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Completeness of Randomized Kinodynamic Planners with State-based Steering

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Abstract

Probabilistic completeness is an important property in motion planning. Although it has been established with clear assumptions for geometric planners, the panorama of completeness results for kinodynamic planners is still incomplete, as most existing proofs rely on strong assumptions that are difficult, if not impossible, to verify on practical systems. In this paper, we focus on an important class of kinodynamic planners, namely those that interpolate trajectories in the state space. We provide a proof of probabilistic completeness for such planners under assumptions that can be readily verified from the system’s equations of motion and the user-defined interpolation function. Our proof relies crucially on a property of interpolated trajectories, termed second-order continuity (SOC), which we show is tightly related to the ability of a planner to benefit from denser sampling. We analyze the impact of this property in simulations on a low-torque pendulum. Our results show that a simple RRT using a second-order continuous interpolation swiftly finds solution, while it is impossible for the same planner using standard Bezier curves (which are not SOC) to find any solution.\footnote{This paper is a revised and expanded version of \cite{1}, which was presented at the International Conference on Robotics and Automation, 2014. The proof (Section 3) has been rewritten using Landau notation for easier reading, and a new evaluation on the low-torque pendulum has been appended, including a proof of incompleteness for fixed-time Bezier interpolation and empirical evaluation in simulations (Section 4).}

Keywords: kinodynamic planning, probabilistic completeness

1. Introduction

A deterministic motion planner is said to be complete if it returns a solution whenever one exists \cite{2}. A randomized planner is said to be probabilistically complete if the probability of returning a solution, when there is one, tends to one as execution time goes to infinity \cite{3}. Although these two notions might seem theoretical, they are of notable practical interest, as proving completeness requires one to formalize the problem by hypotheses on the robot, the environment, etc. While
experiments can show that a planner works for a given robot, in a given environment, for a given query, etc., a proof of completeness is a certificate that the planner works for a precise set of problems. The size of this set depends on how strong the assumptions required to make the proof are: the weaker the assumptions, the larger the set of solvable problems.

Probabilistic completeness has been established for systems with geometric constraints \[4, 3\] such as e.g. obstacle avoidance \[5\]. However, proofs for systems with kinodynamic constraints \[6, 7, 8, 9\] have yet to reach the same level of generality. Proofs available in the literature often rely on strong assumptions that are difficult to verify on practical systems (as a matter of fact, none of the previously mentioned works verified their hypotheses on non-trivial systems). In this paper, we establish probabilistic completeness (Section \[5\]) for a large class of kinodynamic planners, namely those that interpolate trajectories in the state space. Unlike previous works, our assumptions can be readily verified from the system’s equations of motion and the user-defined interpolation function.

The most important of these properties is second-order continuity (SOC), which states that the interpolation function varies smoothly and locally between states that are close. We evaluate the impact of this property in simulations (Section \[4\]) on a low-torque pendulum. Experiments validate our completeness theorem, and suggest that SOC is an important design guideline for kinodynamic planners that interpolate in the state space.

2. Background

2.1. Kinodynamic Constraints

Motion planning was first concerned only with geometric constraints such as obstacle avoidance or those imposed by the kinematic structures of manipulators \[10, 4, 8, 6\]. More recently, kinodynamic constraints, which stem from differential equations of dynamic systems, have also been taken into account \[11, 6, 5, 9\]. Kinodynamic constraints are more difficult to deal with than geometric constraints because they cannot in general be expressed using only configuration-space variables – such as the joint angles of a manipulator, the position and the orientation of a mobile robot, etc. Rather, they involve higher-order derivatives such as velocities and accelerations. There are two types of kinodynamic constraints:

**Non-holonomic constraints:** non-integrable equality constraints on higher-order derivatives, such as found in wheeled vehicles \[12\], under-actuated manipulators \[13\] or space robots.
**Hard bounds:** *inequality* constraints on higher-order derivatives such as torque bounds for manipulators [14], support areas [15] or wrench cones for humanoid stability [16], etc.

Some authors have considered systems that are subject to both types of constraints, such as under-actuated manipulators with torque bounds [13].

### 2.2. Randomized Planners

Randomized planners such as such as Probabilistic Roadmaps (PRM) [4] or Rapidly-exploring Random Trees (RRT) [6] build a roadmap on the state space. Both rely on repeated random sampling of the free state space, *i.e.* states with non-colliding configurations and velocities within the system bounds. New states are connected to the roadmap using a *steering* function, which is a method used to drive the system from an initial to a goal configuration. The steering method may be imperfect, *e.g.* it may not reach the goal exactly, not take environment collisions into account, only apply to states that are sufficiently close, etc. The objective of the motion planner is to overcome these limitations, turning a local steering function into a global planning method.

PRM builds a roadmap that is later used to generate motions between many initial and final states (multiple queries). When new samples are drawn, they are connected to *all* neighboring states in the roadmap using the steering function, resulting in a connected graph. Meanwhile, RRT focuses on driving the system from *one* initial state $x_{init}$ towards a goal area (single query). It grows a tree by connecting new samples to *one* neighboring state, usually their closest neighbor.

Both PRM’s and RRT’s *extension* step are represented by Algorithm 1, which relies on the following sub-routines (see Fig. 1 for an illustration):

- **SAMPLE($S$):** randomly sample an element from a set $S$;
- **PARENTS($x$, $V$):** return a set of states in the roadmap $V$ from which steering towards $x$ will be attempted;
- **STEER($x$, $x'$):** generate a system trajectory from $x$ towards $x'$. If successful, return a new node $x_{steer}$ ready to be added to the roadmap. Depending on the planner, the success criterion may be “reach $x'$ exactly” or “reach a vicinity of $x'$”.

The design of each sub-routine greatly impacts the quality and even the completeness of the resulting planner. In the literature, **SAMPLE($S$)** is usually implemented as uniform random sampling over $S$, but some authors have suggested adaptive sampling as a way to improve planner performance [17]. In geometric
Algorithm 1 Extension step of randomized planners (PRM or RRT)

Require: initial node $x_{init}$, number of iterations $N$

1: $(V, E) \leftarrow (\{x_{init}\}, \emptyset)$
2: for $N$ steps do
3: $x_{rand} \leftarrow \text{SAMPLE}({\mathcal X}_{\text{free}})$
4: $X_{\text{parents}} \leftarrow \text{PARENTS}(x_{rand}, V)$
5: for $x_{\text{parent}}$ in $X_{\text{parents}}$ do
6: $x_{\text{steer}} \leftarrow \text{STEER}(x_{\text{parent}}, x_{\text{rand}})$
7: if $x_{\text{steer}}$ is a valid state then
8: $V \leftarrow V \cup \{x_{\text{steer}}\}$
9: $E \leftarrow E \cup \{(x_{\text{parent}}, x_{\text{steer}})\}$
10: end if
11: end for
12: end for
13: return $(V, E)$

This choice results in the so-called Voronoi bias of RRTs [6]. Both experiments and theoretical analysis support this choice for geometric planning, however it becomes inefficient for kinodynamic planning, as was showed by Shkolnik et al. [18] on systems as simple as the torque-limited pendulum.

2.3. Steering Methods

This paper focuses on steering functions. These can be classified into three categories: analytical, state-based and control-based steering.

**Analytical steering.** This category corresponds to the ideal case when one can compute analytical trajectories respecting the system’s differential constraints, which are usually called (perfect) steering functions in the literature [6, 19]. Unfortunately, it only applies to a handful of systems. Reeds and Shepp curves for cars are a notorious example of this [12].

**Control-based steering.** Generate a control $u : [0, \Delta t] \rightarrow {\mathcal U}_{\text{adm}}$, where ${\mathcal U}_{\text{adm}}$ denotes the set of admissible controls, and compute the corresponding trajectory by forward dynamics. This approach has been called incremental simulation [20], control application [6] or control-space sampling [19, 21] in the literature. It is
Figure 1: Illustration of the extension routine of randomized planners. To grow the roadmap toward the sample $x'$, the planner selects a number of parents $\text{PARENTS}(x') = \{P_1, P_2, P_3\}$ from which it applies the $\text{STEER}(P_i, x')$ method.

widely applicable, as it only requires forward-dynamic calculations, but usually results in weak steering functions as the user has no or limited control over the destination state. In works such as [6, 5], random functions $u$ are sampled from a family of primitives (e.g. piecewise-constant functions), a number of them are tried and only the one bringing the system closest to the target is retained. Linear-Quadratic Regulation (LQR) [22, 23] also qualifies as control-based steering: in this case, $u$ is computed as the optimal policy for a linear approximation of the system given a quadratic cost function.

State-based steering. Interpolate a trajectory $\gamma_{int} : [0, \Delta t] \rightarrow C$, for instance a Bezier curve matching the initial and target configurations and velocities, and compute a control that makes the system track that trajectory. For fully-actuated system, this is typically done using inverse dynamics. An interpolated trajectory is rejected if no suitable control can be found. Algorithm 2 gives the prototype of state-based steering functions.

Algorithm 2 Prototype of state-based steering functions $\text{STEER}(x, x')$

1: $\gamma_{int} \leftarrow \text{INTERPOLATE}(x, x')$
2: $u_{int} := \text{INVERSE_DYNAMICS}(\gamma_{int}(t), \dot{\gamma}_{int}(t), \ddot{\gamma}_{int}(t))$
3: if $\forall t \in [0, \Delta t], u_{int}(t) \subset U_{adm}$ then
4: return the last state of $\gamma_{int}$
5: end if
6: return failure

Compared to control-based steering, this approach seems to apply to a more
limited range of systems (those where inverse dynamic solvers are readily available). However, it delivers control over the destination state, which is crucial, for instance, in any robotic application where contacts need to be actively maintained. Humanoids provide a canonical example for this: their high degree of freedom and their need to maintain geometric constraints while moving (e.g. feet on the floor) make them particularly unsuitable to control sampling. As a matter of fact, to the best of our knowledge, all humanoid motion planners reported so far in the literature (see e.g. [24, 25, 26]) are based on trajectory interpolation.

2.4. Previous works

Randomized planners such as RRT and PRM are both simple to implement yet efficient for geometric planning. The completeness of these planners has been established for geometric planning in [6,7,8]. In their proof, Hsu et al. [8] quantified the problem of narrow passages in configuration space with the notion of \((\alpha, \beta)\)-expansiveness. The two constants \(\alpha\) and \(\beta\) express a geometric lower bound on the rate of expansion of reachability areas.

There is, however, a gap between geometric and kinodynamic planning in terms of proving probabilistic completeness. When Hsu et al. extended their solution to kinodynamic planning [5], they applied the same notion of expansiveness, but this time in the \(X \times T\) (state and time) space with control-based steering. Their proof states that, when \(\alpha > 0\) and \(\beta > 0\), their planner is probabilistically complete. However, whether \(\alpha > 0\) or \(\alpha = 0\) in the non-geometric space \(X \times T\) remains an open question. As a matter of fact, the problem of evaluating \((\alpha, \beta)\) has been deemed as difficult as the initial planning problem [8]. In a parallel line of work, LaValle et al. [6] provided a completeness argument for kinodynamic planning, based on the hypothesis of an attraction sequence, i.e. a covering of the state space where two major problems of kinodynamic planning are already solved: steering and antecedent selection. Unfortunately, the existence of such a sequence was not established.

In the two previous examples, completeness is established under assumptions whose verification is at least as difficult as the motion planning problem itself. Arguably, too much of the complexity of kinodynamic planning has been abstracted into hypotheses, and these results are not strong enough to hold the claim that their planners are probabilistically complete in general. This was exemplified recently when Kunz and Stilman [27] showed that RRTs with control-based steering were not probabilistically complete for a family of control inputs (namely, those

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2 For instance, the RRT used in the simulations of this paper was implemented in less than a hundred lines of Python code.
with fixed time step and best-input extension). At the same time, Papadopoulos et al. [19] established probabilistic completeness for the same planner using a different family of control inputs (randomly sampled piecewise-constant functions). The picture of completeness for kinodynamic planners therefore seems to be a nuanced one.

Karaman et al. [7] introduced the RRT* path planner and extended it to kinodynamic planning with differential constraints in [28], providing a sketch of proof for the completeness of their solution. However, they assumed that their planner had access to the optimal cost metric and optimal local steering, which restricts their analysis to systems for which these ideal solutions are known. The same authors tackled the problem from a slightly different perspective in [29] where they supposed that the PARENTS function had access to \( w \)-weighted boxes, an abstraction of the system’s local controllability. However, they did not show how these boxes can be computed in practice\(^3\) and did not prove their theorem, arguing that the reasoning was similar to the one in [7] for kinematic systems.

While proof of completeness have been given for kinodynamic planners using control-based steering (e.g. [19, 21]), to the best of our knowledge, the present paper is the first to provide a proof of probabilistic completeness for kinodynamic planners using state-based steering.

2.5. Terminology

A function is smooth when all its derivatives exist and are continuous. Let \( \| \cdot \| \) denote the Euclidean norm. A function \( f : A \to B \) between metric spaces is Lipschitz when there exists a constant \( K_f \) such that

\[
\forall (x, y) \in A, \| f(x) - f(y) \| \leq K_f \| x - y \|.
\]

The (smallest) constant \( K_f \) is called the Lipschitz constant of the function \( f \).

Let \( \mathcal{C} \) denote \( n \)-dimensional configuration space, where \( n \) is the number of degrees of freedom of the robot. The state space \( \mathcal{X} \) is the \( 2n \)-dimensional manifold of configuration and velocity coordinates \( x = (q, \dot{q}) \). A trajectory is a continuous function \( \gamma : [0, \Delta t] \to \mathcal{C} \), and the distance of a state \( x \in \mathcal{X} \) to a trajectory \( \gamma \) is

\[
dist_\gamma(x) := \min_{t \in [0, \Delta t]} \| (\gamma, \dot{\gamma})(t) - x \|.
\]

A kinodynamic system can be written as a time-invariant differential system:

\[
\dot{x}(t) = f(x(t), u(t)),
\]

---

\(^3\) The definition of \( w \)-weighted boxes is quite involved: it depends on the joint flow of vector fields spanning the tangent space of the system’s manifold.
where \( u \in \mathcal{U} \) denotes the control input and \( x(t) \in \mathcal{X} \). Let \( \mathcal{U}_{adm} \subset \mathcal{U} \) denote the subset of admissible controls. (For instance, \( \mathcal{U}_{adm} = [\tau_{\min}, \tau_{\max}] \subset \mathcal{U} = \mathbb{R} \) represents bounded torques for a single joint.) A trajectory \( \gamma \) that is solution to the differential system \(^2\) using only controls \( u(t) \in \mathcal{U}_{adm} \) is called an admissible trajectory. The kinodynamic motion planning problem is to find an admissible trajectory from \( q_{\text{init}} \) to \( q_{\text{goal}} \).

A control function \( u : [0, \Delta t] \to \mathcal{U} \) has \( \delta \)-clearance when its image is in the \( \delta \)-interior of \( \mathcal{U}_{adm} \), i.e. for any time \( t \), \( B(u(t), \delta) \subset \mathcal{U}_{adm} \). By extension, we will say that a trajectory \( \gamma \) has \( \delta \)-clearance in control space when the control function \( u(t) \) computed by inverse dynamics has such clearance.

### 3. Completeness Theorem

#### 3.1. System assumptions

Our model for an \( \mathcal{X} \)-state randomized planner is given by Algorithm \(^1\) using state-based steering. We first assume that:

**Assumption 1.** The system is fully actuated.

Full actuation restricts our analysis to a range of systems that includes e.g. industrial manipulators (with or without kinematic redundancy) or space robots with thrusters. It allows us to write the equations of motion of the system in generalized coordinates as:

\[
M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = u, \tag{3}
\]

where \( u \in \mathcal{U}_{adm} \) and we assume that the set of admissible controls \( \mathcal{U}_{adm} \) is compact with non-empty interior. Since torque constraints are our main concern, we will focus on

\[
\mathcal{U}_{adm} := \{ u \in \mathcal{U}, \ |u| \leq \tau_{\max} \}, \tag{4}
\]

which is indeed compact\(^4\) (Vector comparisons are component-wise.) Finally, we suppose similarly to \(^{21}\) that forward and inverse dynamics mappings have Lipschitz smoothness:

**Assumption 2.** The forward dynamics function \( f \) is Lipschitz continuous in both of its arguments, and its inverse \( f^{-1} \) (the inverse dynamics function \( u = f^{-1}(x, \dot{x}) \)) is Lipschitz in both of its arguments.

\(^4\) The application of our proof of completeness to an arbitrary compact set presents no technical difficulty: one can just replace \( |u| \leq \tau_{\max} \) with \( d(u, \partial \mathcal{U}_{adm}) \), with \( \partial \mathcal{U}_{adm} \) the boundary of \( \mathcal{U}_{adm} \). Using Equation \(^4\) avoids this level of verbosity.
These two assumptions are satisfied when \( f \) is given by (3) as long as the matrices \( M(q) \) and \( C(q, \dot{q}) \) are bounded and the gravity term \( g(q) \) is Lipschitz. Indeed, for a small displacement between \( x \) and \( x' \),

\[
\| u' - u \| \leq \| M \| \| \dot{q}' - \dot{q} \| + \| C(q, \dot{q}) \| \| q' - q \| + K_g \| q' - q \| \tag{5}
\]

Let us illustrate this on the double pendulum depicted in Figure 2. When both links have mass \( m \) and length \( l \), the gravity term

\[
g(\theta_1, \theta_2) = \frac{mgl}{2} \left[ \sin \theta_1 + \sin(\theta_1 + \theta_2) \sin(\theta_1 + \theta_2) \right]
\]

is Lipschitz with constant \( K_g = 2mgl \), while the inertial term is bounded by \( \| M \| \leq 3ml^2 \). When joint angular velocities are bounded by \( \omega \), the norm of the Coriolis tensor is bounded by \( 2\omega ml^2 \). Using (5), one can therefore derive the Lipschitz constant \( K_{f^{-1}} \) of the inverse dynamics function.

3.2. Interpolation assumptions

We also require smoothness for the interpolated trajectories:

**Assumption 3.** The interpolation \( \gamma_{\text{int}} : [0, \Delta t] \rightarrow \mathcal{C} \) returned by \text{INTERPOLATE}(x, x') is a smooth Lipschitz function, and its time-derivative \( \dot{\gamma}_{\text{int}} \) is also Lipschitz. It satisfies the boundary conditions \( (\gamma(0), \dot{\gamma}(0)) = x \) and \( (\gamma(\Delta t), \dot{\gamma}(\Delta t)) = x' \).
The following two assumptions ensure a continuous behavior of the interpolation procedure:

**Assumption 4** (Local boundedness). \textit{Interpolated trajectories stay within a neighborhood of their start and end states, i.e. there exists a constant }\( \eta \)\textit{ such that, for any }\((x, x') \in \mathcal{X}^2\), \textit{the interpolated trajectory }\( \gamma_{\text{int}} : [0, \Delta t] \to \mathcal{C} \)\textit{ resulting from }\text{INTERPOLATE}(x, x')\textit{ is included in a ball of center }\( x \)\textit{ and radius }\( \eta \|x' - x\| \).

Let us denote by \( \Delta x = (\Delta q, \Delta \dot{q}) = x' - x \) the displacement between two states \( x = (q, \dot{q}) \) and \( x' = (q', \dot{q}') \). We define the \textit{average velocity} and \textit{discrete time step} between the two states as:

\[
\dot{q}_{\text{avg}} := \frac{\dot{q} + \dot{q}'}{2}, \quad \Delta t_{\text{disc}} := \frac{\Delta q \cdot \dot{q}_{\text{avg}}}{\|\dot{q}_{\text{avg}}\|^2}.
\]

(6)

In order for \( \Delta t_{\text{disc}} \) to be well defined, one needs to ensure that:

\textbf{P1.} the scalar product \( \Delta q \cdot \dot{q}_{\text{avg}} \) is positive: this reflects the fact that displacements and velocities should be aligned as the former gets small;

\textbf{P2.} the velocity \( \dot{q}_{\text{avg}} \) is non-zero: in what follows, we will further assume that \( \|\dot{q}_{\text{avg}}\| \geq \epsilon \), where \( \epsilon \) is a positive constant. The value of this parameter will be derived thereafter.

We say that a pair of state \((x, x')\) is \textit{locally consistent} when it satisfies the two prerequisites above, as well as the following:

\textbf{P3.} the velocity \( \dot{q}_{\text{avg}} \) aligns with the displacement \( \Delta q \) as \( x' \) and \( x \) are close from each other: formally, \( \sin(\Delta q, \dot{q}_{\text{avg}}) = O(\Delta x) \).

The \textit{discrete acceleration} between two locally-consistent states \( x \) and \( x' \) is then defined by \( \ddot{q}_{\text{disc}}(x, x') := \frac{\Delta \dot{q}}{\Delta t_{\text{disc}}} \), which can also be written:

\[
\ddot{q}_{\text{disc}}(x, x') = \frac{\Delta \dot{q} \|\dot{q}_{\text{avg}}\|^2}{\|\Delta q\|\|\dot{q}_{\text{avg}}\| \sqrt{1 - \sin^2(\Delta q, \dot{q}_{\text{avg}})}} = \frac{\Delta \dot{q} \|\dot{q}_{\text{avg}}\|}{\|\Delta q\|} + O(\Delta x) \quad (7)
\]

The rationale behind these expressions is that the continuous analog of \( \ddot{q}_{\text{disc}} \) when \( x' \to x \) is 

\[
\frac{\|\dot{q}\| \frac{d\dot{q}}{dt}}{\|\dot{q}\|} = \frac{\|\dot{q}\|}{\|\dot{q}\|} \frac{d\dot{q}}{dt} = \frac{\dot{q}}{dt}, \text{ i.e. the continuous acceleration.}
\]

Prerequisite P2 is a loose restriction: The probability that uniform random sampling returns a pair \((x, x')\) violating it is on the scale of \( \epsilon \) (divided by the maximum velocity), which can be chosen arbitrarily small. Meanwhile, P1 delimits a half-space where the angle between \( \Delta q \) and \( \dot{q}_{\text{avg}} \) is acute, thus non-empty and sampled with 1/2 probability[^5]. Also, note that local consistency does not preclude \( \dot{q} \) or \( \dot{q}' \)

[^5]: The sampling routine could also be adapted to always return new samples \( x' \) locally consistent with a given \( x \) in the roadmap.
from being zero, as usually happens at $x_{init}$ and $x_{goal}$. The only case excluded is when both are zero, or $\dot{q}' = -\dot{q}$, which would imply that the trajectory halts at some intermediate state. As we will see, such halts can always be excluded by leveraging the fact that $U_{adm}$ has non-empty interior.

**Assumption 5 (Discrete-acceleration convergence).** When start and end states become close, accelerations of interpolated trajectories uniformly converge to the discrete acceleration between them, i.e. there exists some $\nu > 0$ such that, if $\gamma_{int} : [0, \Delta t] \to C$ results from $\text{INTERPOLATE}(x, x')$ for a locally consistent pair $(x, x')$, then

$$\forall \tau \in [0, \Delta t], \|\ddot{\gamma}_{int}(\tau) - \ddot{\dot{q}}_{disc}(x, x')\| \leq \nu \|\Delta x\|.$$

These three assumptions ensure that the planner interpolates trajectories locally and “continuously” when $x$ and $x'$ are close and locally consistent (we will see that the subset of pairwise locally-consistent states is sufficient to establish completeness). We will call them altogether second-order continuity, where “second-order” refers to the discrete acceleration (7) used in Assumption 5. This continuous behavior plays a key role in our proof of completeness, as it ensures that denser sampling will allow finding arbitrarily narrow state-space passages.

Let us consider the example of a single pendulum with joint angle $q$ and state $x = (q, \dot{q}) \in \mathbb{R}^2$, for the interpolation function $\gamma = \text{INTERPOLATE}(x, x')$ given by

$$\gamma_s : [0, \Delta t] \rightarrow [-\pi, \pi] \quad t \mapsto \gamma_s(t) = \frac{\Delta \dot{q}}{2 \Delta t} t^2 + \dot{q} t + q. \quad (8)$$

for $\Delta t = \Delta t_{disc} = (\Delta q \dot{q}_{avg}) / (\dot{q}_{avg}^2) = \Delta q / \dot{q}_{avg}$. One can then check that $\gamma_s(0) = \dot{q}$, $\gamma_s(\Delta t) = \dot{q}'$, $\gamma_s(0) = q$ and

$$\gamma_s(\Delta t) = \frac{1}{2} (2 \dot{q} + \Delta \dot{q}) \Delta t + q = \dot{q}_{avg} \Delta t + q = \Delta q + q = q'.$$

The function being a second-order polynomial, itself and all of its derivatives are smooth and Assumption 3 is verified. One can check Assumption 4 by:

$$\|\gamma_s(t) - q\| \leq \frac{|\Delta \dot{q}| + 2 \dot{q} \Delta t}{2} \leq \frac{|\Delta \dot{q}| + 2 \dot{q} |\Delta q|}{2 \varepsilon} \leq \frac{1 + 2 \omega}{2 \varepsilon} \|x' - x\|$$

where $\omega$ is again the maximum angular velocity, while $\|\gamma_s(t) - \dot{q}\| \leq |\Delta \dot{q}| \leq \|x' - x\|$; the assumption follows by taking e.g. $\eta = 1 + (1 + 2 \omega) / 2 \varepsilon$. Finally, $\ddot{\gamma}_s(t) = \frac{\Delta \ddot{q}}{\Delta t} = \ddot{\dot{q}}_{disc}$ by construction, so that Assumption 5 is also verified. The function $\gamma_s$ given by Equation (8) provides a valid interpolator satisfying all our assumptions for the simple pendulum.
3.3. Completeness theorem

In order to prove the theorem, we will use the following two lemmas, which are proved in Appendix A.

**Lemma 1.** Let $g : [0, \Delta t] \to \mathbb{R}^k$ denote a smooth Lipschitz function. Then, for any $(t, t') \in [0, \Delta t]^2$,

$$
\left\| \dot{g}(t) - g(t') - g(t) \right\| \leq \frac{K_g}{2} |t' - t|.
$$

**Lemma 2.** If there exists an admissible trajectory $\gamma$ with $\delta$-clearance in control space, then there exists $\delta' < \delta$ and a neighboring admissible trajectory $\gamma'$ with $\delta'$-clearance in control space whose acceleration and velocity never vanish, i.e. such that $\|\dot{\gamma}\|$ (resp. $\|\dot{\gamma}'\|$) is always greater than some constant $\bar{m} > 0$ (resp. $\bar{m'} > 0$).

We can now state our main theorem:

**Theorem 1.** Consider a time-invariant differential system (2) with Lipschitz-continuous $f$ and full actuation over a compact set of admissible controls $U_{adm}$. Suppose that the kinodynamic planning problem between two states $x_{init}$ and $x_{goal}$ admits a smooth Lipschitz solution $\gamma : [0, T] \to C$ with $\delta$-clearance in control space. A randomized motion planner (Algorithm 1) using a second-order continuous interpolation is probabilistically complete.

**Proof.** Let $\gamma : [0, T] \to C, t \mapsto \gamma(t)$ denote a smooth Lipschitz admissible trajectory from $x_{init}$ to $x_{goal}$, and $u : [0, T] \to U_{adm}$ its associated control trajectory with $\delta$-clearance in control space. Consider two states $x = (q, \dot{q})$ and $x' = (q', \dot{q}')$. Given a sufficiently dense sampling of the state space, we suppose that $\text{dist}_a(x) \leq \rho$ and $\text{dist}_a(x') \leq \rho$, where $\rho$ is the radius of a tube around the trajectory $\gamma$. Given the minimum velocity $\bar{m} > 0$ of the trajectory $\gamma$, we take $\epsilon$ as $\bar{m}/2$ and enforce that $\rho < \bar{m}/3$. Assuming again a sufficiently dense sampling, the two states can be close enough so that (1) $\|\Delta \dot{q}\| < \bar{m}/3$ and (2) $\|\Delta \ddot{q}\| < 2\|\ddot{q}\|$.

Then:

$$
\|\dot{q}\| \geq \|\dot{q} - \dot{\gamma}\| - \|\dot{\gamma}\| \geq \|\dot{\gamma}\| - \rho \geq \bar{m} - \rho \geq 2\bar{m}/3
$$

$$
\|\dot{q}_{avg}\| \geq \|\dot{q}_{avg} - \dot{q}\| - \|\dot{q}\| \geq \|\dot{q}\| - \|\Delta \ddot{q}\|/2 \geq 2\bar{m}/3 - \bar{m}/6 = \epsilon.
$$

All states sampled in the tube of radius $\rho$ therefore satisfy property P2. Consider their corresponding time instants on the trajectory:

$$
t := \arg\min_t \|(\gamma(t), \dot{\gamma}(t)) - x\|,
$$

$$
t' := \arg\min_t \|(\gamma(t), \dot{\gamma}(t)) - x'\|.
$$
Supposing without loss of generality that \( t' > t \), we denote by \( \Delta t = t' - t > 0 \). Next, we enforce that \( \rho/\Delta t = O(\Delta t) \), i.e. the radius \( \rho \) is at most quadratic in the time difference. Then,

\[
\Delta q \cdot \dot{q}_{\text{avg}} = (\Delta q - \Delta \gamma) \cdot \dot{q}_{\text{avg}} + \Delta \gamma \cdot (\dot{q}_{\text{avg}} - \dot{\gamma}_{\text{avg}}) + \Delta \gamma \cdot \dot{\gamma}_{\text{avg}}.
\]

The first two terms are \( O(\rho) = O(\Delta t^2) \), while the third term \( \Delta \gamma \cdot \dot{\gamma}_{\text{avg}} = \|\dot{\gamma}\|^2 \Delta t + O(\Delta t^2) \). Therefore,

\[
\Delta q \cdot \dot{q}_{\text{avg}} > \hat{m}^2 \Delta t + r(\Delta t)
\]

where the residual \( r(\Delta t) = O(\Delta t^2) \). Relying again on dense sampling, we assume that \( \Delta t \) is small enough so that \( r(\Delta t) < \hat{m}^2 \Delta t/2 \). It follows that \( \Delta q \cdot \dot{q}_{\text{avg}} > 0 \), thus the pair \((x, x')\) satisfies P1. Next,

\[
|\sin(\Delta q, \dot{q}_{\text{avg}})| \leq |\sin(\Delta q, \Delta \gamma)| + |\sin(\Delta \gamma, \dot{\gamma}_{\text{avg}})| + |\sin(\dot{\gamma}, \dot{q}_{\text{avg}})|.
\]

Using the inequality \( |\sin(\lambda v + u, v)| \leq \|u\|/\lambda \|v\| \) for any \( \lambda > 0 \), we can upper-bound the last two terms by

\[
|\sin(\Delta \gamma, \dot{\gamma}_{\text{avg}})| = |\sin(\dot{\gamma} \Delta t + O(\Delta t^2), \dot{\gamma})| = O(\Delta t^2)/\|\dot{\gamma}\| \Delta t = O(\Delta t)/\hat{m},
\]

\[
|\sin(\dot{\gamma}, \dot{q}_{\text{avg}})| = |\sin(\dot{q}_{\text{avg}} + O(\rho), \dot{q}_{\text{avg}})| = O(\rho)/\|\dot{q}_{\text{avg}}\| = O(\rho)/\hat{m} \Delta t.
\]

Both right-hand side expressions are \( O(\Delta t) \). Meanwhile, simple vector geometry shows that

\[
\sin(\Delta q, \Delta \gamma) \leq \frac{\text{dist}_x(x) + \text{dist}_x(x')}{\|\Delta \gamma\|} \leq \frac{\rho}{\hat{m} \Delta t},
\]

so that the first term of the inequality is \( O(\Delta t) \) as well. Finally, note that \( \|\Delta q\| = \|\Delta \gamma\| + O(\rho) = \|\dot{\gamma}\| \Delta t + O(\Delta t^2) \geq \hat{m} \Delta t + O(\Delta t^2) \). Since \( \hat{m} > 0 \), this implies that \( \Delta t = O(\Delta q) \), from which we conclude that the sine above is \( O(\Delta q) \) and the couple \((x, x')\) satisfies P3. To summarize, when sampling is dense enough with the conditions that we have defined so far, we have seen how the states \( x \) and \( x' \) are locally consistent.

We can now define \( \Delta t_{\text{disc}} = (\Delta q \cdot \dot{q}_{\text{avg}})/\|\dot{q}_{\text{avg}}\|^2 > 0 \) the discrete time step between two locally consistent states \( x \) and \( x' \). We further assume that \( \rho/\Delta t_{\text{disc}} = O(\Delta t) \), a quadratic upper bound similar to that already enforced on \( \rho \), which will prove useful thereafter.

Let \( \gamma_{\text{int}} : [0, \Delta t] \to C \) denote the result of the interpolation between \( x \) and \( x' \). For \( \tau \in [0, \Delta t] \), the torque required to follow the trajectory \( \gamma_{\text{int}} \) is \( u_{\text{int}}(\tau) := f(\gamma_{\text{int}}(\tau), \dot{\gamma}_{\text{int}}(\tau), \ddot{\gamma}_{\text{int}}(\tau)) \). Since \( u \) has \( \delta \)-clearance in control space,

\[
|u_{\text{int}}(\tau)| \leq |u_{\text{int}}(\tau) - u(t)| + |u(t)|
\]

\[
\leq |f(\gamma_{\text{int}}(\tau), \dot{\gamma}_{\text{int}}(\tau), \ddot{\gamma}_{\text{int}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t))| + (1 - \delta) \tau_{\text{max}},
\]
(As previously, vector inequalities are component-wise.) Let us denote by $|u_{\text{int}}|$ the first term of this inequality. We will now show that $|u_{\text{int}}| = O(\Delta t) \to 0$ when $\Delta t \to 0$, and therefore $|u_{\text{int}}(\tau)| \leq \tau_{\text{max}}$ for a small enough $\Delta t$ (i.e. when sampling density is high enough). Let us first rewrite it as follows:

$$
|u_{\text{int}}| = |f(\gamma_{\text{int}}(\tau), \dot{\gamma}_{\text{int}}(\tau), \ddot{\gamma}_{\text{int}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t))|
$$

$$
\leq \|f(\gamma_{\text{int}}(\tau), \dot{\gamma}_{\text{int}}(\tau), \ddot{\gamma}_{\text{int}}(\tau)) - f(\gamma(t), \dot{\gamma}(t), \ddot{\gamma}(t))\|_{\infty}
$$

$$
\leq K_f \| (\gamma_{\text{int}}(\tau), \dot{\gamma}_{\text{int}}(\tau)) - (\gamma(t), \dot{\gamma}(t)) \| + K_f \| \ddot{\gamma}_{\text{int}}(\tau) - \ddot{\gamma}(t) \|
$$

$$
\leq K_f \left[ (\eta + \nu) \| \Delta x \| + \text{dist}_\gamma(x) \right] + K_f \left\| \frac{\| q \|}{\| \Delta q \|} \Delta \hat{q} - \hat{\gamma}(t) \right\|
$$

The replacement of the norm $\| \cdot \|$ by $\| \cdot \|_{\infty}$ is possible because all norms of $\mathbb{R}^n$ are equivalent (a change in norm will be reflected by a different constant $K_f$). The transition from the second to the third row uses Lipschitz smoothness of $f$, as well as the triangular inequality to separate position-velocity and acceleration coordinates. The transition from the third to the fourth row relies on the two interpolation assumptions: local boundedness (yields the $\eta$ factor in the distance term) and, since $x$ and $x'$ are locally consistent, convergence to the discrete-acceleration (yields the $\nu$ factor in the distance term, as well as the acceleration term–note how, for the latter, we used the expression from Equation (7), which is possible since $x$ and $x'$ are locally consistent).

The position-velocity term (PV) satisfies:

$$(\text{PV}) \leq (2\rho + \| \Delta \gamma \|)(\eta + \nu) + \rho \leq \frac{1}{2} K_\gamma (\eta + \nu) \Delta t + (1 + 2(\eta + \nu)) \rho.
$$

Since $\rho = O(\Delta t)$, we have $(\text{PV}) = O(\Delta t)$ and thus $|\tilde{u}| \leq (A) + O(\Delta t)$. Next, the difference ($A$) can be bounded as:

$$(A) \leq \left\| \frac{\Delta \hat{q}}{\| \Delta q \|} - \frac{\| \Delta \hat{\gamma} \|}{\| \Delta \gamma \|} \right\| + \left\| \frac{\| \Delta \hat{\gamma} \|}{\| \Delta \gamma \|} \frac{\| \hat{\gamma}(t) \|}{\| \Delta \gamma \|} - \frac{\| \Delta \gamma \|}{\Delta t} \right\|$$

$$
+ \left\| \frac{\Delta \hat{\gamma}}{\Delta t} - \hat{\gamma}(t) \right\|.
$$

From Lemma [1], the two terms ($A'$) and ($A''$) satisfy:

$$(A') \leq \frac{K_\gamma}{2} \| \Delta \hat{\gamma} \| \Delta t = O(\Delta t),$$

$$(A'') \leq \frac{K_\gamma}{2} \Delta t = O(\Delta t),$$

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where the first upper bound \( O(\Delta t) \) comes from the fact that \( \| \Delta \dot{\gamma} \| \| \Delta \ddot{\gamma} \| \Delta t \to 0 \Delta t \). We now have \( \bar{u} \leq (\Delta) + O(\Delta t) \). The term \( \Delta t \) can be seen as the deviation between the discrete accelerations of \( \gamma_{\text{int}} \) and \( \dot{\gamma} \). Let us decompose it in terms of norm and angular deviations:

\[
(\Delta) \leq \left( \frac{\| \Delta \dot{\gamma} \|}{\| \Delta \ddot{\gamma} \|} - \frac{\| \Delta \dddot{\gamma} \|}{\| \Delta \ddot{\gamma} \|} \right) \frac{\| \Delta \dot{\gamma} \|}{\| \Delta \gamma \|} + \frac{\| \Delta \dddot{\gamma} \|}{\| \Delta \ddot{\gamma} \|} \left( \frac{\| \Delta \dot{\gamma} \|}{\| \Delta \gamma \|} - \frac{\| \Delta \dddot{\gamma} \|}{\| \Delta \ddot{\gamma} \|} \right)
\]

The factor \( \frac{2\| \Delta \dot{\gamma} \|}{\| \Delta \gamma \|} \) before \( (\theta) \) is \( O(1) \) when \( \Delta t \to 0 \), while simple vector geometry then shows that

\[
\sin (\Delta \dot{\gamma}, \Delta \ddot{\gamma}) \leq \frac{\text{dist}_\gamma(x) + \text{dist}_\gamma(x')}{\| \Delta \dot{\gamma} \|} \leq \frac{\rho}{\bar{m} \Delta t},
\]

where \( \bar{m} := \min_t \| \dot{\gamma}(t) \| \). From Lemma 2 we can assume this minimum acceleration to be strictly positive. Then, it follows from \( \rho = O(\Delta t^2) \) that the sine above is \( O(\Delta t) \). Recalling the fact that \( 1 - \cos \theta < \sin \theta \) for any \( \theta \in [0, \pi/2] \), we have

\( (\theta) = O(\Delta t) \).

Finally,

\[
(N) \leq \frac{\| \Delta \dot{\gamma} \|}{\| \Delta \gamma \|} \| \dot{\gamma} \| + \| \dot{\gamma} \| \frac{\| \Delta \dot{\gamma} \|}{\| \Delta \gamma \|} \frac{\| \Delta \ddot{\gamma} \|}{\| \Delta \ddot{\gamma} \|} + \frac{\| \Delta \ddot{\gamma} \|}{\| \Delta \ddot{\gamma} \|} \frac{\| \Delta \dddot{\gamma} \|}{\| \Delta \dddot{\gamma} \|} \frac{\| \Delta \dddot{\gamma} \|}{\| \Delta \dddot{\gamma} \|}
\]

Where we used the fact that \( \| \Delta \gamma \| \leq \text{dist}_\gamma(x) + \| \Delta q \| + \text{dist}_\gamma(x') = \| \Delta q \| + O(\rho) \), and similarly for \( \| \Delta \dot{\gamma} \| \). Because \( \| \Delta \dot{\gamma} \| = \| \dot{q} \| \| \Delta t \text{disc} \| + O(\| \Delta t \text{disc} \|) \) and \( \rho/\Delta t \text{disc} = O(\Delta t) \), the last two fractions are \( O(\Delta t) \), so our last term \( (N) = O(\Delta t) \).

Overall, we have derived an upper bound \( |u(\tau)| \leq (1 - \delta)\tau_{\text{max}} + O(\Delta t) \).

As a consequence, there exists a constant \( \delta t > 0 \) such that, whenever \( \Delta t \leq \delta t \), interpolated torques satisfy \( |u| \leq \tau_{\text{max}} \) and the interpolated trajectory \( \gamma_{\text{int}} = \text{INTERPOLATE}(x, x') \) is admissible. Note that the constant \( \delta t \) is uniform, in the sense that it does not depend on the index \( t \) on the trajectory.
Conclusion of the Proof. We have effectively constructed the attraction sequence conjectured in [6]. We can now conclude the proof similarly to the strategy sketched in that paper. Let us denote by $B_t := B((\gamma, \dot{\gamma})(t), \delta \rho)$, the ball of radius $\delta \rho$ centered on $(\gamma, \dot{\gamma})(t) \in \mathcal{X}$, where $\delta \rho = O(\delta t^2)$ as before. Suppose that the roadmap contains a state $x \in B_t$, and let $t' := t + \delta t$. If the planner samples a state $x' \in B_{t'}$, the interpolation between $x$ and $x'$ will be successful and $x'$ will be added to the roadmap. Since the volume of $B_{t'}$ is non-zero, the event $\{\text{SAMPLE}(\mathcal{X}_{\text{free}}) \in B_{t'}\}$ will happen with probability one as the number of extensions goes to infinity. At the initialization of the planner, the roadmap is reduced to $x_{\text{init}} = (\gamma(0), \dot{\gamma}(0))$. Therefore, using the property above, by induction on the number of time steps $\delta t$, the last state $x_{\text{goal}} = (\gamma(T), \dot{\gamma}(T))$ will be eventually added to the roadmap with probability one, and the planner will find an admissible trajectory connecting $x_{\text{init}}$ to $x_{\text{goal}}$. ■

4. Completeness and state-based steering in practice

Shkolnik et al. [18] showed how RRTs could not be directly applied to kinodynamic planning due to their poor expansion rate at the boundaries of the roadmap. They illustrated this phenomenon on the planning problem of swinging up a (single) pendulum vertically against gravity. Let us consider the same system, i.e. the 1-DOF single pendulum depicted in Figure 2(A), with length $l = 20$ cm and mass $m = 8$ kg. It satisfies the system assumptions of Theorem 1 a fortiori, as we saw that they apply to the double pendulum.

We assume that the single actuator of the pendulum, corresponding to the joint angle $\theta$ in Figure 2, has limited actuation power: $|\tau| \leq \tau_{\text{max}}$. The static equilibrium of the system requiring the most torque is given at $\theta = \pm \pi/2$ with $\tau = \frac{1}{2}lmg \approx 7.84$ Nm. Therefore, when $\tau_{\text{max}} < 7.84$ Nm, it is impossible for the system to raise upright directly, and the pendulum rather needs to swing back and forth to accumulate kinetic energy before it can swing up. For any $\tau_{\text{max}} > 0$, the pendulum can achieve the swingup in a finite number of swings $N$, with $N \to \infty$ as $\tau_{\text{max}} \to 0$.

4.1. Bezier interpolation

A common solution [30, 31, 32] to connect two states $(q, \dot{q})$ and $(q', \dot{q}')$ is the cubic Bezier curve (also called “Hermit curve”) which is the quadratic function $B(t)$ such that $B(0) = q$, $\dot{B}(0) = \dot{q}$, $B(T) = q'$ and $\dot{B}(T) = \dot{q}'$, where $T$ is the fixed duration of the interpolated trajectory. Its expression is given by:

$$B(t) = \frac{-2\Delta q + T(\dot{q} + \dot{q}')}{T^3} t^3 + \frac{3\Delta q - 2\dot{q} - \dot{q}'}{T^2} t^2 + \dot{q} t + q$$
This interpolation is straightforward to implement, however it does not verify our Assumption 5, as for instance
\[ \ddot{B}(0) = \frac{6\Delta q - 4\dot{q} - 2q'}{T^2} \xrightarrow{\Delta x \to 0} \frac{-6\dot{q}}{T^2} \neq 0. \] (9)

Our proof of completeness does not apply to such interpolators: even though a feasible trajectory is sampled as closely as possible \((\Delta x \to 0)\), the interpolated acceleration will not approximate the smooth acceleration underlying the feasible trajectory.

**Proposition 1.** A randomized motion planner interpolating pendulum trajectories by Bezier curves with a fixed duration \(T\) cannot find non-quasi static solutions by increasing sampling density.

**Proof.** When actuation power decreases, the pendulum needs to store kinetic energy in order to swing up, which implies that all swingup trajectories go through velocities \(|\dot{\theta}| > \dot{\theta}_{\text{swingup}}(\tau_{\text{max}})\). The function \(\dot{\theta}_{\text{swingup}}\) increases to a positive limit \(\dot{\theta}_{\text{lim}}\) as \(\tau_{\text{max}} \to 0\), where \(\dot{\theta}_{\text{lim}} > \sqrt{8g/l}\) from energetic considerations.\(^6\) Yet, feasible accelerations are also bound by \(|\ddot{\theta}| \leq K\tau_{\text{max}}\) for some constant \(K > 0\). Combining both observations in (9) yields:

\[ K\tau_{\text{max}} \geq 6\frac{\dot{\theta}}{T^2} > 6\frac{\dot{\theta}_{\text{swingup}}(\tau_{\text{max}})}{T^2} \Rightarrow \dot{\theta}_{\text{swingup}}(\tau_{\text{max}}) \leq \frac{KT^2}{6}\tau_{\text{max}}. \]

Since the planner uses a constant \(T\) and \(\dot{\theta}_{\text{swingup}}\) increases to \(\dot{\theta}_{\text{lim}}\) when \(\tau_{\text{max}} \to 0\), this inequality cannot be satisfied for arbitrary small actuation power \(\tau_{\text{max}}\). Hence, even with an arbitrarily high sampling density around a feasible trajectory \(\gamma(t)\), the planner will not be able to reconstruct a feasible approximation \(\gamma_{\text{int}}(t)\).

4.2. Second-order continuous interpolation

Since the system has only one degree of freedom, one can interpolate trajectories that comply with our Assumption 5 using constant accelerations by selecting

\[^6\text{The expression } \dot{\theta} = \sqrt{8g/l} \text{ corresponds to the kinetic energy } \frac{1}{2}ml\dot{\theta}^2 = mgl, \text{ the latter being the (potential) energy of the system at rest in the upward equilibrium. During a successful last swing, the kinetic energy at } \theta = 0 \text{ is } \frac{1}{2}ml\dot{\theta}_{\text{swingup}}^2 + W_g + W_f = mgl, \text{ with } W_g < 0 \text{ the work of gravity and } W_f \text{ the work of actuation forces between } \theta = 0 \text{ and } \theta = \pi. \text{ The work } W_f \text{ vanishes when } \tau_{\text{max}} \to 0. \]
a suitable trajectory duration. We propose here the example previously given in Equation (8):

\[ \gamma_s : [0, \Delta t] \rightarrow [\pi - \pi] \]

\[ t \mapsto \gamma_s(t) = \frac{\Delta \dot{q}}{2\Delta t} t^2 + t \dot{q} + q. \]  

We showed in Section 3.2 how this interpolation satisfies all Assumptions 3, 4 and 5. Note that it only applies to single-DOF systems. For multi-DOF systems, the correct duration \( \Delta t_C \) used to transfer from one state to another is different for each DOF, hence constant accelerations cannot be used. One can then apply optimization techniques [22, 9] or use a richer family of curves such as piecewise linear-quadratic segments [33].

4.3. Comparison in simulations

According to Theorem 1 and our previous discussion, a randomized planner based on Bezier interpolation is not expected to be probabilistically complete as \( \tau_{\text{max}} \to 0 \), while the same planner using the SOC interpolation will be complete at any rate. We asserted this statement in simulations of the pendulum with RRT [34].

Our implementation of RRT is that described in Algorithm 1, with the addition of the steer-to-goal heuristic: every \( m = 100 \) steps, the planner tries to steer to \( x_{\text{goal}} \) rather than \( x_{\text{rand}} \). This extra step speeds up convergence when the system reaches the vicinity of the goal area. We use uniform random sampling for \( \text{SAMPLE}(S) \), while for \( \text{PARENTS}(x', V) \) returns the \( k = 10 \) nearest neighbors of \( x' \) in the roadmap \( V \). All the source code used in these experiments can be accessed at [35].

We compared the performance of RRT with the Bezier and SOC interpolations, all other parameters being the same, on a single pendulum for a range of maximum torque values. For a meaningful comparison, we set up a benchmark where both RRTs are run on the same random samples \( x_{\text{rand}} \). The benchmark consists of four settings \( \tau_{\text{max}} \in \{5, 6, 7, 8\} \) Nm, each setting comprising 20 runs in total, each run starting from a random state \( x = (\theta, \dot{\theta}) \) with \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) (we are interested in swingup trajectories). We measure the performance of a planner over time by its distance to goal, \( i.e. \) the distance from the closest roadmap node to the goal, where our distance function is:

\[ d(x_0, x_1) = \sqrt{1 - \cos(\theta_0 - \theta_1)} + \left| \frac{\dot{\theta}_0 - \dot{\theta}_1}{4 \theta_{\text{max}}} \right|. \]  

Figure [3] shows the results of the benchmark, with green and red trajectories corresponding respectively to the Bezier and SOC RRTs. Vertical lines mark times at which a planner finds a solution. There are no green vertical line for \( \tau_{\text{max}} = 5 \) Nm as no RRT using Bezier interpolation succeeded in reaching the goal, which concurs with Proposition 1. The data generated in this benchmark is released at:
Figure 3: Results of the benchmark comparing the performance of RRT on a kinodynamic planning task (swinging up a pendulum) when using either Bezier (in green) or Second-Order Continuous (in red) interpolation functions. The benchmark consists of 20 runs for each setting $\tau_{\text{max}} \in \{5, 6, 7, 8\}$ Nm. Each run extending an RRT-Bezier and an RRT-SOC on the same random samples. The curves above show the minimum distance-to-goal of roadmaps over time. Average values over all runs of the benchmark are drawn in bold. The area between worst and best cases are filled in light green and red for Bezier and SOC, respectively. Red (resp. green) vertical lines indicate times when an RRT-SOC (resp. RRT-Bezier) successfully connected to the goal state. When it is possible to reach the goal in a single swing ($\tau_{\text{max}} \geq 7$ Nm), RRT-Bezier finds solutions faster. However, for $\tau_{\text{max}} = 6$ Nm, more than half RRT-Bezier instances fail to find a solution after $10^2$ iterations, and all of them fail for $\tau_{\text{max}} = 5$ Nm. Meanwhile, the RRT-SOC variant with suitable interpolations always finds solutions (Theorem 1).
To understand where RRT-Bezier failures stem from, let us compare the impact of the two interpolation functions on a particular run. In the roadmap depicted in Figure 4, the RRT-SOC combo found a four-swing solution after 8,800 RRT extensions. Meanwhile, even after more than 100,000 RRT extensions, the RRT-Bezier combo does not find any solution. Its roadmap, depicted in Figure 5, counts 26,663 roughly distributed between two zones. The first one is a dense, diamond-shape area near the downward equilibrium $\theta = 0$. It corresponds to states that are straightforward to connect by Bezier interpolation, and as expected from Proposition 1, velocities $\dot{\theta}$ in this area decrease sharply with $\theta$. The second one consists of two cones directed towards the goal. Both areas exhibit a higher density near the axis $\dot{\theta} = 0$, which is also consistent with Proposition 1.

The comparison of the two roadmaps is clear: with a second-order continuous interpolation, the RRT-SOC planner leverages additional sampling into exploration of the state space. Conversely, RRT-Bezier lacks this property (Proposition 1), and its roadmap stays confined to a subset of the pendulum’s reachable space.

5. Conclusion

In this paper, we provided the first “operational” proof of probabilistic completeness for a large class of randomized kinodynamic planners, namely those that interpolate state-space trajectories. We observed that an important ingredient for completeness is the “continuity” of the interpolation procedure, which we characterized by the second-order continuity (SOC) property. In particular, we found in simulation experiments that this property is critical to planner performances: a standard RRT with second-order continuous interpolation has no difficulty finding swingup trajectories for a low-torque pendulum, while the same RRT with Bezier interpolation (which are not SOC) could not find any solution. This experimentally confirms our completeness theorem and suggests that second-order continuity is an important design guideline for kinodynamic planners with state-based steering.

References


Figure 4: Phase-space portrait of a roadmap constructed by RRT using the second-order continuous (SOC) interpolation. In this run, the planner found a successful trajectory (thick red curve) after 8,800 extensions. This planner is probabilistically complete (Theorem 1) thanks to the fact that SOC curves satisfy Assumption 5.

Figure 5: Roadmap constructed by RRT after 100,000 extensions using the Bezier interpolation. Reachable states are distributed in two major areas: a central, diamond shape corresponding to the states that the planner can connect at any rate, and two cones directed towards the goal ($\theta = \pi$ or $\theta = -\pi$). Even after several days of computations, this planner could not find a successful motion plan in this run. Our completeness theorem does not apply to this planner because Bezier curves do not comply with discrete accelerations (Assumption 5).


Appendix A. Proofs of the lemmas

**Lemma 1.** Let $g : [0, \Delta t] \rightarrow \mathbb{R}^k$ denote a smooth Lipschitz function. Then, for any $(t, t') \in [0, \Delta t]^2$,

$$
\left\| \dot{g}(t) - \frac{g(t') - g(t)}{|t' - t|} \right\| \leq \frac{K_g}{2} |t' - t|.
$$
Proof. For $t' > t$,
\[
\left\| \dot{g}(t) - \frac{g(t') - g(t)}{t' - t} \right\| \leq \frac{1}{t' - t} \left\| \int_t^{t'} (\dot{g}(t) - \dot{g}(w)) \, dw \right\|
\leq \frac{1}{t' - t} \int_t^{t'} \| \ddot{g}(t) - \ddot{g}(w) \| \, dw
\leq \frac{K_g}{t' - t} \int_t^{t'} |t - w| \, dw
\leq \frac{K_g}{2} (t' - t). \quad \Box
\]

Lemma 2. If there exists an admissible trajectory $\gamma$ with $\delta$-clearance in control space, then there exists $\delta' < \delta$ and a neighboring admissible trajectory $\gamma'$ with $\delta'$-clearance in control space whose acceleration and velocity never vanish, i.e. such that $\|\ddot{\gamma}'\|$ (resp. $\|\dot{\gamma}'\|$) is always greater than some constant $\tilde{m} > 0$ (resp. $\tilde{m} > 0$).

Proof. The main idea behind the proof is to leverage the non-empty interior $U_{admn}$ to add arbitrarily small controls around all zeros of the trajectory $\gamma$. First, imagine that there is a time interval $[t, t']$ on which $\ddot{\gamma} \equiv 0$. Then, suffices to add a wavelet function $\delta\ddot{\gamma}_i$ of arbitrary small amplitude $\delta\ddot{a}_i$ and zero integral over $[t, t']$ to generate a new trajectory $\ddot{\gamma} + \delta\ddot{\gamma}$ where the acceleration cancels on at most a discrete number of time instants. Adding accelerations $\delta\ddot{\gamma}_i$ directly is possible thanks to full actuation. The amplitude of the wavelet $\delta\ddot{\gamma}_i$ needs to be sufficiently small to achieve $\delta'$-clearance in $U_{admn}$ for some $\delta' < \delta$ (a solution always exists thanks to the openness and non-emptiness of its interior).

Suppose now that the roots of $\ddot{\gamma}$ form a discrete set $\{t_0, t_1, \ldots, t_m\}$. Let $t_0$ be one of the first of these roots, and let $[t, t']$ denote a neighborhood of $t_0$. We now add two wavelet functions $\delta\ddot{\gamma}_i$ and $\delta\ddot{\gamma}_j$ of zero integral over $[t, t']$ and arbitrary small amplitude to two coordinates $i$ and $j$, but this time enforcing that the sum of the two wavelets satisfies $|\delta\ddot{\gamma}_i + \delta\ddot{\gamma}_j| \geq \epsilon_{ij} > 0$. This method ensures that the root $t_0$ is eliminated (either $\ddot{\gamma}_i(t_0) \neq 0$ or $\ddot{\gamma}_j(t_0) \neq 0$) while the positivity of $|\delta\ddot{\gamma}_i + \delta\ddot{\gamma}_j|$ ensures that no new root is created at the zero-crossing points of $\ddot{\gamma}_i$ or $\ddot{\gamma}_j$. We conclude by iterating the process on the finite set of roots.

Once $\|\ddot{\gamma}\| \geq \tilde{m}$ for some $\tilde{m} > 0$, the velocity $\ddot{\gamma}$ cannot be uniformly zero over any interval. Let us suppose that $\ddot{\gamma}(t_0) = 0$ for some time $t_0$, and take an interval $[t, t']$ around it. Again, we add two wavelet functions $\delta\ddot{\gamma}_i$ and $\delta\ddot{\gamma}_j$ of zero integral over $[t, t']$, arbitrary small amplitude $\delta a_i, \delta a_j$ and such that $|\delta\ddot{\gamma}_i + \delta\ddot{\gamma}_j| \geq \epsilon_{ij} > 0$. (An example of such function is $\ddot{\gamma}_i(w) = \delta a_i(w) \sin \left(\pi \frac{w - t}{t' - t}\right)$.) For $\|\delta a_i\|$ and $\|\delta a_j\|$ small enough, the wavelets can be added to the trajectory $\gamma$ while enforcing $\delta'$-clearance for some $\delta' < \delta$, which concludes the argument. \quad \Box