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The discrete-time second-best dynamic road pricing scheme

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Abstract

Congestion pricing is widely recognized as an efficient instrument for alleviating road congestion. Most of existing road pricing scheme are developed based on traffic equilibrium model. The equilibrium can be looked as a traffic state desired by traffic management authorities. However, when the traffic system has multiple equilibria if the initial traffic state falls beyond the attraction domain, it may not converge to the desired equilibrium under the pricing based on the traffic equilibrium through a day-to-day adjustment process (Bie and Lo, 2010). In this study, we aim to develop a more practical second-best dynamic road pricing scheme implemented on a subset of links, which can drive the traffic evolution towards the desired second-best traffic user equilibrium state from any initial traffic state. This second-best dynamic pricing scheme has the following characteristics: (i) the dynamic pricing is discrete-time scheme, as opposed to most of existing continuous dynamic pricing scheme; (ii) the derivation of the dynamic pricing is applicable to very general day-to-day traffic dynamic model; (iii) the dynamic pricing can direct the system to converge to the desired equilibrium from any initial traffic state even multiple equilibria exist. This study also presents rigorous proofs and numerical tests to verify above these characteristics of our dynamic pricing scheme.

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Keywords: second-best road pricing; day-to-day traffic dynamic model; dynamic road pricing; multiple equilibria

1. Introduction

Roadway congestion is one of the most challenging problems faced by major cities, which has not only brought about enormous economic losses, but environmental pollution. These losses can be avoided in principle, because they mostly result from socially inefficient choices by individual drivers. It has been recognized that road pricing is an effective instrument for alleviating roadway congestion through implementing influence on the traffic demand and traffic flow assignment. The suitable tolls charged on certain links can make the limited road capacity efficiently utilized. The initial idea comes from Pigou (1924), who used an example of congestion road to show the externalities and optimal congestion charges. Walters (1961), Beckmann (1967) and Vickrey (1969) are also the representative and classical works on both intellectual and practical developments of road pricing.

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The first-best pricing or the marginal cost pricing is the theoretical basis of road pricing. This theory states that the user of road should pay toll to internalize the congestion externality or additional cost that the road user imposes on other users. By doing so, each road users will bear the marginal social cost of road use other than the marginal private costs (Yang and Huang, 2005), and the system optimum (SO) can be obtained. There are various extensions of the first-best pricing, e.g., multiclass, multicriteria traffic equilibrium, link flow interactions, and stochastic user equilibrium (Dafermos, 1973; Yang, 1999; Yang and Huang, 2004).

Although the first-best pricing scheme has perfect theoretical justification, it is impractical to levy road pricing on each network link in view of the operating cost and public acceptance. Therefore, the first-best pricing scheme is of little practical interests. As a result, various second-best pricing scheme have been proposed. For example, cordon based road pricing scheme, which is one type of second-best pricing scheme, has been put into practice for managing road congestion in congested urban areas in cities like Singapore, Trondheim and Bergen. Most researches in the context of the second-best pricing problems investigated the determination of toll levels for given charge locations. A classic version of this type of problem studies the two route problem, in which an untolled alternative road is available paralleling with the toll road. This problem was first studied by Levy-Lambert (1968) and Marchand (1968), and more recently by Braid (1996), Verhoef et al. (1996), Liu and McDonald (1998), De Palma and Lindsey (2000) and Verhoef and Small (1999). For a general analytical solution for the second-best problem where not all links of a congested transportation network can be tolled, interested readers may refer to Verhoef (2002), Yang and Huang (2005) and Lawphongpanich et al. (2006).

Most previous studies of the either first-best or second-best road congestion pricing employed approaches that are based on a stationary equilibrium traffic flow pattern in traffic networks to decide how much congestion pricing should be levied at the traffic equilibrium state. From the angle of traffic management, the traffic equilibrium state of first or second best road pricing scheme can be looked as a desired objective of traffic agencies, which should be achieved after certain time period if road pricing scheme is implemented. In fact, travelers adjust their traveling routes on a day-to-day basis and result in link or route flows evolution over time before reaching an equilibrium state. So, it is very natural to ask whether the traffic system can eventually reach the desired objective traffic equilibrium state under the fixed road pricing obtained by equilibrium model. As was pointed out by Horowitz (1984), even for a well-behaved system whose equilibrium solution was known to exist, depending on the dynamic route adjustment process, the system might still fail to converge to equilibrium. Even if a traffic system has already been staying at an equilibrium state, the traffic flow pattern would probably fall into a disequilibrium state because of the perturbation of controlled inputs, exogenous information, or random events, and then start to adjust again toward a new equilibrium (Guo et al., 2015a). Therefore, it is imperative to develop dynamic congestion management measures, which will direct the traffic system to reach and stay at certain equilibrium state desired by traffic authorities. Moreover, the advent of new technologies for the provision of information to automobile drivers and for the tracking of individual vehicles make it possible to implement dynamic congestion management. Dynamic pricing can be classified into two categories, i.e., day-to-day dynamic and within-day pricing according to the time scale of its application. In this study, we would focus on the day-to-day time scale dynamic pricing.

Compared to congestion pricing based on traffic equilibrium, the research works on day-to-day dynamic congestion pricing are much less. Sandholm (2002) applied evolutionary game approach to study the day-to-day dynamic congestion pricing which can force traffic system to reach system optimal state. In their study, the variable congestion pricing is based on marginal travel cost and the elastic and inelastic traffic demand are both considered. Friesz et al. (2004) proposed a disequilibrium day-to-day dynamic pricing scheme, which maximizes the net social benefit over the planning period, considering drivers’ day-to-day behavior articulated in continuous time, revenue constraints and taking the form of ordinary differential equations. However, the proposed dynamic congestion pricing scheme using the continuous time optimal control theory does not guarantee that the traffic system will evolve to an objective state (e.g., SO) of traffic system that may be more desirable for traffic management agencies to achieve. Moreover, they made strong plausible regularity assumptions in their study; for example, the path cost function is assumed to be convex and monotonically increasing. Taking into account a more general behavior adjustment process and making less restrictive assumptions, Yang and W. Y. (2006) developed a day-to-day dynamic congestion pricing strategies that can force the traffic system to evolve from the status quo to SO. They showed that under, their dynamic pricing scheme, the steady state would be SO when the day-to-day dynamic traffic flow becomes stationary. Moreover, they claimed that the evolution to SO may be independent of the underlying day-to-day behavior adjustment process. More
specifically, the traffic system can reach SO while the underlying process takes many forms of the widely used day-to-day dynamical systems in the literatures, including those proposed by Smith (1984), Friesz et al. (2004), Nagurney and Zhang (1997), and Yang (2005). Further, Yang et al. (2007) extended their work and proposed a steepest descent day-to-day dynamic toll which can force the traffic system to evolve from the status quo to a stationary state of the traffic system of system optimum, rather than user equilibrium, and minimizes the derivative of the total system cost with regard to day \( t \) or reduces the total system cost the most for each day. Tan et al. (2015) investigated evolutionary implementation of congestion pricing schemes to minimize the system cost and time, measured in monetary and time units, respectively, with the travelers’ day-to-day route adjustment behavior and their heterogeneity.

It can be observed that above mentioned day-to-day dynamic congestion pricing schemes are all developed based on continuous day-to-day traffic dynamic model, which is conventionally expressed by ordinary differential equations. Although continuous time day-to-day traffic dynamic models have good mathematical properties in traffic evolution, Watling and Hazelton (2003) have pointed two major defects of continuous day-to-day approaches:

1. continuous-time trip adjustment is not plausible in reality.
2. homogeneous population assumptions in these approaches require additional dispersion modules.

Therefore, discrete-time versions of day-to-day traffic dynamic models are more suitable to describe travelers’ routing choice adjustment behavior, which is assumed to be repeated daily, in accordance with daily changes in traffic flows. Therefore, the above mentioned day-to-day dynamic congestion pricing scheme are not appropriate for real application.

To our best knowledge, the first study in the literature on the discrete-time day-to-day dynamic road pricing scheme is Guo et al. (2015a). Their discrete-time dynamic pricing scheme can be applied to very general discrete-time day-to-day traffic dynamic model, which belongs to rational behavior adjustment process (RBAP) (Guo et al., 2015b; Yang and Zhang, 2009). It can drive traffic system to reach a deterministic user equilibrium state satisfying a set of an upper bound of link flows. From the theoretical proof of Guo et al. (2015a), it is known that their dynamic pricing scheme can force the traffic system to converge to the objective traffic equilibrium state by setting the upper bound of link flow equivalent to the objective traffic equilibrium flow. However, this conclusion is obtained based on the assumption of the uniqueness of user equilibrium (Assumption 1 and Assumption 2 of Guo et al. (2015a)). Therefore, the dynamic pricing scheme of Guo et al. (2015a) cannot be applied to general traffic conditions (e.g. traffic network with asymmetric travel cost function generally leading to multiple traffic equilibrium states). Meanwhile, Guo et al. (2015a) did not verify whether their pricing scheme can be applied to the second-best pricing cases, in which the pricing only be levied on a subset of links. For practical application, it is necessary to develop the second-best discrete-time day-to-day dynamic road pricing scheme for general traffic conditions.

This study aims to develop a second-best discrete-time day-to-day dynamic road pricing scheme which is implemented on a subset of links of traffic network. Through the second-best dynamic pricing scheme, the traffic system can converge to a desired traffic equilibrium, which is a second-best traffic equilibrium and is obtained by traffic management implemented on a subset of links. For example, the objective traffic state can be an equilibrium under the traditional second-best pricing scheme, which is usually obtained by solving a bi-level mathematical programming. Specifically, this paper first proves that the dynamic scheme of Guo et al. (2015a) levied on a fixed subset of links can be applied to the second-best day-to-day dynamic road pricing cases if the traffic system has a unique traffic equilibrium. However, if multiple traffic equilibria exist, we present a small example to show that dynamic road pricing levied on a fixed subset links cannot drive the traffic system to converge to the objective traffic equilibrium. Therefore, this study develops a second-best dynamic road pricing scheme implemented on a dynamic subset of links. Through rigorous proof, it can be verified that the second-best dynamic road pricing scheme implemented on variable subset of links can direct the traffic system to reach the objective traffic equilibrium from any initial traffic state. For traffic management, to drive the traffic evolution towards the desired traffic equilibrium from any initial state is essential, as it has been pointed out in Bie and Lo (2010) that only states within its attraction domain are attracted to the equilibrium if multiple equilibria exist.

This paper is organized as follows: the notations and the formulation of day-to-day dynamic traffic model are given in Section 2. Section 3 first shows the dynamic pricing scheme of Guo et al. (2015a) can be applied to the second-best cases if the traffic system has unique equilibrium. The problem of second-best pricing implemented on a fixed subset
of links and our second-best dynamic pricing scheme applied to the general day-to-day traffic dynamic model are also presented in Section 3. In Section 4, for real application, we discuss how to develop the second-best dynamic pricing scheme for a specific day-to-day dynamic traffic model developed by He et al. (2010). Numerical tests are showed in Section 5. Finally, Section 6 presents the conclusions of this study and our future works.

2. Model

2.1. Notation

Let $G(N,A)$ denote a general traffic network with a set $N$ of nodes and a set $A$ of directed links. $W$ is the set of OD pairs. Let $\bar{A}$ denote the fixed subset of links used to charge the toll. $\bar{A}_{t+1}$ is the set of links charged on day $t+1$, which varies with the day-to-day traffic state. $R_w$ is the set of feasible acyclic routes between OD pair $w \in W$, and $R = \bigcup_{w \in W} R_w$. The cardinality of $R_w$ is assumed to be finite. $d = (d_w, w \in W)$ denotes the traffic demand column vector, whose element $d_w(\geq 0)$ is the travel demand between OD pair $w \in W$ and fixed. Let $f_{rw}(\geq 0)$ be the traffic flow on route $r \in R_w$ and $x_a(\geq 0)$ be the flow on link $a \in A$ at time $t$, and $f^t = (f_{rw}, r \in R_w, w \in W)$, and $x_a^t = (x_a^t, a \in A)$ denote the vectors of route flows and link flows, respectively. $\delta_{w,a}$ is the link-route incidence matrix, where $\delta_{w,a} = 1$ if route $r$ uses link $a$ and 0 otherwise. Let $\Lambda = (\Lambda_{rw}, r \in R_w, w \in W)$ be the OD-route incidence matrix, where $\Lambda_{rw} = 1$ if route $r$ connects OD pair $w$ and 0 otherwise. Naturally, we have $x_a^t = \Lambda f^t$ and $d = \Lambda f^t$. $c_a(x)$ is the travel cost function of link $a \in A$. $c(x_a^t) = (c_a(x_a^{t-1}), a \in A)$ is the corresponding link travel cost vector at time $t$. Furthermore, let $C_{rw}^t(f^t)(>0)$ be the travel cost of route $r \in R_w$, and $C(f^t) = (C_{rw}^t(f^t), r \in R_w, w \in W)$ be the corresponding route travel cost vector at time $t$. Then, $C(f^t) = \Lambda^T c(x_a^t) = \Lambda^T c(\Lambda f^t)$. $\tau^{t+1} = (\tau^t, a \in A)$ is the vector of link-specific toll charged on day $t+1$ ($t = 0, 1, 2, \ldots)$, in which

$$\tau_a^{t+1} = \begin{cases} 
\geq 0, & a \in \bar{A}, \\
0, & a \in A/\bar{A}, 
\end{cases}$$

(2.1)

where $\bar{A}_{t+1} = \bar{A}$ or $\bar{A}_{t+1}$ is used to levy the second-best pricing on day $t+1$. Finally, it is supposed that the traffic network $G$ is strongly connected. i.e., each OD pair is connected by at least one route in the network. Let the set $\Omega_x$ be the feasible link flow and $\Omega_t$ be the feasible path flow.

$$\Omega_f = \{f|d = \Lambda f, f \geq 0\}$$

(2.2)

$$\Omega_x = \{x|x = \Delta f, f \in \Omega_f\}$$

(2.3)

In this study, the objective traffic state $x_{obj}$ is a traffic equilibrium, which is obtained under a certain second-best traffic management measure implemented on a given subset $\bar{A} \subset A$ of links. It can be expressed as follows:

$$(c(x) + \tau^*)^T (x - x_{obj}) \geq 0, \forall x \in \Omega_x$$

(2.4)

where $\tau^* = (\tau^*_a, a \in A)$ is the travel cost induced by the second-best traffic management measure, and

$$\tau^*_a = \begin{cases} 
\geq 0, & a \in \bar{A}, \\
0, & a \in A/\bar{A}, 
\end{cases}$$

(2.5)

The objective traffic equilibrium state should be an asymptotical stable state since there are many disturbances in real traffic network, i.e., it is desired that the traffic system still can revert to the objective equilibrium state after the interferences. Therefore, it is assumed that the objective traffic state $x_{obj}$ is an asymptotical stable equilibrium, and its attraction domain is denoted as $B(x_{obj})$, which is defined as $B(x_{obj}) = \{x^t \in \Omega_x | \lim_{t \to \infty} x^t = x_{obj}\}$. The discussions on asymptotic stability and attraction of domain of equilibrium can be referred to Bie and Lo (2010) and Watling (1999).

This study assumes that, before departure from their origins on each day, each traveler has the complete information of link flows, travel cost on the previous day and the intraday link tolls charged on the given subset of links. Each
traveler will try to minimize his or her travel cost when traveling from an origin to a destination, and the traffic state is user equilibrium (UE) when the traffic system reaches a steady state.

2.2. Day-to-day Traffic Dynamic Adjustment under Dynamic Pricing Model

The implementation process of the second-best day-to-day dynamic road pricing, which controls the evolution of traffic state, can be described as in Figure 1. In the evolution process of traffic state, the traffic state $x^{(t+1)}$ on day $t + 1$ is determined by the traffic state $x^{(t)}$ on day $t$ and the congestion control (road pricing) $\tau^{(t+1)}$ on day $t + 1$. Therefore, this study applies the traffic dynamic adjustment model of Guo et al. (2015a) to describe traffic state evolution under the dynamic pricing. The model is expressed as follows:

$$x^{(t+1)} = (1 - \alpha^{(t)})x^{(t)} + \alpha^{(t)}y^{(t)}, \quad t = 0, 1, 2, \ldots,$$

where $0 < \alpha^{(t)} \leq 1$ is the parameter of step size or adjustment ratio of link flow, and $y^{(t)} = y(x^{(t)}, \tau^{(t+1)})$ is the objective adjustment vector of link flow at time $t + 1$ and satisfies

$$y^{(t)}\begin{cases} \in \Psi^{(t)}, & \text{if } \Psi^{(t)} \neq \emptyset, \\ = x^{(t)}, & \text{if } \Psi^{(t)} = \emptyset. \end{cases}$$

where

$$\Psi^{(t)} = \{y | y \in \Omega_x, (y - x^{(t)})^T(c(x^{(t)}) + \tau^{(t+1)}) < 0\}$$

It should be noted that above model is a link-based day-to-day traffic dynamic model and also belongs to the so-called rational behavior adjustment process (RBAP) of flows in a traffic network with fixed demand (Zhang et al., 2001). The day-to-day traffic dynamic adjust model of RBAP, can be grouped into two classes: path-based and link-based models. The path-based dynamical models are built upon route flow variables, and traffic system reaches steady state when the route flows remain unchanged from day to day. As summarized by Yang and Zhang (2009), there exists five major categories of path based day-to-day dynamical model, i.e., the simplex gravity flow dynamics (Smith, 1983), proportional-switch adjustment process (Smith, 1984; Smith and Wisten, 1995; Huang and Lam, 2002; Peeta and Yang, 2003; Mounce, 2006; Mounce and Carey, 2011), network tatonnement process (Friesz et al., 1994; Jin, 2007; Guo and Huang, 2009), projected dynamical system (Zhang and Nagurney, 1996; Nagurney and Zhang, 1997), and evolutionary traffic dynamics (Sandholm, 2001; Yang, 2005). This study only discusses day-to-day dynamic road pricing scheme levied on a given subset of links. Therefore, the link-based day-to-day traffic dynamic model is considered. The first link-based day-to-day traffic dynamic model is presented by He et al. (2010). A very general formation of link-based day-to-day traffic dynamic model is given by Guo et al. (2015b).

3. Dynamic pricing for general traffic conditions

3.1. Dynamic Pricing Scheme of Guo et al. (2015a)

As mentioned in Section 1, the dynamic pricing scheme of Guo et al. (2015a) is implemented on all links. In this section, we will verify that the dynamic pricing scheme of Guo et al. (2015a) can also be implemented on a fixed subset of links to drive the traffic system to converge to the objective equilibrium $x^{ob}$ under the assumption of
uniqueness of UE. The dynamic road pricing scheme of Guo et al. (2015a) implemented on a fixed subset of links can be described as follows:

\[
\tau^{(t+1)} = \begin{cases} Z(t), & \text{if } Z(t) \neq 0, \\ t^{(t)}, & \text{if } Z(t) = 0, \end{cases}
\]

(3.1)

where

\[
Z(t) = \{\tau | \tau \geq 0, (\tau - t^{(t)})^T (x_{\text{obj}} - x(t)) < 0\},
\]

(3.2)

and

\[
\tau_a^{(t+1)} = \begin{cases} \geq 0, & a \in \bar{A}; \\ = 0, & a \in A/\bar{A}. \end{cases}
\]

(3.3)

For convenience to expression, some assumptions of Guo et al. (2015a) are presented as follows:

**Assumption 1.** For any link \( a \in A \), the link travel cost function \( c_a(x_a) \) is positive, continuously differentiable, and strictly increasing with link flow \( x_a \).

**Assumption 2.** \( \alpha^{(t)} (t = 0, 1, \ldots) \) is a sequence satisfying

\[
\sum_{t=0}^{+\infty} \alpha^{(t)} = +\infty, \quad \sum_{t=0}^{+\infty} (\alpha^{(t)})^2 < +\infty.
\]

(3.4)

**Assumption 3.** \( \sum_{t=1}^{+\infty} (\tau^{(t+1)} - \tau^{(t)})^T x^{(t)} < \infty \)

The Assumption 1 can be found in many studies on static traffic equilibrium model to ensure the uniqueness of equilibrium solution of link flow. Assumption 2 on stepsize ensures the convergence of the dynamic process. Assumption 3 sets the constraint on the link tolls throughout the whole implementation horizon rather than on each individual day. It indicates that the total daily toll on the basis of last link flows cannot be increased unlimitedly. From assumptions 1-3, some conclusions can be obtained as follows:

**Proposition 3.1.** The combined dynamic system in equations (2.4)-(2.6) and (3.1)-(3.3) has unique equilibrium state \((\bar{x}, \bar{\tau})\), which satisfies the following conditions:

\[
\bar{\tau}_a(\bar{x}_a - x_{\text{obj}}) = 0, \quad a \in \bar{A}
\]

and

\[
\bar{x}_a \leq x_{\text{obj}}^a, \quad a \in \bar{A},
\]

and the stationary link flow \( \bar{x} \) and the road pricing \( \bar{\tau} \) are both bounded.

**Proposition 3.2.** The combined dynamic system in equations (2.4)-(2.6) and (3.1)-(3.3) can converge to its unique equilibrium state \((\bar{x}, \bar{\tau})\).

Propositions 3.1 and 3.2 can be easily obtained by the proof of Guo et al. (2015a). Their proof only verifies that the traffic flow equilibrium \( \bar{x}_a \leq x_{\text{obj}}^a \) for link \( a \in \bar{A} \), however, we do not know whether \( \bar{x}_a = x_{\text{obj}}^a \) for link \( a \in A/\bar{A} \). So we present the following proposition:

**Proposition 3.3.** The flow equilibrium \( \bar{x} \) of the dynamic system in equations (2.4)-(2.6) and (3.1)-(3.3) is equivalent to the objective traffic equilibrium state \( x_{\text{obj}} \).

Before the proof of Proposition 3.3, a lemma is firstly given:

**Lemma 1.** If \( \bar{x}_a = x_{\text{obj}}^a \), \( \forall a \in \bar{A} \), then \( \bar{x} = x_{\text{obj}} \).

**Proof.** The proof is by contradiction. Suppose there is a link \( \hat{a} \in A/\bar{A} \), on which \( \bar{x}_{\hat{a}} \neq x_{\text{obj}}^\hat{a} \). As the link travel cost \( c_a(x_a) \) is separable and strictly increasing, then

\[
(c(\bar{x}) + \bar{\tau})^T (x - \bar{x}) \geq 0, \quad \forall x \in \Omega_\bar{x}.
\]
and

\[(c(x^{\text{obj}}) + \tau^*)^T(x - x^{\text{obj}}) \geq 0, \forall x \in \Omega_x.\]

Therefore,

\[(c(\bar{x}) + \bar{\tau})^T(x^{\text{obj}} - \bar{x}) \geq 0, \quad (3.5)\]

\[(c(x^{\text{obj}}) + \tau^*)^T(\bar{x} - x^{\text{obj}}) \geq 0. \quad (3.6)\]

From equations (3.5)-(3.6), it can be found that

\[
\sum_{a \in A / \bar{A}} c_a(\bar{x}_a)(x_a^{\text{obj}} - \bar{x}_a) \geq \sum_{a \in \bar{A}} (c_a(\bar{x}_a) + \tau_a)(\bar{x}_a - x_a^{\text{obj}})
\]

\[
\sum_{a \in \bar{A} / A} c_a(x_a^{\text{obj}})(\bar{x}_a - x_a^{\text{obj}}) \geq \sum_{a \in \bar{A}} (c_a(x_a^{\text{obj}}) + \tau_a^*)(x_a^{\text{obj}} - \bar{x}_a)
\]

Because \(\bar{x}_a = x_a^{\text{obj}}\) (\(\forall a \in A\) and \(a \neq \bar{a}\))

\[c_a(\bar{x}_a)(x_a^{\text{obj}} - \bar{x}_a) \geq 0\]

\[c_a(x_a^{\text{obj}})(\bar{x}_a - x_a^{\text{obj}}) \geq 0\]

Further,

\[(c_a(\bar{x}_a) - c_a(x_a^{\text{obj}}))(x_a^{\text{obj}} - \bar{x}_a) \geq 0\]

i.e.,

\[(c_a(\bar{x}_a) - c_a(x_a^{\text{obj}}))(\bar{x}_a - x_a^{\text{obj}}) \leq 0 \quad (3.7)\]

Obviously, (3.7) contradicts with the strict monotonicity of \(c_a(x_a)\). The proof is completed. \(\square\)

Next, a proof based on Lemma 1 is presented to verify Proposition 3.3.

**Proof.** Combining inequalities (3.5)-(3.6) in the proof of Lemma 1, we can obtain an inequality:

\[
(x^{\text{obj}} - \bar{x})^T(c(\bar{x}) - (c(x^{\text{obj}})) \geq -(x^{\text{obj}} - \bar{x})^T \bar{\tau} - (\bar{x} - x^{\text{obj}})^T \tau^*
\]

\[= -\sum_{a \in A}(\alpha_a^{\text{obj}} - \bar{\tau}_a)\bar{\tau}_a - \sum_{a \in \bar{A}}(\bar{x}_a - x_a^{\text{obj}})\tau_a^* \quad (3.8)\]

By Proposition 3.1, it can be found that,

\[(\bar{\tau})^T(x^{\text{obj}} - \bar{x}) = 0, \quad (3.9)\]

and for any \(x_a^{\text{obj}}\) (\(a \in \bar{A}\))

\[
x_a^{\text{obj}} \begin{cases} = \bar{x}_a, & \text{if } \bar{\tau}_a > 0 \\ > \bar{x}_a, & \text{if } \bar{\tau}_a = 0 \end{cases} \quad (3.10)\]

By contradiction, suppose that there is a link \(\bar{a} \in \bar{A}\) which \(x_{\bar{a}}^{\text{obj}} \neq \bar{x}_{\bar{a}}\), then equation (3.8) can be transferred as

\[(x^{\text{obj}} - \bar{x})^T(c(\bar{x}) - (c(x^{\text{obj}})) \geq (x_{\bar{a}}^{\text{obj}} - \bar{x}_{\bar{a}})\tau_{\bar{a}}^* \geq 0.\]

From equation (3.10), the above inequality is equivalent to the following:

\[(\bar{x} - x^{\text{obj}})^T(c(\bar{x}) - (c(x^{\text{obj}})) \leq 0 \quad (3.11)\]

This contradicts with the Assumption 1 on the monotonicity of \(c(x)\). Therefore, \(x_a^{\text{obj}} = \bar{x}_a\) (\(a \in \bar{A}\)). Further, by the Lemma 1, it can be found that \(\bar{x} = x^{\text{obj}}\). The proof is completed. \(\square\)

By Proposition 3.2 and 3.3, it can be concluded that the traffic dynamic system in equations (2.4)-(2.6) can converge to the given desired traffic equilibrium \(x^{\text{obj}}\) under the dynamic road pricing scheme in equations (3.1)-(3.3).
3.2. Problem of Existing Dynamic Pricing Scheme for Second-best Cases

In Section 3.1, we have proved that the dynamic road pricing scheme proposed in Guo et al. (2015a) can be implemented on a fixed subset and drive the traffic system to converge to the given second-best objective traffic equilibrium state when the traffic system has a unique traffic equilibrium state. Next, we will give an example to explain why the dynamic road pricing scheme Guo et al. (2015a) is no longer applicable for the case when multiple traffic equilibria exist. Consider a simple network with only one OD pair and two links between the OD pair. The link travel cost are respectively:

\[ c_1 = f_1 + 3f_2 + 1, \quad c_2 = 2f_1 + f_2 + 2. \] (3.12)

Let traffic demand be 1 unit. The road pricing is implemented on link 1 and is 1 unit. It can be found that, under the fixed road pricing, there are three user equilibria:

\[ f'_I = (1, 0), f'_II = (0, 1), f'_III = \left( \frac{1}{3}, \frac{2}{3} \right). \] (3.13)

Supposing \( f'_I = (1, 0) \) be the objective traffic equilibrium state, and the another equilibrium flow \( f'_II = (0, 1) \) is the initial flow pattern. The initial travel cost corresponding to the initial flow without road pricing is

\[ c_1(f'_I) = 4, \quad c_2(f'_II) = 3. \] (3.14)

This means that the initial state \( f'_II = (0, 1) \) also is a UE flow without any road pricing. So, for any road pricing \( \tau = (\tau_1 \geq 0, \tau_2 = 0) \), \( f'_II = (0, 1) \) is a UE equilibrium flow, i.e., for any feasible flow \( f \) of this example,

\[ (f - f'_II)^T C(f'_II) \geq 0, \] (3.15)

and

\[ c_1(f'_II) > c_2(f'_II). \] (3.16)

Therefore, for any nonnegative second-best road pricing \( \tau = (\tau_1 \geq 0, \tau_2 = 0) \) implemented on link 1, \( c_1(f'_II) + \tau_1 > c_2(f'_II) + \tau_2 \), and

\[ (f - f'_II)^T (C(f'_II) + \tau) \geq 0. \] (3.17)

Further, there is no feasible flow \( y \) defined in equation (3.10) which satisfies

\[ (y - f'_II)^T (C(f'_II) + \tau) < 0, \] (3.18)

In the dynamic process in equations (2.6)-(2.8), the flow \( f(t) \) is always equal to \( f'_II \) instead of converging to the objective traffic equilibrium (1, 0).

From the above simple example, one can find that road pricing implemented on the fixed subset of links cannot drive the traffic system to converge to the objective traffic equilibrium state for multiple traffic equilibria case. In next section, we will discuss whether a dynamic road pricing scheme, which is implemented on a variable subset of links, can direct the traffic system to reach the objective traffic equilibrium when the network has multipute traffic equilibria.

It should be noted that the dynamic pricing scheme of Guo et al. (2015a) even levied on a dynamic subset of links also cannot guarantee that the traffic system must be driven to converge to the objective equilibrium if the traffic system has multiple equilibria. From the results of Guo et al. (2015a), one can observe that, if day-to-day traffic dynamics under their dynamic pricing scheme has steady point \((\bar{x}, \bar{\tau})\), then the charged links set should be \( \bar{A} \) in Eq.(2.5), and

\[ (x^{obj} - \bar{x})^T \bar{\tau} = 0, \]

and

\[ x^{obj}_a - \bar{x}_a \geq 0, \quad \forall a \in \bar{A}. \]

From the proof in Section 3.1, one can find that, without the assumption of monotonicity of link travel cost \( c(x) \), the conclusion of \( x^{obj} = \bar{x} \) cannot be guaranteed. To describe the problem more clearly, we consider an example with simple network which has only one OD pair. There are three links between the OD pair. The travel cost of each link is:
on link 2, there are three user equilibrium states: \((1,0,1), (0,1,1)\) and \((1/2,1/4,5/4)\). Assume the objective traffic state to be achieved is \((0,1,1)\). The specific pricing scheme applied is the Walrasian scheme of \textit{Garcia et al. (2012)}\footnote{See reference here}, which also belongs to the dynamic pricing scheme of \textit{Guo et al. (2015b)}:

\[
\tau^{t+1} = [\tau^{t} + \rho^{(n)}(x^{(t)} - \bar{x})]_{+}, \quad t = 0, 1, 2, \ldots
\]

where \(\rho^{(t)}\) is the sensitive parameter and determines the speed at which the tolls are updated. According to the assumption in \textit{Garcia et al. (2012)}, let \(\rho^{(t)} = 1/(t + 1)^{1/4}\) and \(\alpha^{(t)} = 1/(t + 1)^{0.51}\) in Eq.(2.6). The dynamic pricing is implemented on the dynamic charged link set \(\bar{A}^{(t+1)}\) as defined by this study in Eq.(3.19). The evolution process of the path flows under the pricing scheme of \textit{Garcia et al. (2012)} is simulated by the traffic dynamic model of \textit{He et al. (2010)} with \(\lambda = 0.6\) and shown in Figure 2 with initial state \((0,0,2)\) and objective equilibrium \((0,1,1)\). The toll at each time gained by the pricing scheme of \textit{Garcia et al. (2012)} is shown in Figure 3.

\[\text{Fig. 2. The dynamics of flow with initial state (0,0,2) and objective equilibrium (0,1,1)}\]

\[\text{Fig. 3. The dynamics of toll with initial state (0,0,2) and objective equilibrium (0,1,1)}\]

From Figure 2 and 3, one can find that the traffic dynamic system under the dynamic pricing scheme of \textit{Garcia et al. (2012)} with dynamic charged link set converges to the steady flow pattern \((0,1.25,0.75)\) rather than the objective
equilibrium flow pattern (0, 1, 1). The steady toll is 0.329, under which the steady flow (0, 1.25, 0.75) is a UE flow pattern.

From the above discussions, it is obvious that, in some cases, existing dynamic pricing cannot drive the traffic dynamic system to converge to a given objective equilibrium when traffic system has multiple equilibria. Therefore, it is necessary to develop the specific second-best dynamic pricing scheme for multiple equilibria case.

3.3. The second-best dynamic pricing scheme for multiple equilibria case

From the example mentioned above, the second-best dynamic road pricing scheme is implemented on a variable subset of links. This means that, in the dynamic process, the subset of links $\overline{A}^{(t)}$ used to implement road pricing on day $t$ may be not identical with $\overline{A}^{(t+1)}$ used on day $t + 1$. In this study, the $\overline{A}^{(t+1)}$ is determined by the traffic flow $x^{(t)}$, and its expression can be described as follows:

$$\overline{A}^{(t+1)} \subseteq \overline{A}^{(t+1)}_+,$$

where

$$\overline{A}^{(t+1)}_+ = \{ a \in A | x_a^{\text{obj}} - x_a^{\text{obj}} > 0 \}$$

By the road pricing levied on the variable subset $\overline{A}^{(t+1)}$ of links, we can obtain the following conclusion:

**Proposition 3.4.** For any $x^{(t)} \neq x^{\text{obj}}$ (t = 0, 1, ...), there is a road pricing $\tau^{(t+1)} = (\tau_a^{(t+1)}, a \in A)$, in which

$$\tau_a^{(t+1)} = \begin{cases} 
\geq 0, \text{if } a \in \overline{A}^{(t+1)}_+ \\
= 0, \text{otherwise,}
\end{cases}$$

(3.21)

can make $x^{\text{obj}}$ satisfying

$$(x^{\text{obj}} - x^{(t)})^T (c(x^{(t)}) + \tau^{(t+1)}) < 0.$$  

(3.22)

**Proof.** For any $x^{(t)} \neq x^{\text{obj}}$, by the definition of $\overline{A}^{(t+1)}$ in equations (3.19) and (3.20), one can observe that, for $a \in \overline{A}^{(t+1)}_+$,

$$x_a^{\text{obj}} - x_a^{(t)} < 0.$$

For any $\tau \geq 0$ satisfying equation (3.21), one can find that

$$\sum_{a \in A} (c_a(x^{(t)})^T \tau_a) (x_a^{\text{obj}} - x_a^{(t)}) = \sum_{a \in \overline{A}^{(t+1)}} (c_a(x^{(t)}) + \tau_a) (x_a^{\text{obj}} - x_a^{(t)}) + \sum_{a \in A \setminus \overline{A}^{(t+1)}} c_a(x^{(t)}) (x_a^{\text{obj}} - x_a^{(t)})$$

$$= \sum_{a \in \overline{A}^{(t+1)}} \tau_a (x_a^{\text{obj}} - x_a^{(t)}) + \sum_{a \in A} c_a(x^{(t)}) (x_a^{\text{obj}} - x_a^{(t)}),$$

and

$$\sum_{a \in \overline{A}^{(t+1)}} \tau_a (x_a^{\text{obj}} - x_a^{(t)}) < 0.$$

One also can find that $\sum_{a \in A} c_a(x^{(t)}) (x_a^{\text{obj}} - x_a^{(t)})$ is fixed since the $x^{(t)}$ and $x^{\text{obj}}$ are both given at time $t$. Therefore, there exists a $\tau^{(t+1)} \geq 0$ satisfying equation (3.21) and the following condition:

$$\sum_{a \in \overline{A}^{(t+1)}} \tau_a^{(t+1)} (x_a^{\text{obj}} - x_a^{(t)}) < -\sum_{a \in A} c_a(x^{(t)}) (x_a^{\text{obj}} - x_a^{(t)}).$$

Further,

$$\sum_{a \in A} (c_a(x^{(t)}) + \tau_a^{(t+1)}) (x_a^{\text{obj}} - x_a^{(t)}) = \sum_{a \in \overline{A}^{(t+1)}} \tau_a^{(t+1)} (x_a^{\text{obj}} - x_a^{(t)}) + \sum_{a \in A \setminus \overline{A}^{(t+1)}} c_a(x^{(t)}) (x_a^{\text{obj}} - x_a^{(t)}) < 0$$

The proof is completed.
By Proposition 3.4, a second-best dynamic road pricing scheme can be defined: for the given objective traffic equilibrium state $x^{obj}$, the road pricing $\tau^{(t+1)}$ in equation (3.21) on day $t+1$ is determined by the traffic flow $x^{(t)}$ on day $t$, and is expressed as follows:

$$
\tau^{(t+1)} = \begin{cases} 
0, & \text{if } a \in A^{(t+1)} \\
0, & \text{otherwise}, 
\end{cases} \quad (3.23)
$$

where $A^{(t+1)} = \bar{A}^{(t+1)}$, if $\|x^{obj} - x^{(t)}\| \geq \delta$

and

$$
\tau^{(t+1)} = \bar{A}^{(t+1)}, \quad (3.24)
$$

and

$$
\tau^{(t+1)} = \tau^*, \quad (3.25)
$$

where

$$
Q^{(t+1)} = \{ \tau | \tau \geq 0, (c(x^{(t)}) + \tau)^T(x^{obj} - x^{(t)}) < 0 \}. \quad (3.26)
$$

where $\tau^*$ is the road pricing corresponding to the second-best objective traffic equilibrium state $x^{obj}$ in equation (2.4) and (2.5). $\| \cdot \|$ is the Euclidean metric to measure the distance between $x^{(t)}$ and $x^{obj}$. $\delta > 0$ is the given distance level to make sure that the neighborhood $H(x^{obj}, \delta) = \{ x | x \in \Omega_x, \| x - x^{obj} \| < \delta \}$ is a subset of the attraction domain of $x^{obj}$, i.e., $H(x^{obj}, \delta) \subseteq B(x^{obj})$.

In practice, it may be difficult to determine the attraction domain of $x^{obj}$. However, by applying numerical method (e.g., the method discussed in Bie and Lo (2010)), it is not hard to find a proper value of $\delta > 0$ to guarantee $H(x^{obj}, \delta) \subseteq B(x^{obj})$. For example, the following iterative process can be used to get a proper $\delta$.

Step 1: Given initial value of $\delta^n > 0$, parameter $0 < \alpha < 1$, and let iterative number $n = 0$;

Step 2: Randomly produce $N$ (500 or larger positive number) feasible flows belong to $H(x^{obj}, \delta^n)$;

Step 3: For each example feasible flow, judge whether it can converge to the objective traffic equilibrium state under the road pricing $\tau^*$ by the traffic dynamic model in equations (2.6)-(2.8).

Step 4: If all $N$ example flows all can converge to $x^{obj}$, then stop; otherwise, let $\delta^{n+1} = \alpha \delta^n$, $n=n+1$. Go to step 2.

Before we can prove that the second-best dynamic pricing scheme in equations (3.23)-(3.26) can direct the traffic system to reach the objective traffic equilibrium state, a assumption is presented as follows:

**Assumption 4.** For any $x \in \Omega_x \setminus H(x^{obj}, \delta)$, there exist a pricing $\tau$ in Eq.(3.25) and Eq.(3.26) which can drive the function $y = y(x, \tau)$ in equation (2.7) to satisfy the following inequality:

$$
(x - x^{obj})^T(y(x, \tau) - x) < 0. \quad (3.27)
$$

**Theorem 3.1.** With Assumption 2 and Assumption 4, the dynamic second-best dynamic congestion pricing scheme in equations (3.23)-(3.26) can drive the traffic dynamic system in equations (2.4)-(2.6) to converge to the objective traffic equilibrium $x^{obj}$.

**Proof.** First, we prove that the day-to-day traffic dynamic system eventually enter into the neighborhood $H(x^{obj}, \delta)$ of the objective traffic equilibrium state $x^{obj}$. The proof is by contradiction. For any $x^{(t)} \in \Omega_x \setminus H(x^{obj}, \delta)$, then the distance $D(x^{(t+1)}, x^{obj})$ between $x^{(t+1)}$ and the objective traffic equilibrium $x^{obj}$ can be described by Euclidean norm as follows:

$$
D(x^{(t+1)}, x^{obj}) = \|x^{(t+1)} - x^{obj}\|^2. \quad (3.28)
$$

Substituting equation (2.6) into equation (3.28), it can be obtained that

$$
D(x^{(t+1)}, x^{obj}) = \|x^{(t)} - x^{obj}\|^2 + 2\alpha^t(x^{(t)} - x^{obj})^T(y^{(t)} - x^{(t)}) + (\alpha^t)^2\|y^{(t)} - x^{(t)}\|^2,
$$
i.e.,
\[
D(x^{t+1}, x^{obj}) - D(x^t, x^{obj}) = 2 \alpha^t(x^t - x^{obj})^T(y^t - x^t) + (\alpha^t)^2 ||y^t - x^t||^2.
\] (3.29)

From equation (3.29), it can be obtained that
\[
D(x^{t+1}, x^{obj}) - D(x^0, x^{obj}) = 2 \sum_{n=0}^{t} \alpha^n(x^n - x^{obj})^T(y^n - x^n) + (\alpha^n)^2 ||y^n - x^n||^2.
\] (3.30)

Since \(\Omega_\delta/H(x^{obj}, \delta)\) is a compact set, and \(D(x, x^{obj})\) is continuous with respect to \(x \in \Omega_\delta\), the distance function \(D(x, x^{obj})\) is bounded. Therefore, the right-hand side of equation (3.30) and \(||y-x||^2\) are both bounded in \(\Omega_\delta/H(x^{obj}, \delta)\). Further, from Assumption 2 and Assumption 4, it can be derived that
\[
\sum_{n=0}^{+\infty} (\alpha^n)^2 ||y^n - x^n||^2 < +\infty
\] (3.31)
and,
\[
\sum_{n=0}^{+\infty} \alpha^n(x^n - x^{obj})^T(y^n - x^n) > -\infty.
\] (3.32)

Let
\[
Z(x^t, y^t) = (x^t - x^{obj})^T(y^t - x^t).
\] (3.33)

Then, from Assumption 4, we have \(Z(x^t, y^t) < 0\) for any \(x^t \in \Omega_\delta/H(x^{obj}, \delta)\). Thus, by Theorem 5.2.12 in Trench (2013), there is a constant \(h < 0\) such that \(Z(x^t, y^t) \leq h < 0\) for \(\forall x^t \in H(x^{obj}, \delta)\). As \(0 < \alpha^{t_0} \leq 1\),
\[
Z(x^t, y^t) = (x^t - x^{obj})^T(y^t - x^t) \leq \alpha^{t_0}Z(x^{t_0}, y^{t_0}).
\] (3.34)

If the day-to-day traffic dynamic system under the second-best dynamic pricing scheme cannot enter into the neighborhood \(H(x^{obj}, \delta)\), then we have
\[
\sum_{n=0}^{+\infty} \alpha^n(x^n - x^{obj})^T(y^n - x^n) \leq \sum_{n=0}^{+\infty} \alpha^n h < -\infty
\] (3.35)

This contradicts with equation (3.32). Therefore, the dynamic pricing scheme can drive the traffic dynamic system enter into the neighborhood \(H(x^{obj}, \delta) \subset B(x^{obj})\).

By the definition of attraction domain of a stable equilibrium in Bie and Lo (2010), the attraction domain \(B(x^{obj})\) of \(x^{obj}\) can be described as follows:
\[
B(x^{obj}) = \{x^t \in \Omega_\delta | \lim_{t \to +\infty} x^t = x^{obj}\}.
\]

Because the neighborhood \(H(x^{obj}, \delta) \subset B(x^{obj})\), one can derive that, for any \(x^t \in H(x^{obj}, \delta)\), \(\lim_{t \to +\infty} x^t = x^{obj}\). Therefore, the dynamic second-best dynamic congestion pricing scheme in equations (3.23)-(3.26) can drive the traffic dynamic system in equations (2.4)-(2.6) to converge to the objective traffic equilibrium \(x^{obj}\). The proof is completed.

Through the proof of Theorem 3.1, one can observe that, for \(x^t \in \Omega_\delta/H(x^{obj}, \delta) (t = 0, 1, \ldots)\), under the dynamic pricing \(t^{t+1} (t = 0, 1, \ldots)\), if the function \(y^t = y(x^t, t^{t+1})\) of a certain specific formulation of the day-to-day traffic dynamic system in equations (2.4)-(2.6) satisfies equation (3.27):
\[
(x^t - x^{obj})^T(y^t - x^t) < 0,
\]
the dynamic pricing sequence \( \{\tau^{i+1}, t = 0, 1, \ldots\} \) can drive the specific formulation of the day-to-day traffic dynamic system in equations (2.4)-(2.6) following Assumption 2 to enter into the neighborhood \( H(x^{ob}, \delta) \) of the objective traffic equilibrium \( x^{ob} \), and finally motivates the day-to-day traffic dynamic system converge to the objective traffic equilibrium \( x^{ob} \). However, equation (3.27) is not intuitive so that one may doubt whether there is a specific day-to-day traffic dynamic model following Assumption 4. Therefore, the next section will discuss how to develop the dynamic pricing scheme \( \{\tau^{i+1}, t = 0, 1, \ldots\} \) following (3.21) and (3.22) which can drive the function \( y^{i}(t) = y(x^{i}(t), \tau^{i+1}(1) = 0, 1, \ldots) \) of some existing day-to-day traffic dynamic models satisfying equation (3.27) for any \( x^{i}(t) \). Further, the rationality of Assumption 4 is verified.

4. A Specific Case

4.1. The Dynamic Pricing Scheme for A Specific Traffic Dynamic Model

He et al. (2010) presented the first link-based day-to-day traffic model. The formulation of the function \( y^{i}(t) = y(x^{i}(t), \tau^{i+1}(1) = 0, 1, \ldots) \) of He et al. (2010) is formulated as follows:

\[
y^{i}(t) = \arg\min_{y \in \Omega_{x}} \lambda (x^{i}(t))^T y + (1 - \lambda) D(x^{i}(t) - y)
\]

where \( 0 < \lambda < 1 \). \( D(x^{i}(t) - y) \) is a function which can measure the distance of \( x^{i}(t) \) and \( y \). From the results of (Guo et al., 2015b), it is known that the link-based network tatonnement process and link-based projected dynamic model can be described by a unified expression:

\[
x^{i}(t+1) - x^{i}(t) = \alpha^{i}(t)y^{i}(t) - x^{i}(t),
\]

where \( y^{i}(t) \) is the projection of \( x^{i}(t) - \beta(x^{i}(t)) + \tau^{i+1}(1) \) onto the feasible link flow domain \( \Omega_{x} \):

\[
y^{i}(t) = P_{\Omega_{x}}[x^{i}(t) - \beta(x^{i}(t)) + \tau^{i+1}(1)]
\]

\[
= \arg\min_{y \in \Omega_{x}} \parallel y - (x^{i}(t) - \beta(x^{i}(t)) + \tau^{i+1}(1)) \parallel^{2}
\]

where \( \beta \) is a positive constant and determines the flow changing direction. One can find that let \( \beta = \frac{2 - 2 \lambda}{\lambda} \), then equations (4.1) and (4.3) are equivalent when the distance function \( D(x, y) = \parallel x - y \parallel^{2} \). Therefore, above mentioned three link-based day-to-day traffic dynamic models can be looked as one. For this class of function \( y^{i}(t) = y(x^{i}(t), \tau^{i+1}(1)) \) in equation (4.1) or (4.3), we have the following conclusions:

**Theorem 4.1.** The second-best dynamic pricing scheme \( \tau^{i+1}(1) = 0, 1, \ldots) \) in equations (3.23)-(3.26) levied on the dynamic subset \( \tilde{A}^{(i+1)} \) of links can drive the function \( y^{i}(t) = y(x^{i}(t), \tau^{i+1}(1)) \) in equation (4.1) or (4.3) to satisfy equation (3.27). Further, the dynamic pricing scheme can drive the day-to-day traffic dynamic model with the function \( y^{i}(t) = y(x^{i}(t), \tau^{i+1}(1)) \) in equation (4.1) or (4.3) to converge to the objective traffic equilibrium \( x^{ob} \).

**Proof.** First, for any \( x^{i}(t) \in \Omega_{x}/H(x^{ob}, \delta) \), let us prove that, under the dynamic road pricing scheme \( \tau^{i+1}(1) \) in equations (3.12)-(3.15), \( x^{i}(t), x^{ob} \) and the \( y^{i}(t) \) in equation (4.1) or (4.3) have the relation in equation (3.27), i.e.,

\[
(x^{i}(t) - x^{ob})T(y^{i}(t) - x^{i}(t)) < 0.
\]

From equation (4.3), it can be found that, when \( x^{i}(t) \in \Omega_{x}/H(x^{ob}, \delta) \), any feasible flow \( y \in \Omega_{x} \),

\[
\parallel \beta(x^{i}(t)) + \tau^{i+1}(1) - (y^{i}(t) - x^{i}(t)) \parallel^{2} \leq \parallel \beta(x^{i}(t)) + \tau^{i+1}(1) - (y - x^{i}(t)) \parallel^{2}.
\]

Further,

\[
\parallel \beta(x^{i}(t)) + \tau^{i+1}(1) - (y^{i}(t) - x^{i}(t)) \parallel^{2} \leq \parallel \beta(x^{i}(t)) + \tau^{i+1}(1) - (x^{i}(t) - x^{i}(t)) \parallel^{2},
\]

and

\[
(x^{i}(t) + \tau^{i+1}(1))T(y^{i}(t) - x^{i}(t)) < 0.
\]
Therefore, $x^{obj}, y^{i} \in \Psi^{(i)}$ and $x^{(i)} \notin \Psi^{(i)}$. If $x^{obj} = y^{(i)}$, then

$$
(x^{(i)} - x^{obj})^T (x^{(i)} - y^{(i)}) = (x^{(i)} - x^{obj})^T (x^{(i)} - x^{obj}) > 0.
$$

(4.6)

If $x^{obj} \neq y^{(i)}$, let $Q^{(i)} = \{z | z = x^{(i)} - y, y \in \Omega_x\}$. From equation (4.3), one can find that $x^{(i)} - y^{(i)}$ is the projection of $\beta(c(x^{(i)} + \tau^{(i+1)}))$ on set $Q^{(i)}$, and the original point $O$ also belongs set $Q^{(i)}$. Because points $O$, $x^{(i)} - x^{obj}$ and $x^{(i)} - y^{(i)}$ are different, they can make a plane or a line. If these three points make a plane, let $\tilde{Q}^{(i)} = \{z | z = \alpha_1 O + \alpha_2 (x^{(i)} - x^{obj}) + \alpha_3 (x^{(i)} - y^{(i)}), \sum \alpha_i = 1, \alpha_1, \alpha_2, \alpha_3 \geq 0\}$. It can be found that set $\tilde{Q}^{(i)}$ is a subset of the plane, and $\tilde{Q}^{(i)} \subseteq Q^{(i)}$. $\tilde{Q}^{(i)}$ is also a closed convex set. One can also find that $x^{(i)} - y^{(i)}$ is the projection of $\beta(c(x^{(i)} + \tau^{(i+1)}))$ onto $Q^{(i)}$. Let point $\hat{B}$ be the projection of $\beta(c(x^{(i)} + \tau^{(i+1)}))$ onto the plane made by points $O$, $x^{(i)} - x^{obj}$ and $x^{(i)} - y^{(i)}$. Although $\hat{B}$ may be not equal to $x^{(i)} - y^{(i)}$, the minimum distance between point of $\tilde{Q}^{(i)}$ and $\hat{B}$ is $||\hat{B} - (x^{(i)} - y^{(i)})||^2$. Thus, when $O$, $x^{(i)} - x^{obj}$ and $x^{(i)} - \hat{y}^{(i)}$ make a plane, we can use figure 4 to show the position relation among $\beta(c(x^{(i)} + \tau^{(i+1)}))$, $x^{(i)} - x^*$ and $x^{(i)} - \hat{y}^{(i)}$.

From equations (4.4) and (4.5), it can be found that the angle $\gamma \in [0, \pi/2)$ between vectors $c(x^{(i)} + \tau^{(i+1)})$ and $x^{(i)} - y^{(i)}$, and the angle $\zeta \in [0, \pi/2)$ between vectors $c(x^{(i)} + \tau^{(i+1)})$ and $x^{(i)} - x^{obj}$. Thus the angle between vectors $OB$ and $x^{(i)} - y^{(i)}$ is a cute angle, and the angle between vectors $OB$ and $x^{(i)} - x^{obj}$ also is a cute angle. Further, the angle between vectors $x^{(i)} - x^{obj}$ and $x^{(i)} - y^{(i)}$ also is a cute angle. So, when $x^{(i)}$, $x^*$ and $\hat{y}^{(i)} \in \Psi^{(i)}$ make a plane,

$$
(x^{(i)} - x^{obj})^T (x^{(i)} - y^{(i)}) > 0.
$$

(4.7)

![Fig. 4. Relation among $\beta(c(x^{(i)} + \tau^{(i+1)}))$, $x^{(i)} - x^{obj}$ and $x^{(i)} - y^{(i)}$](image)

If points $O$, $x^{(i)} - x^{obj}$ and $x^{(i)} - y^{(i)}$ are collinear, then $x^{(i)}$, $x^{obj}$ and $y^{(i)}$ are collinear. As $x^{(i)} \notin \Psi^{(i)}$, and $\Psi^{(i)}$ is a convex set,

$$
(x^{(i)} - x^{obj})^T (x^{(i)} - y^{(i)}) > 0.
$$

(4.8)

Combining equations (4.6), (4.7) and (4.8), it can be obtained that

$$
(x^{(i)} - x^{obj})^T (x^{(i)} - y^{(i)}) > 0,
$$

i.e.,

$$
(x^{(i)} - x^{obj})^T (y^{(i)} - x^{(i)}) < 0.
$$

(4.9)

By equation (4.9) and the proof of Theorem 3.1, the conclusion of Theorem 4.1 is immediately obtained. The proof is completed.
4.2. Some Discussions on the Subset of Links Charged and the Pricing Scheme

In our second-best dynamic road pricing scheme, the subset $\bar{A}^{(t+1)}$ in equations (3.19)-(3.20), which is used to levy road pricing, varies with day-to-day traffic dynamic when $x^{(t)} \in \Omega_x / \mathbf{H}(x_0^{obj}, \delta)$. The variability of $\bar{A}^{(t+1)}$ is used to guarantee that the objective equilibrium $x_0^{obj}$ is a feasible adjustment of $x^{(t)} \in \Omega_x / \mathbf{H}(x_0^{obj}, \delta)$, i.e.,

$$ (c(x^{(t)}) + r^{(t+1)})(x_0^{obj} - x^{(t)}) < 0. \quad (4.10) $$

It should be noted that set $\bar{A}^{(t+1)}$ may be not identical with the set $\bar{A}_+^{(t+1)}$ in equation $(3.20)$. It may consist of some links of $\bar{A}_+^{(t+1)}$ if there is a road pricing scheme $\tau^{(t+1)}$ levied on its links satisfying equation $(4.10)$. Therefore, we can consider to minimize the number of links charged on day $t + 1$ by solving the following problem:

$$ \begin{align*}
\min_{(\tau,a)} & \sum_{a \in \bar{A}^{(t+1)}} r_a \\
\text{s.t.} & \quad (c(x^{(t)}) + \tau)(x_0^{obj} - x^{(t)}) < 0, \\
& \quad 0 \leq \tau_a \leq \bar{\tau}_a, a \in \bar{A}^{(t+1)}, \\
& \quad \tau_a = 0, \text{ otherwise,} \\
& \quad r_a = 1, \text{ otherwise,} \\
\end{align*} \quad (4.11a) $$

where $\bar{\tau}_a$ is a given boundary of road pricing charged on link $a \in \bar{A}^{(t+1)}$.

For a given subset $\bar{A}^{(t+1)}$ links charged, the pricing $\tau^{(t+1)}$ satisfies condition $(4.10)$ is not unique. So, after the decision on $\bar{A}^{(t+1)}$, we can consider to minimize the total road pricing charged on day $t + 1$. This problem can be formulated as the following minimum problem:

$$ \begin{align*}
\min & \sum_{a \in A} \tau_a \\
\text{s.t.} & \quad \tau_a = 0, \text{ if } a \in A / \bar{A}^{(t+1)}, \\
& \quad 0 \leq \tau_a \leq \bar{\tau}_a, \text{ otherwise;} \\
& \quad (c(x^{(t)}) + \tau)(x_0^{obj} - x^{(t)}) < 0. \\
\end{align*} \quad (4.12a) $$

5. Numerical Test

In this section, some numerical tests are presented to verify our second-best dynamic road pricing scheme. The simple test example only has an OD pair and three links. The link travel cost functions are shown as follows:

$$ c_1 = f_1 + f_2 + 4, \quad c_2 = 2f_1 + f_2 + 1, \quad c_3 = f_3 + 6. \quad (5.1) $$

The traffic demand $d$ is assumed to be 2 units. Under a pricing scheme $(0, 2, 0)$ levied on link 2, the traffic system has three traffic equilibrium states:

$$ f^I = (2, 0, 0), \quad f^{II} = (0, 2, 0), \quad f^{III} = (1, 1, 0). \quad (5.2) $$

The link-based day-to-day traffic dynamic model of He et al. (2010) is applied in this numerical tests, and its specific formulation has been presented as follows:

$$ x^{(t+1)} = x^{(t)} + \alpha^{(t)}(y^{(t)} - x^{(t)}), \quad (5.3) $$
where

$$\mathbf{y}^{(t)} = \arg \min_{\mathbf{y} \in \Omega_{\mathbf{x}}} \lambda (\mathbf{c}(\mathbf{x}^{(t)}) + \tau^{(t+1)})^T \mathbf{y} + (1 - \lambda)\|\mathbf{x}^{(t)} - \mathbf{y}^{(t)}\|^2.$$  \hfill (5.4)

In the following numerical tests, $\delta = 0.1$, the value of $\lambda$ is set as 0.6, and $\alpha^{(t)} = \frac{1}{(t+1)^{0.51}}$ to satisfy Assumption 2.

As the test example has only three links, we can set the set $\mathbf{A}^{(t+1)}$ consisting of only one link. The charging pricing $\tau^{(t+1)}$ can be obtained by the following simple method and satisfies equation (4.10):

step 1 Let $\tau^{(t+1)} = 0$;
step 2 Calculate the value of $(\mathbf{c}(\mathbf{x}^{(t+1)}) + \tau^{(t+1)})^T (\mathbf{x}_{\text{obj}} - \mathbf{x}^{(t)})$;
step 3 If $(\mathbf{c}(\mathbf{x}^{(t+1)}) + \tau^{(t+1)})^T (\mathbf{x}_{\text{obj}} - \mathbf{x}^{(t)}) < 0$, then stop; otherwise, let $\tau^{(t+1)} = \tau^{(t+1)} + 0.1$. Go to step 2.

Figure 5 shows the trajectories of traffic states with different feasible initial states under the dynamic pricing scheme when the objective equilibrium state is $(2, 0, 0)$. To show the dynamics of toll and set $\mathbf{A}^{(t+1)}$, Figure 6 shows the toll charged on every day with the initial flow $(0, 0, 2)$.

![Fig. 5. Flow trajectories with objective equilibrium flow (2, 0, 0)](image1)

![Fig. 6. The dynamics of toll with initial flow (0, 0, 2) and objective equilibrium (2, 0, 0)](image2)
The trajectories of flows for the objective equilibrium (0, 2, 0) are shown in Figure 7. The dynamics of toll charged with initial flow (0, 0, 2) is shown in Figure 8.

From Figure 5 and 7, one can find that, under the second-best dynamic road pricing, the example traffic network can converge to the objective equilibrium from any feasible initial traffic state. From Figure 6 and 8, one can observe that after the first few days without pricing, the pricing \( \tau > 0 \) is charged on link 2.

6. Conclusion

This study develops a second-best dynamic road pricing scheme implemented on a dynamic subset of links, so that the traffic evolution based on day-to-day adjustment process can be directed to converge to a desired second-best objective traffic equilibrium from any initial traffic state. This discrete-time dynamic pricing is essential for traffic management when multiple traffic equilibria exist, as not all initial traffic states can be directed towards the desired traffic equilibrium if they do not fall into certain attraction domain. Before the second-best dynamic pricing scheme is presented, we prove that existing discrete-time day-to-day dynamic pricing scheme can only drive the traffic system with unique traffic equilibrium to converge to the equilibrium. In addition, our second-best dynamic road pricing scheme can be applied to very general day-to-day traffic dynamic model. For the real application, we discuss how to
develop second-best dynamic road pricing scheme based on a specific day-to-day traffic dynamic model developed by He et al. (2010).

As discussed in Section 4, there are still some challenging questions needed to be studied. For example, the pricing \( \tau^{(t+1)} \) is not only determined by the traffic state \( x^{(t)} \) on day \( t \), but also the links that toll charge is levied. In Section 4, we discuss on how to determine the road pricing by minimizing the number of link charged, as well as by minimizing the total charging toll levied on day \( t+1 \) for a given set of links charged. Indeed, it is more proper to determine the set of links charged and the road pricing simultaneously. This will be further studied in our future works. The idea of the second-best dynamic pricing scheme can also be applied to develop other second-best traffic management measures. For example, we want to develop the dynamic speed limit on some links, or the dynamic reversible lanes on some links to manage traffic flow to achieve a desired traffic equilibrium.

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