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Pulse Width Modulation for Multi-Agent Systems ⋆

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Abstract

This paper studies the consensus problem for multi-agent systems. A distributed consensus algorithm is developed by constructing homogenous pulse width modulators for agents in the network. In particular, a certain percentage of the sampling period named duty cycle is modulated according to some state difference with respect to the neighbors at each sampling instant. During each duty cycle, the amplitude of the pulse is fixed. The proposed pulse width modulation scheme enables all agents to sample asynchronously with arbitrarily large sampling periods. It provides an alternative digital implementation strategy for multi-agent systems. We show that consensus is achieved asymptotically under the proposed scheme. The results are compared with the self-triggered ternary controller.

Key words: Sampled-data control; multi-agent systems; pulse width modulation.

1 Introduction

Pulse width modulation (PWM) is one of the most frequently used ways to perform analog-to-digital conversion with applications in diverse areas including signal processing, control, communication, and power electronics (Skoog & Blankenship 1970). Ease of implementation makes the utilization of PWM an attractive alternative in many control systems (Wang, Meng & Chen 2014). PWM uses rectangular pulse waves with fixed amplitude while the pulse width is adjusted during each period. All pulses have the same amplitude during the duty cycle of the period, but the sign is determined at the beginning of each period according to the control objective. PWM shares the same philosophy as event triggered control, which has been shown to be efficient in utilization of communication and computational resources (Sánchez, Guarnes & Dormido 2009), (Meng & Chen 2012), (Ramesh, Sandberg & Johansson 2013). Both PWM and event triggered control can be regarded as state-dependent switching control laws. In the PWM scheme, the time when the control signal switches from "on" to "off" depends on the sampled state at the beginning of each cycle.

A multi-agent system is a system composed of multiple interacting intelligent agents. Typical multi-agent systems include multiple spacecraft, fleets of autonomous rovers, and formations of unmanned aerial vehicles. The research interest in consensus problems for multi-agent systems is evident with recent monographs (Ren & Beard 2008), (Mesbahi & Egerstedt 2010) and papers (Xiao & Wang 2008), (Meng, Ren & You 2010), (Qin, Zheng & Gao 2011), (Liu, Li, Xie, Fu & Zhang 2013). Early control algorithms for consensus problems are based on continuous information exchange with the assumption that the communication bandwidth is sufficiently large. However, the communication bandwidth

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is often limited in reality. Therefore, a digital implementation of multi-agent systems is much desired.

In this paper, we explore the consensus problem for multi-agent systems with PWM. After obtaining neighbors' information, each agent converts the information into the width of a rectangular pulse wave with unit amplitude. Then the pulse wave is applied to the local agent as an input signal. In contrast to existing results on digital control for multi-agent systems, the main contributions lie in the following four aspects: complete distribution, asynchronous sampling, arbitrarily large sampling period, and saturation free. Firstly, the proposed algorithm is completely distributed in the sense that we require only neighbors' information instead of global topology information, such as the largest or the smallest positive eigenvalues of the associated graph Laplacian matrix. This supports a plug-and-play implementation easily handling agents added to or removed from the network. Secondly, we show that asynchronous sampling is possible for the proposed PWM scheme. Thirdly, we demonstrate that the sampling period can be arbitrarily large for asymptotic consensus. Lastly, the PWM algorithm with a fixed amplitude is advantageous to deal with actuator saturation.

Notation: let $\mathbb{Z}^+$ be the set of non-negative integers, that is, $\mathbb{Z}^+ = \{0, 1, 2, \ldots\}$. The sign function is defined as $\mathrm{sgn}(z) = 1$ if $z > 0$, $\mathrm{sgn}(z) = 0$ if $z = 0$, and $\mathrm{sgn}(z) = -1$ if $z < 0$. For a given real number c, $\lceil c \rceil$ denotes the smallest integer larger than or equal to c.

2 Problem Formulation

2.1 Algebraic Graph Theory

Digraphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ are frequently used to model information exchange among agents, where the vertex set $\mathcal{V} = \{1, \ldots, N\}$ represents agents in a network, and the edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ characterizes the connectivity between agents. The set of neighbors of node i is denoted $\mathcal{N}_i := \{j : (j, i) \in \mathcal{E}\}$ and $|\mathcal{N}_i|$ is the neighborhood cardinality. A directed path is a non-empty subgraph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ of $\mathcal{G}$ of the form $\mathcal{V}' = \{i_0, i_1, \ldots i_k\}$, $\mathcal{E}' = \{(i_0, i_1), (i_1, i_2), \ldots, (i_{k-1}, i_k)\}$ where the i_j, $j = 0, 1, \ldots, k$ are all distinct. A (non-empty) directed graph is said to have a directed spanning tree if there exists at least one node having a directed path to all other nodes.

2.2 System Model

The dynamics of each agent obeys a single integrator model

$$\dot{x}_i(t) = u_i(t), \qquad i \in \mathcal{V}, \tag{1}$$

where $x_i(t)$ is a scalar and $u_i(t)$ denotes the control input for each agent. A distributed PWM algorithm is considered here in the sense that each agent receives information only from neighbors. Also note that each agent has access to only the relative state differences from neighbors with respect to its own state. The information from neighbors will be modulated and then applied as a control input. PWM strategy guarantees a strictly positive lower bound of inter-sample periods for each agent and thus rules out Zeno behavior (Johansson, Egerstedt, Lygeros & Sastry 1999).

2.3 Distributed PWM

Let us first define some terminologies. *Sampling instants* $\{kh_i, k \in \mathbb{Z}^+\}$ are the instants when agent i measures the relative differences with respect to all its neighbors $j \in \mathcal{N}_i$ periodically with a fixed *sampling period* h_i. The PWM control scheme can be described as follows. On each period the input u_i for agent i is switched exactly once from either 1 or -1 to 0. The length of the duration of the kth sampling period on which the input holds the fixed value 1 or -1 is known as the *duty cycle* α_i^k and the *duty rate* is denoted α_i^k/h_i. The duty cycle depends on the state, which will be shown later. The PWM control scheme originates from the control of switching power converters, where usually it is reasonable to assume that the switches can be "on" and "off" at any ratio $\alpha_i^k/h_i \in [0, 1)$.

Let us define an indicator function $s_i(t)$ for agent i to describe "on" and "off" times over a sampling period. When $\alpha_i^k = 0$, $s_i(t) = 0$ for $t \in [kh_i, kh_i + h_i)$; when $\alpha_i^k \neq 0$, $s_i(t) = 1$ if $t \in \left[kh_i, kh_i + \alpha_i^k\right)$, and $s_i(t) = 0$ if $t \in \left[kh_i + \alpha_i^k, kh_i + h_i\right)$. The length of the duty cycle for agent i at sampling instant kh_i is defined as $\alpha_i^k = 0$ if $\mathcal{N}_i = \emptyset$ or $z_i(kh_i) = 0$, and

$$\alpha_i^k = \min\left\{\frac{|z_i(kh_i)|}{2|\mathcal{N}_i|}, \quad h_i\right\}, \tag{2}$$

otherwise, where

$$z_i(kh_i) = \sum\nolimits_{j \in \mathcal{N}_i} (x_i(kh_i) - x_j(kh_i)).$$

Intuitively, each agent measures the sum of the disagreement with respect to its neighbors, and sets the length of its duty cycle proportional to the discrepancy. We define the piecewise constant signal

$$\hat{z}_i(t) = z_i(kh_i), \qquad \text{for } t \in [kh_i, kh_i + h_i),$$

and let the control input for agent i be given by

$$u_i(t) = -s_i(t)\,\mathrm{sgn}\hat{z}_i(t). \tag{3}$$

The solution notion for the differential equation (1) with (3) can be defined using the notion of sample-and-hold solution (Clarke, Ledyaev, Sontag & Subbotin 1997).

Remark 1 *The sample pattern here is different from the traditional sample-and-hold case (Xie, Liu, Wang & Jia 2009). Here each agent samples the neighbors' information periodically in an asynchronous way. Note also that the sampling periods for distinct agents are different. The PWM algorithm allows a distributed implementation without using any a priori information about the global topology. Our PWM scheme shares the philosophy of event triggered control since the length of the pulse depends on the sampled state information.*

Remark 2 *The PWM algorithm is similar to the finite time consensus algorithm in Cortés (2006) and the ternary controller in De Persis & Frasca (2013), as it uses* $\{-1, 0, 1\}$ *as the control input set. The PWM algorithm is different from those algorithms in information acquisition and utilization. The finite time consensus algorithm in Cortés (2006) requests neighbors' state and updates the controller continuously, while the ternary controller in De Persis & Frasca (2013) uses self-triggered communication and piecewise constant control between two consecutive sampling instants. The PWM scheme obtains the information periodically, and the control signal is switched once during each period.*

The objective of this paper is to propose a PWM algorithm such that global asymptotic consensus is achieved for the multi-agent system (1).

Definition 3 *The multi-agent system* (1) *with a given PWM algorithm* u_i*, for all* $i \in \mathcal{V}$*, achieves global asymptotic consensus if for all* $x_i(0) \in \mathbb{R}$ *and all* $i \in \mathcal{V}$*, it holds that* $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$*, for all* $i, j \in \mathcal{V}$*.*

3 PWM over Directed Graphs

Without loss of generality, we relabel $\mathcal{V} = \{1, 2, \ldots, N\}$ such that $0 < h_1 \leq h_2 \leq \cdots \leq h_N$. Define $\Phi(x) = \max_{i\in\mathcal{V}} x_i$, $\Psi(x) = \min_{i\in\mathcal{V}} x_i$, and $V(x) = \Phi(x) - \Psi(x)$, where $x = [x_1, x_2, \ldots, x_N]^{\mathrm{T}}$. In addition, denote $\Phi^* = \Phi(x(0))$, $\Psi^* = \Psi(x(0))$. Before giving the main result, we first present two supporting lemmas. The following lemma shows that the states of all agents of the system (1) with the control law (3) remain bounded for all $t \geq 0$, where the proof is given in Appendix A.

Lemma 4 *Consider the multi-agent system (1) with the PWM control law (3). It follows that* $\Psi^* \leq x_i(t) \leq \Phi^*$*, for all* $t \geq 0$ *and all* $i \in \mathcal{V}$*.*

The following lemma shows that the state of an agent is strictly less than an explicit upper bound as long as it is initially strictly less than this bound, where the proof is given in Appendix B.

Lemma 5 *Consider the multi-agent system (1) with the PWM control law (3). Suppose that* $x_p(k^*h_p) \leq \Phi^* - \varsigma$ *for some* $k^* \in \mathbb{Z}^+$ *and some* $p \in \mathcal{V}$*, where* $0 < \varsigma < \Phi^*$ *is a constant. Then,* $x_p(t) \leq \Phi^* - \varsigma/2^{k-k^*}$*, for all* $t \in [k^*h_p, kh_p)$ *and for all* $k \in \mathbb{Z}^+$ *satisfying* $k > k^*$*.*

Next we give the main result of this paper.

Theorem 6 *Consider the multi-agent system (1) with the PWM control law (3) and suppose that the communication graph* $\mathcal{G}$ *is directed. Global asymptotic consensus is achieved if and only if* $\mathcal{G}$ *contains a directed spanning tree.*

PROOF. Necessity: consider that there are two groups $\mathcal{V}_1$ and $\mathcal{V}_2$, where there are no links between $\mathcal{V}_1$ and $\mathcal{V}_2$. Choose $x_i(0) = c_1$, for all $i \in \mathcal{V}_1$ and $x_j(0) = c_2$, for all $j \in \mathcal{V}_2$ and let $c_1 \neq c_2$. According to the control law (3), we know that $x_i(t) = c_1$ for all $t \geq 0$ and $i \in \mathcal{V}_1$, and $x_j(t) = c_2$ for all $t \geq 0$ and $j \in \mathcal{V}_2$. Therefore, global asymptotic consensus cannot be achieved.

Sufficiency: We use $V(x) = \Phi(x) - \Psi(x)$ as a Lyapunov function candidate.

Suppose that $V(x(0)) \neq 0$ (otherwise, $x_1(t) \equiv x_2(t) \equiv \cdots \equiv x_N(t)$, for all $t \geq 0$ according to (3)). It follows from Lemma 4 that $\Psi^* \leq x_i(t) \leq \Phi^*$, for all $t \geq 0$ and all $i \in \mathcal{V}$. We will show that $V(x(t))$ is strictly decreasing after a sufficiently long time.

Since $\mathcal{G}$ contains a directed spanning tree, we choose any root node p and suppose that the root node satisfies $x_p(0) \leq \Phi^* - \varsigma$ (the opposite case will be discussed later), where $\varsigma = (\Phi^* - \Psi^*)/2 = V(x(0))/2 > 0$. Consider the time interval $[0, \overline{N}h_N)$, where $\overline{N} = 2N - 2$. Define $\overline{N}_p = \lceil \frac{\overline{N}h_N}{h_p} \rceil$. It follows from Lemma 5 that $x_p(t) \leq \Phi^* - \varsigma/2^{\overline{N}_p}$, for all $t \in [0, \overline{N}h_N)$. Since agent p is a root agent, we know that there exists a path of length one from agent p to agent $i_1 \in \mathcal{V}\backslash\{p\}$. We next analyze the trajectory of agent i_1.

Case I: $z_{i_1}(0) \leq 0$. It follows that $0 \leq u_{i_1}(t) \leq 1$, for all $t \in [0, \alpha_{i_1}^0)$ according to (3). Therefore,

$$\begin{aligned}
x_{i_1}(t) &\leq x_{i_1}(0) + \alpha_{i_1}^0 \\
&\leq x_{i_1}(0) + \frac{\frac{\sum_{j\in\mathcal{N}_{i_1}\backslash\{p\}} x_j(0)+x_p(0)}{|\mathcal{N}_{i_1}|} - x_{i_1}(0)}{2} \\
&= \frac{(\sum_{j\in\mathcal{N}_{i_1}\backslash\{p\}} x_j(0) + x_p(0))/|\mathcal{N}_{i_1}| + x_{i_1}(0)}{2} \\
&\leq \Phi^* - \varsigma/(2^{\overline{N}_p+1}N),
\end{aligned}$$

for all $t \in [0, h_{i_1})$.

Case II: $z_{i_1}(0) > 0$. It follows that $u_{i_1}(t) = -1$, for all $t \in [0, \alpha_{i_1}^0)$ according to (3). For the case of $|z_{i_1}(0)|/|\mathcal{N}_{i_1}| \leq$

$2h_{i_1}$, we know that

$$
\begin{aligned}
x_{i_1}(h_{i_1}) &= x_{i_1}(0) - \frac{x_{i_1}(0) - \frac{\sum_{j\in\mathcal{N}_{i_1}\setminus\{p\}} x_j(0)+x_p(0)}{|\mathcal{N}_{i_1}|}}{2} \\
&= \frac{(\sum_{j\in\mathcal{N}_{i_1}\setminus\{p\}} x_j(0)+x_p(0))/|\mathcal{N}_{i_1}| + x_{i_1}(0)}{2} \\
&\le \Phi^* - \varsigma/(2^{\overline{N}_p+1}N).
\end{aligned}
$$

For the case of $|z_{i_1}(0)|/|\mathcal{N}_{i_1}| > 2h_{i_1}$, it follows that $x_{i_1}(h_{i_1}) = x_{i_1}(0) - h_{i_1} \le \Phi^* - h_{i_1} \le \Phi^* - h_1$.

Combining these two cases, we know that $x_{i_1}(h_{i_1}) \le \Phi^* - \chi_1$, where $\chi_1 = \min\{h_1, \varsigma/(2^{\overline{N}^*+1}N)\}$ with $\overline{N}^* = \lceil \overline{N}h_N/h_1\rceil$. It thus follows from Lemma 5 that $x_{i_1}(t) \le \Phi^* - \chi_1/2^{\overline{N}_{i_1}}$ for all $t \in [h_{i_1}, (\overline{N}_{i_1}+1)h_{i_1})$, where $\overline{N}_{i_1} = \lceil \overline{N}h_N/h_{i_1}\rceil - 1$. Then, $x_{i_1}(t) \le \Phi^* - \chi_1/2^{\overline{N}^*}$ for $t \in [h_N, \overline{N}h_N)$ since $[h_N, \overline{N}h_N) \subseteq [h_{i_1}, (\overline{N}_{i_1}+1)h_{i_1})$ and $\overline{N}_{i_1} \le \overline{N}^*$.

We next focus on the time interval $[h_N, 2h_N)$. We know that there exists a path of length one from $\{p, i_1\}$ to i_2. It is not hard to show that there exists a sampling instant $\overline{k}h_{i_2} \in [h_N, 2h_N)$ for agent i_2. We next analyze the trajectory of agent i_2 after $\overline{k}h_{i_2}$.

Case I: there exists an edge from agent p to agent i_2. Then, following the same analysis as for agent i_1, we have that $x_{i_2}((\overline{k}+1)h_{i_2}) \le \Phi^* - \chi_1$.

Case II: there exists an edge from agent i_1 to agent i_2. Similar to the analysis for agent i_1, we have $x_{i_2}((\overline{k}+1)h_{i_2}) \le \Phi^* - \chi_2$ for all $t \in [h_N, \overline{N}h_N)$ where $\chi_2 = \min\{h_1/(2^{1+\overline{N}^*}N), \varsigma/(2^{1+\overline{N}^*}N)^2\}$ since $x_{i_1}(t) \le \Phi^* - \chi_1/2^{\overline{N}^*}$. It thus follows from Lemma 5 that $x_{i_2}(t) \le \Phi^* - \chi_2/2^{\overline{N}_{i_2}}$ for all $t \in [(\overline{k}+1)h_{i_2}, (\overline{k}+1+\overline{N}_{i_2})h_{i_2})$, where $\overline{N}_{i_2} = \lceil(\overline{N}h_N - (\overline{k}+1)h_{i_2})/h_{i_2}\rceil$. Then $x_{i_2}(t) \le \Phi^* - \chi_2/2^{\overline{N}^*}$ for all $t \in [3h_N, \overline{N}h_N)$ since $[3h_N, \overline{N}h_N) \subseteq [(\overline{k}+1)h_{i_2}, (\overline{k}+1+\overline{N}_{i_2})h_{i_2})$ and $\overline{N}_{i_2} \le \overline{N}^*$.

By repeating the above process, it is not hard to show that $x_i(t) \le \Phi^* - \chi_{N-1}/2^{\overline{N}^*}$ for all $t \in [(2N-3)h_N, \overline{N}h_N)$ and all $i \in \mathcal{V}$, where $\chi_{N-1} = \min\{\varsigma/(2^{\overline{N}^*+1}N)^{N-1}, h_1/(2^{\overline{N}^*+1}N)^{N-2}\}$. This implies that $\Phi(\overline{N}h_N) \le \Phi^* - \chi_N$, where $\chi_N = \chi_{N-1}/2^{\overline{N}^*}$. Note that this conclusion is based on the assumption $x_p(0) \le \Phi^* - \varsigma$. Instead, now consider the case $x_p(0) > \Phi^* - \varsigma = \Psi^* + \varsigma$. Doing analogous analysis for $\Psi(x(t))$, we have $x_i(t) \ge \Psi^* + \chi_N$, for all $t \in [(2N-3)h_N, \overline{N}h_N)$ and all $i \in \mathcal{V}$. Therefore, it follows that $\Psi(\overline{N}h_N) \ge \Psi^* + \chi_N$.

Combining the analysis above for Φ and Ψ, we have $V(x(t)) \le V(x(0)) - \chi_N$ since either $x_i(t) \le \Phi^* - \chi_N$ or $x_i(t) \ge \Psi^* + \chi_N$ holds for all $t \in [(2N-3)h_N, \overline{N}h_N)$. Consider the case of $\chi_N = \varsigma/(2^{N\overline{N}^*+N-1}N^{N-1})$. It follows that $V(x(\overline{N}h_N)) \le V(x(0)) - \varsigma/(2^{N\overline{N}^*+N-1}N^{N-1}) = \alpha V(x(0))$, where $\alpha = 1 - 1/(2^{N\overline{N}^*+N}N^{N-1})$. Without loss of generality, we assume that $x_p(0) \le \Phi^* - \varsigma$. We can then find a sampling instant $\widetilde{k}h_p \in [(2N-3)h_N, (2N-2)h_N)$ such that $x_p(\widetilde{k}h_p) \le \Phi^* - \varsigma/(2^{N\overline{N}^*+N-1}N^{N-1})$. Using a similar analysis to the time interval $[0, \overline{N}h_N)$, it is not hard to show that $V(x(2\overline{N}h_N)) \le \alpha^2 V(x(0))$. Therefore, we obtain $V(x(r\overline{N}h_N)) \le \alpha^r V(x(0))$. By noting that $0 < \alpha < 1$ is a constant, we have $\lim_{t\to\infty} V(x(t)) = 0$. For the case of $\chi_N = h_1/(2^{N\overline{N}^*+N-\overline{N}^*-2}N^{N-2})$, the analysis is similar to that of $\chi_N = \varsigma/(2^{N\overline{N}^*+N-1}N^{N-1})$. Overall, we know that $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$, for all $i, j \in \mathcal{V}$ and therefore global asymptotic consensus is achieved. ■

4 Simulation Example

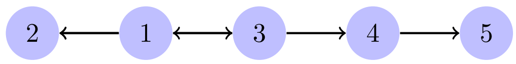

Fig. 1. A graph contains a directed spanning tree

Consider the multi-agent system with the topology shown in Fig. 1 to illustrate Theorem 6. There exists a spanning tree in the communication topology shown in Fig. 1. We choose a random initial condition which is generated from the uniform distribution on the interval $[-1, 1]$. The control signal $u_i(t)$ is modulated according to the algorithm (3) to control each agent. The sampling period for each agent is generated randomly. Fig. 2 shows the evolution of all agents and we see that rendezvous is achieved. The control input of agent 5 is shown in Fig. 3 for the first 4 seconds. We see that the duration of pulses becomes shorter and shorter at the beginning, which indicates agent 5 is closer and closer to its neighbors. After a while, the sign of the pulse becomes negative, which indicates that the state of agent 5 is larger than the average of its neighbors.

The proposed PWM algorithm is a distributed digital algorithm for multi-agent systems. There is another class of digital algorithms for multi-agent systems, called event/self-triggered control (Dimarogonas, Frazzoli & Johansson 2012), (Fan, Feng, Wang & Song 2013), (Garcia, Cao, Yu, Antsaklis & Casbeer 2013), (Seyboth, Dimarogonas & Johansson 2013), (Meng & Chen 2013), (De Persis & Frasca 2013) and (Xiao, Meng & Chen 2015). Note that Dimarogonas et al. (2012), Garcia et al. (2013), Seyboth et al. (2013), Meng & Chen (2013), and Xiao et al. (2015) use the broadcasting way to communicate with neighbors to guarantee the average consensus. Therefore, the comparisons here are made with Protocol A in De Persis & Frasca (2013)

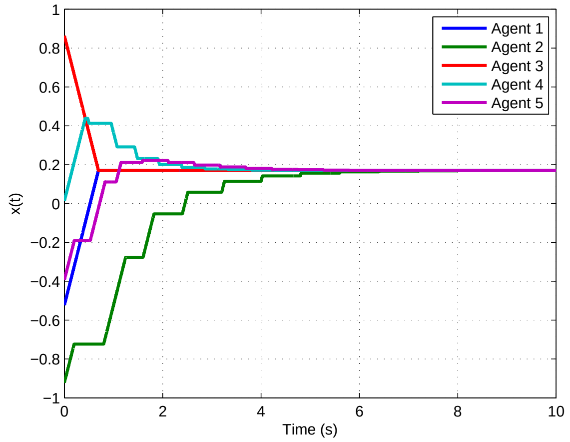

Fig. 2. Evolution of all agents

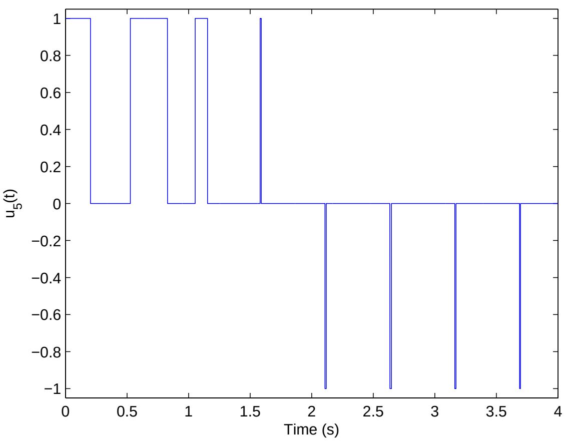

Fig. 3. Control input of agent 5

Table 1
Comparison with self-triggered ternary controllers

	De Persis & Frasca (2013)	PWM
P1	Bounded & Finite-time	Asymptotic
P2	Self-triggered	Periodic
P3	Undirected	Directed
P4	2.1167×10^{-4}	1.1748×10^{-7}
P5	2.1394	1.5504
P6	0.0013	0.5481

only because of the similarities between the proposed PWM method and self-triggered ternary controllers. They share the same sensing model. In addition, average consensus is not preserved in De Persis & Frasca (2013) and herein.

Now let us compare the proposed methods with self-triggered ternary controllers from six different perspectives (P1: Convergence to consensus; P2: Triggering method; P3: Communication Graph; P4: State disagreement; P5: Energy consumption; P6: Average communication time). The comparison is summarized in Table 1. Bounded consensus is achieved within a finite time in De Persis & Frasca (2013), while the proposed PWM algorithm ensures global asymptotic consensus (P1). The work in De Persis & Frasca (2013) uses a self-triggered way to request information from neighboring agents, while the proposed PWM algorithm obtains neighbors' information periodically (P2). The result in De Persis & Frasca (2013) is based on the assumption of undirected connected graphs. However, our result applies to directed graphs containing a directed spanning tree, which include the undirected connected graph as a special case (P3).

Let us perform numerical simulations to compare properties (P4–P6). Note that the comparison is done under the same initial condition which is generated randomly and under the network topology used in Dimarogonas et al. (2012). The simulation time is $T = 10$ seconds. The sampling period of each agent for the PWM method is chosen randomly in the interval (0,1) and $\epsilon = 0.01$ for the ternary controller in De Persis & Frasca (2013). Note that a different choice of ϵ may lead to different numerical values listed in Table 1. For P4, the state disagreement is defined as $\sum_{i=1}^{N}(x_i(T) - \bar{x}(T))^2$, where $\bar{x}(T) = \sum_{i=1}^{N} x_i(T)/N$ is the average of all states. The numerical values shown in Table 1 confirm that consensus is reached for the PWM method, and bounded consensus by the self-triggered ternary controller in De Persis & Frasca (2013). For P5, the energy consumption is defined as $J = \int_0^T u'(\tau)u(\tau)d\tau$. The energy consumed by the PWM approach is less than the self-triggered ternary controller. At the same time, the average communication period of the PWM algorithm is much larger than the ternary controller, which means less communication cost for the PWM method (P6).

5 Conclusion

In this paper, a PWM method was introduced to control a multi-agent system with the objective of reaching consensus. It was shown that no global knowledge about the topology was needed to guarantee asymptotic consensus. The PWM scheme allows all agents to sample asynchronously with arbitrarily large sampling periods. The magnitude of the control signal can be easily chosen by a practitioner based upon actuator saturation constraints as the control signal has fixed amplitude. The efficiency of the algorithm was demonstrated by simulations. For future work, we would like to consider agents with general linear dynamics. We would also like to take into account measurement noise and disturbances.

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A Proof of Lemma 4

We first show that for the sampling interval $[kh_i, kh_i + h_i), k \in \mathbb{Z}^+$ of agent $i \in \mathcal{V}$, the state remains bounded and the bound is determined by the states of all agents at sampling instant kh_i, that is, $\Psi(x(kh_i)) \leq x_i(t) \leq \Phi(x(kh_i))$. We prove this fact by considering the following subcases:

Case I: $\mathcal{N}_i = \emptyset$, or $z_i(kh_i) = 0$. It is trivial to show that $x_i(t) = x_i(kh_i)$ and therefore $\Psi(x(kh_i)) \leq x_i(t) \leq \Phi(x(kh_i))$, for all $t \in [kh_i, kh_i + h_i)$.

Case II: $\mathcal{N}_i \neq \emptyset$, and $z_i(kh_i) < 0$. Then, it follows that $u_i(t) = 1$ for all $t \in [kh_i, kh_i + \alpha_i^k)$ according to (3). Therefore, $x_i(kh_i) \leq x_i(t) \leq x_i(kh_i) + \alpha_i^k \leq x_i(kh_i) - 0.5z_i(kh_i)/|\mathcal{N}_i| = [\sum_{j\in\mathcal{N}_i} x_j(kh_i)/|\mathcal{N}_i| + x_i(kh_i)]/2$, for all $t \in [kh_i, kh_i + \alpha_i^k)$. Thus, $\Psi(x(kh_i)) \leq x_i(t) \leq \Phi(x(kh_i))$, for all $t \in [kh_i, kh_i + h_i)$.

Case III: $\mathcal{N}_i \neq \emptyset$, and $z_i(kh_i) > 0$. It can be shown that $\Psi(x(kh_i)) \leq x_i(t) \leq \Phi(x(kh_i))$ for all $t \in [kh_i, kh_i + h_i)$ similar to Case II.

Combining all these subcases, we know that $\Psi(x(kh_i)) \leq x_i(t) \leq \Phi(x(kh_i))$ for any $i \in \mathcal{V}$ and $t \in [kh_i, kh_i + h_i]$ due to the continuity of $x_i(t)$. We also know that there exists an agent $j \in \mathcal{V}$ such that $x_j(kh_i) = \Phi(x(kh_i))$

and $kh_i \in [\tilde{k}h_j, \tilde{k}h_j + h_j]$ with $\tilde{k}h_j < kh_i$. Based on the above fact we obtain $\Phi(x(kh_i)) \leq \Phi(x(\tilde{k}h_j))$. By repeating this process, we have $\Phi(x(kh_i)) \leq \Phi^*$ for any $k \in \mathbb{Z}^+$. We can similarly show that $\Psi(x(kh_i)) \geq \Psi^*$ for any $k \in \mathbb{Z}^+$. Therefore, we have that $\Psi^* \leq x_i(t) \leq \Phi^*$ for all $t \geq 0$ and all $i \in \mathcal{V}$.

B Proof of Lemma 5

We first consider the time interval $t \in [k^*h_p, k^*h_p + h_p)$ and show that $x_p(t) \leq \Phi^* - \varsigma/2$ for all $t \in [k^*h_p, k^*h_p + h_p)$.

Case I: $\mathcal{N}_i = \emptyset$. It is trivial to show that $x_p(t) = x_p(k^*h_p)$ and therefore $x_p(t) \leq \Phi^* - \varsigma/2$ for all $t \in [k^*h_p, k^*h_p + h_p)$.

Case II: $\mathcal{N}_i \neq \emptyset$, and $z_p(k^*h_p) < 0$. It follows that $u_p(t) = 1$, for all $t \in [k^*h_p, k^*h_p + \alpha_p^{k^*})$ according to (3). Therefore, $x_p(t) \leq x_p(k^*h_p) + \alpha_p^{k^*} \leq x_p(k^*h_p) - 0.5z_i(k^*h_p)/|\mathcal{N}_p| \leq \Phi^* - \varsigma/2$ for all $t \in [k^*h_p, k^*h_p + h_p)$, where we have used the fact that $x_j(k^*h_p) \leq \Phi^*$, for all $j \in \mathcal{V}$ based on Lemma 4.

Case III: $\mathcal{N}_i \neq \emptyset$, and $z_p(k^*h_p) > 0$. The relationship $x_p(t) \leq x_p(k^*h_p) \leq \Phi^* - \varsigma/2$ for all $t \in [k^*h_p, k^*h_p + h_p)$ can be proved since $u_p(t) \leq 0$ for all $t \in [k^*h_p, k^*h_p + h_p)$.

Then, by repeating the above analysis, it follows that $x_p(t) \leq \Phi^* - \varsigma/2^2$ for all $t \in [(k^*+1)h_p, (k^*+2)h_p)$ and therefore $x_p(t) \leq \Phi^* - \varsigma/2^2$, for all $t \in [k^*h_p, (k^*+2)h_p)$. Finally, we have that $x_p(t) \leq \Phi^* - \varsigma/2^{k-k^*}$, for all $t \in [k^*h_p, kh_p)$ for all $k \in \mathbb{Z}^+$ satisfying $k > k^*$.