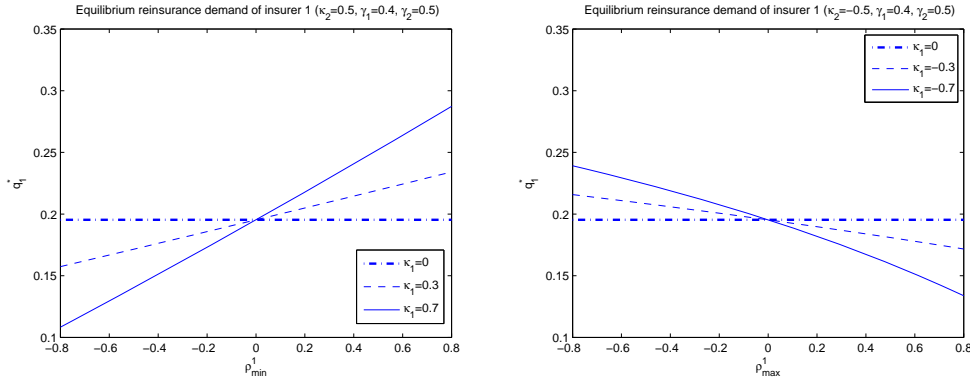


Figure 1: Effect of κ_1 on the equilibrium reinsurance demand of insurer 1,. The left figure (resp. right figure) corresponds to the case when both insurers are competitive, i.e. $\kappa_1 = 0.8, \kappa_2 = 0.5$ (resp. cooperative, i.e. $\kappa_1 = -0.8, \kappa_2 = -0.5$), with $[\underline{\rho}_2, \bar{\rho}_2] = [0.1, 0.6]$.

Figure 1 studies the impact of the ambiguous correlation, which is measured by the confidence interval $[\underline{\rho}_1, \bar{\rho}_1]$, to the equilibrium reinsurance demand of insurer 1. Since insurer 1 optimally chooses his reinsurance strategy under the worst-case scenario of ρ , Theorem 3.3 states that the

equilibrium reinsurance demand of insurer 1, q_1^* , is sensitive only to her own lowest correlation estimate, $\underline{\rho}_1$ (resp. her highest correlation estimate, $\bar{\rho}_1$), when both insurers are competitive, i.e. $\kappa_1, \kappa_2 > 0$ (resp. cooperative, i.e. $\kappa_1, \kappa_2 < 0$). On the left-hand side of Figure 1, we see that q_1^* increases as $\underline{\rho}_1$ increases across different values of the sensitivity parameter κ_1 when both insurers are competitive. As $\underline{\rho}_1$ increases, insurer 1 becomes more confident on her correlation estimation and that she is more positively correlated with her competitor. Hence increasing reinsurance protection would not improve her relative terminal surplus with respect to that of her competitor. Therefore, she would spend less money on the reinsurance protection $1 - q_1^*$, which is consistent with the result in [2] when there is no ambiguous correlation. On the other hand, when both insurers are cooperative, i.e. $\kappa_1, \kappa_2 < 0$, the right-hand side of Figure 1 shows that q_1^* decreases as the $\bar{\rho}_1$ increases. When both insurers are cooperative, each would choose her reinsurance strategy so as to maximize the sum of her and her cooperator's terminal surpluses. Therefore, as $\bar{\rho}_1$ increases, insurer 1 becomes less confident on her correlation with her cooperator, and therefore would increase her purchase of the reinsurance protection, i.e. $1 - q_1^*$.

4.2. Nature of externalities of $[\underline{\rho}_m, \bar{\rho}_m]$ on insurer k , for $k \neq m = 1, 2$.

Perhaps the most relevant question in the non-zero-sum game subject to the ambiguous correlation is whether the quality of counterparty's correlation estimates would induce positive or negative externalities to the insurer. We study this question for the cases when both insurers are competitive (see Figure 2) and when both insurers are cooperative (see Figure 3). Note that the case when insurer 1 is certain on her correlation estimate i.e. $\underline{\rho}_1 = \bar{\rho}_1$, but insurer 2 remains uncertain with the true correlation is non-existent. This is because insurer 2 would also have no ambiguity in correlation as she can observe the correlation from insurer 1. To facilitate the discussion, we shall consider the following two cases:

Case A “No Ambiguity in both insurers”, i.e. when both insurers are certain about the correlation (ρ). In this case, we shall let $\rho = 0.35$ in Figure 2 and $\rho = -0.35$ in Figure 3;

Case B “Ambiguity in both insurers”, i.e. when both insurers are uncertain about ρ .

Few words are in place to facilitate the following discussions. Recall that the true correlation

is assumed to be $\rho = 0.35$ in Figure 2. In addition, since $[0.2, 0.6] \subset [-0.2, 0.6]$, the case of $[\underline{\rho}_2, \bar{\rho}_2] = [0.2, 0.6]$ can be interpreted as the scenario in which insurer 2 has better information on the correlation estimates than the case of $[\underline{\rho}_2, \bar{\rho}_2] = [-0.2, 0.6]$. On the other hand, we assume the true correlation to be $\rho = -0.35$ in Figure 3. Since $[-0.5, -0.2] \subset [-0.5, 0.2]$, the case of $[\underline{\rho}_2, \bar{\rho}_2] = [-0.5, -0.2]$ constitutes the case when insurer 2 has better information on the correlation than the case of $[\underline{\rho}_2, \bar{\rho}_2] = [-0.5, 0.2]$.

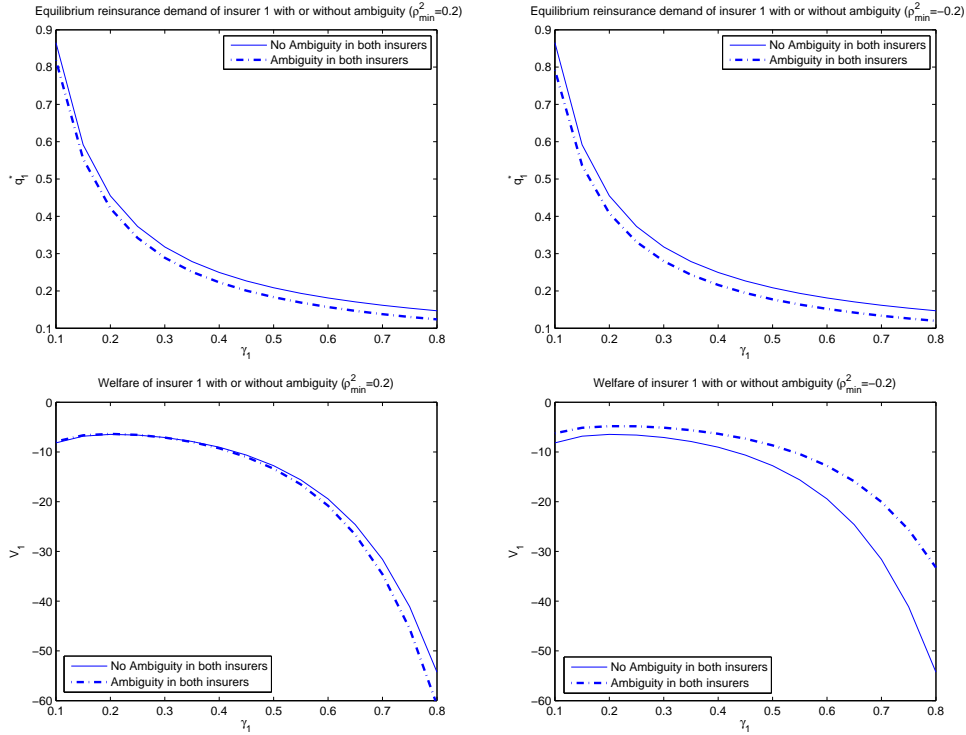


Figure 2: Effect of γ_1 on the equilibrium reinsurance demand and value function of insurer 1 ($t = 6, T = 10, [\underline{\rho}_1, \bar{\rho}_1] = [0.2, 0.5]$). The figures in the first column (resp. second column) correspond to the case when both insurers are competitive, i.e. ($\kappa_1 = 0.8, \kappa_2 = 0.5$), when $[\underline{\rho}_2, \bar{\rho}_2] = [0.2, 0.6]$ (resp. $[\underline{\rho}_2, \bar{\rho}_2] = [-0.2, 0.6]$).

Figure 2 reveals the impact of the insurer 2's confidence interval $[\underline{\rho}_2, \bar{\rho}_2]$ on insurer 1 for Cases A and B when both insurers are competitive, i.e. $\kappa_1, \kappa_2 > 0$. We first discuss the effect of the quality of insurer 2's correlation estimates on the equilibrium reinsurance demand of insurer 1. The first row of Figure 2 shows that insurer 1 always spends more on the reinsurance protection $1 - q_1^*$ at equilibrium when facing ambiguous correlation for both cases when $[\underline{\rho}_2, \bar{\rho}_2] = [0.2, 0.6]$

and $[\underline{\rho}_2, \bar{\rho}_2] = [-0.2, 0.6]$. This can be interpreted from from (13) that the relative performance concerns take effect on the insurer 1's strategy through the term $\rho_1^* \kappa_1 \frac{\sigma_2}{\sigma_1} q_2^*$ and that ρ_1^* is chosen such that $\rho \kappa_1$ is minimized.

More importantly, the second row of Figure 2 also indicates that the welfare, which is measured by her value function (V_1) of insurer 1 deteriorates as insurer 2 improves her correlation estimates from $[\underline{\rho}_2, \bar{\rho}_2] = [-0.2, 0.6]$ to $[\underline{\rho}_2, \bar{\rho}_2] = [0.2, 0.6]$. To see this, observe that the value function of insurer 1 (V_1) in Case B lies below that in Case A when $[\underline{\rho}_2, \bar{\rho}_2] = [0.2, 0.6]$. On the other hand, the value function of insurer 1 (V_1) in Case B lies above that in Case A when $[\underline{\rho}_2, \bar{\rho}_2] = [-0.2, 0.6]$. In other words, the ambiguous correlation can induce *negative* externalities to insurer 1 in the case when insurer 2 improves her correlation estimates, i.e. $[\underline{\rho}_2, \bar{\rho}_2] = [0.2, 0.6]$. This can be explained by the fact that two insurers are assumed to be competitive, i.e. $\kappa_1, \kappa_2 > 0$. Poorer correlation estimates from insurer 2 proves to be advantageous to insurer 1 as he has better information $[\underline{\rho}_1, \bar{\rho}_1] = [0.2, 0.5]$ to achieve higher relative terminal surplus.

Figure 3 studies the impact of the insurer 2's confidence interval $[\underline{\rho}_2, \bar{\rho}_2]$ on insurer 1 for Cases A and B when both insurers are cooperative, i.e. $\kappa_1, \kappa_2 < 0$. As in Figure 2, the first row of Figure 3 shows that ambiguous correlation leads to higher demand for reinsurance protection $1 - q_1^*$ at equilibrium even when both insurers are cooperative. The second row of Figure 3 shows the welfare of insurer 1 improves when insurer 2 improves her correlation estimates from $[\underline{\rho}_2, \bar{\rho}_2] = [-0.5, 0.2]$ to $[\underline{\rho}_2, \bar{\rho}_2] = [-0.5, -0.2]$, which can be seen by the gap between the welfare of insurer 1 in Case A and that in Case B is narrowed when $\bar{\rho}_2$ decreases from 0.2 to -0.2 , *ceteris paribus*. When insurer 2 has better information on the correlation ρ , she can therefore increase her relative terminal surplus, which in turn also increases the relative terminal surplus of insurer 1, as they are cooperative. In other words, the improvement of the correlation estimates of insurer 2 can induce *positive* externalities to insurer 1 when two insurers are cooperative, which is in contrast with the results for the case of competitive insurers in Figure 2.

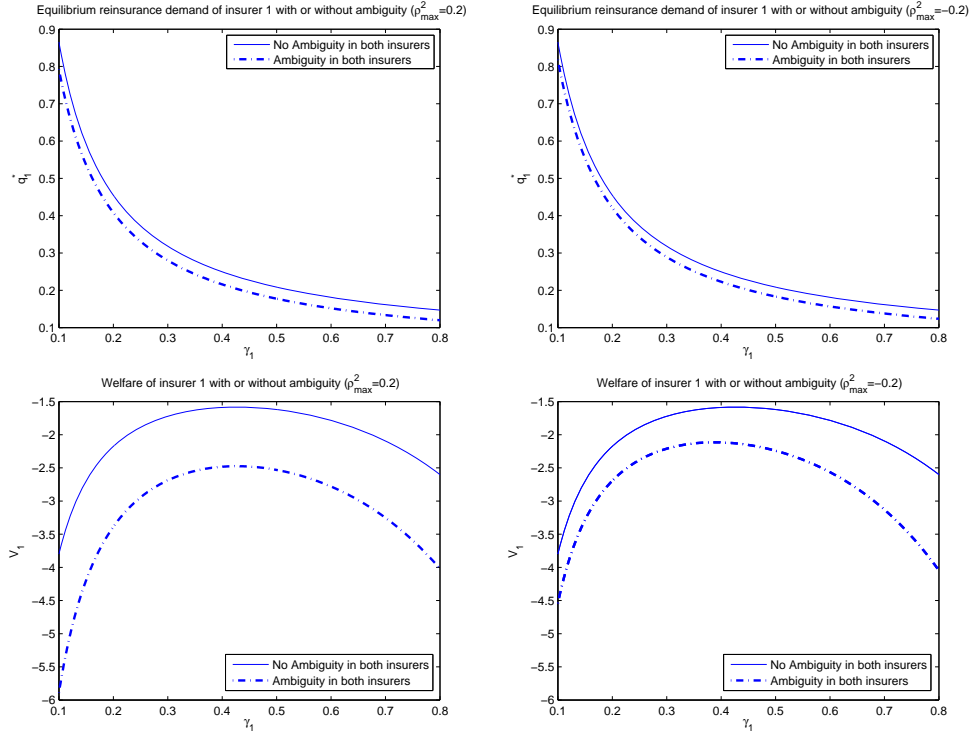


Figure 3: Effect of γ_1 on the equilibrium reinsurance demand and value function of insurer 1 ($T = 10$, $[\rho_1, \bar{\rho}_1] = [-0.5, -0.2]$). The figures in the first column (resp. second column) correspond to the case when both insurers are cooperative, i.e. ($\kappa_1 = -0.8$, $\kappa_2 = -0.5$), when $[\rho_2, \bar{\rho}_2] = [-0.5, 0.2]$ (resp. $[\rho_2, \bar{\rho}_2] = [-0.5, -0.2]$).

5. Conclusion

We essentially set the risk-free interest rate to be zero for simplicity. In fact, the extension to the case of non-zero risk-free interest rate is trivial. Although we focus on the reinsurance games with uncertain correlation between two insurers, it is possible to allow the insurers invest their surplus in financial market, especially when the financial market is statistically independent of the claim processes. In this special case, the investment-reinsurance game is actually a combination of a reinsurance game and an independent investment game, as pointed out in [14]. See [2, 10, 13] for the extensions to including investment opportunities. However, when the financial market is dependent of the insurance market, the correlation structure of the diffusions processes could be complicated. It would be an interesting future research. This paper concerns the robustness on the correlation. We refer readers to [13, 14] for the robustness on the drift terms of the diffusion processes.

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