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Integrity of Systems under Decentralized Integral Control

Vinay Kariwala 2 J. Fraser Forbes 3 and Edward S. Meadows

Department of Chemical & Materials Engineering, University of Alberta, Edmonton, Canada T6G 2G6

Abstract

A decentralized controller that maintains closed loop stability, when the individual controllers fail or are taken out of service, provides fault tolerance and flexibility in operation. Recently, necessary and sufficient conditions [12] for the existence of a block decentralized controller with integral action for a system to possess integrity against controller failures were proposed. In this paper, these conditions are equivalently expressed using the well-known notions of Niederlinski index (NI) and block relative gain (BRG). The alternate representation implies that under minor assumptions, the available necessary conditions based on NI and BRG are actually both necessary and sufficient. We also show that confirming the existence of a block decentralized controller with integral action such that the system has integrity is NP-hard.

Key words: Block relative gain, Computational complexity, Decentralized control, Niederlinski index, NP-hard, Reliable control.

1 Introduction

This paper deals with reliable stabilization of linear rational stable systems using a block decentralized controller with integral action. A system is said to possess integrity, if there exists a block diagonal controller with integral action in every output channel such that the closed loop stability is maintained when any combination of the individual controllers fails [3]. It is assumed that a controller that fails is immediately taken out of service, i.e. the corresponding entries in the block diagonal controller matrix are replaced by zero. Some researchers have considered the problem of checking whether the closed loop system is reliably stable for a given controller; see [1] for a review. The focus of this work is on deriving controller-independent conditions which can establish the existence or non-existence of a controller such that the system possesses integrity.

Because of its practical implications, the integrity problem has been studied widely by researchers, particularly in the area of process control. For fully decentralized control, a well-known result that relates reliable stability with relative gain array (RGA) [2] is provided by Grosdidier et. al. [11]. It is shown that a system has integrity only if all the corresponding relative gains of the steady state gain matrix are positive. Similar to fully decentralized control, a system with specified block pairings has integrity only if the determinant of all the corresponding block relative gains (BRG) [15] of the steady state gain matrix are positive [10]. Grosdidier and Morari [9] generalized the concept of Niederlinski index (NI) to block pairings to derive similar necessary conditions. Chiu and Arkun [4] have further suggested that the necessary conditions based on BRG and NI be evaluated for all principal block sub-matrices of the system. These necessary conditions based on BRG and NI are useful for eliminating alternatives for input-output pairings. It is not apparent whether the system with the pairings chosen based on these necessary conditions, will have integrity.

Recently, Gündes and Kabuli [12] presented necessary and sufficient conditions for integrity of the system partitioned into 4 or fewer blocks. In this note, we show that these conditions can be equivalently expressed using NI and, when the individual blocks are square, also using BRG. In general, satisfying the conditions of [12] does not guarantee that the decentralized controller will have no unstable poles other than at the origin of the complex plane, as is assumed in the derivation of available necessary conditions based on NI and BRG in [9,10]. When the controller is allowed to have any number
of unstable poles, the alternative representation implies that the conditions based on BRG and NI, traditionally believed to be only necessary, are actually both necessary and sufficient. The expressions presented by Gündes and Kabuli [12] become increasingly complex with the number of blocks. Then, an additional advantage of the alternative representation in terms of NI and BRG is that the extension to the general case, where the system is partitioned into any number of blocks, is simple.

For fully decentralized control, we also show that the necessary and sufficient conditions due to Gündes and Kabuli [12] are satisfied if and only if (iff) NI calculated based on steady state gain matrix is a P-matrix [16]. This observation suggests that verifying the existence of a block diagonal controller with integral action such that the system has integrity is NP-hard unless P = NP [8].

2 Necessary and sufficient conditions

In this note, we denote the linear rational stable system as G(s) and the gain matrix as G. We consider that G(s) is partitioned into M non-overlapping subsystems such that, \( G_{ij} \in \mathbb{R}^{m_i \times m_j}, m_i \leq m_j, i = 1, 2, \ldots, M. \) The matrix containing diagonal blocks of G is represented as \( \tilde{G} \). The block diagonal controller \( K(s) \) with integral action is expressed as \( K(s) = (1/s) \cdot \tilde{C}(s) \), where \( \tilde{C}(s) = \text{diag}(C_{ii}(s)) \) and \( C_{ii}(s) \) is a \( m_i \times m_i \) dimensional transfer matrix (see Figure 1). Here, we allow \( C(s) \) to be improper, provided \( K(s) \) is proper. When \( \text{rank}(G_{ii}) = m_i \), the right inverse of \( G_{ii} \) is denoted as \( G_{ii}^\dagger \). Note that the existence of \( G_{ii}^\dagger \) is necessary for the \( i^{th} \) loop to have integral action. We call a square real matrix \( A \) a positive definite matrix (denoted as \( A > 0 \)) if all the eigenvalues of its symmetric part \( (A + A^T) \) are positive [13].

Next, we define integrity formally and present the necessary and sufficient conditions of Gündes and Kabuli [12] for integrity of \( G(s) \).

**Definition 1** The system \( G(s) \) is said to have integrity, if there exists a block diagonal controller \( K(s) = EK(s) \) with integral action, which stabilizes \( G(s) \) for all \( E \in \mathcal{E} \), where

\[
\mathcal{E} = \{ E = \text{diag}(\epsilon_i \cdot I_{m_i}) \mid \epsilon_i = \{ 0, 1 \}, i = 1, \ldots, M \} \tag{1}
\]

In the literature, a system possessing integrity has also been referred as reliably stable with integral action; see e.g. [12]. The existence of a block diagonal controller such that \( G(s) \) has integrity depends on the chosen input-output pairings. In the remaining discussion, we assume that \( G(s) \) has been permuted such that the subsystems corresponding to the chosen pairings lie along the diagonal blocks of \( G(s) \).

To present the conditions of Gündes and Kabuli [12], we need the following additional notation. For \( j = 2, \ldots, M, i = 1, \ldots, j - 1, \) define

\[
X_{ij} = G_{jj} - G_{jG} G_{ii}^\dagger G_{ij} \tag{2}
\]

When \( M \geq 3, \) for \( k = 1, \ldots, M - 2 \) and \( \ell, m = k + 1, \ldots, M, \ell \neq m, \)

\[
Y_{\ell m} = G_{\ell m} - G_{\ell k} G_{kk}^\dagger G_{km} \tag{3}
\]

and for \( v = 3, \ldots, M, q = 1, \ldots, v - 2 \) and \( r = q + 1, \ldots, v - 1, \)

\[
Z_{rq}^v = X_{qv} Y_{rr} G_{rr}^\dagger (X_{qv} G_{rr}^\dagger)^{-1} Y_{rv} \tag{4}
\]

When \( M = 4 \), define

\[
W = Z_{42}^1 - (Y_{41}^1 - Y_{42}^1 G_{22}^1 (X_{12}^1 G_{22}^1)^{-1} Y_{43}^1) G_{33}^1.
(Z_{23}^1 G_{33}^1)^{-1} (Y_{44}^1 - Y_{42}^1 G_{22}^1 (X_{12}^1 G_{22}^1)^{-1} Y_{44}^1) \tag{5}
\]

**Theorem 2** [12] Let \( \text{rank}(G_{ii}) = m_i \) for all \( i = 1, \ldots, M. \) There exists a block diagonal controller with
integral action such that $G(s)$ has integrity, if

$$\det(X_{ij} G_{jj}^\dagger) > 0$$  \hspace{1cm} (6)
$$\det(Z_{eq} G_{v^e}^\dagger) > 0$$  \hspace{1cm} (7)
$$\det(WG_{44}^\dagger) > 0$$  \hspace{1cm} (8)

where $j = 2, \ldots, M, i = 1, \ldots, j - 1$ and $v = 3, \ldots, M, q = 1, \ldots, v - 2, r = q + 1, \ldots, v - 1$. Further, if any $M - 1$ controllers are strictly proper, or when $G_{ij}$ or $G_{ji}, j = 2, \ldots, M, i = 1, \ldots, j - 1$ are strictly proper or when any of these transfer matrices have real blocking zeros, (6)-(8) are also necessary.

The proof of Theorem 2 can be found in [12]. Some remarks that are relevant to the rest of the discussion in this note are in order.

• Even though the off-diagonal blocks of $G(s)$ may not be strictly proper or may not have real blocking zeros, the controllers can always be designed to be strictly proper. When all controllers are strictly proper, (6)-(8) are both necessary and sufficient for existence of a block diagonal controller with integral action such that $G(s)$ has integrity. We note that a similar assumption is made during the proof of the necessary conditions based on BRG in [10].

• When (6)-(8) are satisfied, existence of a controller with integral action is guaranteed such the system has integrity. This controller, however, may have additional unstable poles other than at the origin of the complex plane. The existence of pure integral action controllers is guaranteed by the following proposition.

• When the individual blocks are multi-input single-output (MISO), $X_{ij} G_{jj}^\dagger > 0$, $Z_{eq} G_{v^e}^\dagger > 0$ and $WG_{44}^\dagger > 0$.

Gündes and Kabuli [12] also presented a method for controller design such that $G(s)$ has integrity, when $X_{ij} G_{jj}^\dagger$, $Z_{eq} G_{v^e}^\dagger$ and $WG_{44}^\dagger$ are positive definite.

### 3 Simplified representation

In this section, we show that the conditions in Theorem 2 can be equivalently represented in terms of BRG and NI. For this purpose, we require evaluation of BRG and NI on the principal block sub-matrices of $G$. We define $\psi$ as the ordered subset of the first $M$ positive integers, consisting of at least 2 elements, and $\Psi$ as the ensemble of all such sets $\psi$. For example, when $M = 2$, $\Psi = \{(1, 2)\}$ and when $M = 3$, $\Psi = \{(1, 2), (1, 3), (2, 3), (1, 2, 3)\}$. With this representation, $G_{\psi \psi}$ represents a principal submatrix of $G$ made up of blocks of $G$ indexed by $\psi$, for any $\psi \in \Psi$. Similarly, $[GH]_{\psi \psi}$ represents a principal submatrix of the product of the matrices $G$ and $H$ indexed by $\psi$. Note that when $\psi = (1, 2, \ldots, M)$, $G_{\psi \psi}$ represents the matrix $G$.

**Definition 3** Let $\tilde{G} = diag(G_{ii})$, where $G_{ii} \in \mathbb{R}^{m_i \times m_j}$, $m_i \leq m_j$ and rank($G_{ii}$) = $m_i$ for all $i = 1, \ldots, M$. The Niedrelinks index (NI) of $G$ is defined as

$$NI(G) = \det(\tilde{GG}^\dagger)$$  \hspace{1cm} (9)

Definition 3 is a generalization of NI defined by Grosdidier and Morari [9] for systems partitioned into square blocks. The next proposition relates NI with the existence of a controller such that the system has integrity.

**Proposition 4** Let rank($G_{ii}$) = $m_i$ for all $i = 1, \ldots, M$. Then the conclusions of Theorem 2 hold iff,

$$NI(G) > 0 \hspace{1cm} \forall \psi \in \Psi$$  \hspace{1cm} (10)

**PROOF.** By repeated use of Schur complement lemma (see e.g. [13]), it can be shown that

$$\boxed{\det(X_{ij} G_{jj}^\dagger) = \det([GG^\dagger]_{\{i,j\},\{i,j\}})}$$  \hspace{1cm} (11)

$$\boxed{\det(Z_{eq} G_{v^e}^\dagger) = \frac{\det([GG^\dagger]_{\{q,r,v\},\{q,r,v\}})}{\det(X_{qr} G_{rr}^\dagger)}}$$  \hspace{1cm} (12)

$$\boxed{\det(WG_{44}^\dagger) = \frac{\det(\tilde{GG}^\dagger)}{\det(Z_{eq} G_{v^e}^\dagger) \det(X_{ij} G_{jj}^\dagger)}}$$  \hspace{1cm} (13)

where $j = 2, \ldots, M, i = 1, \ldots, j - 1$ and $v = 3, \ldots, M, q = 1, \ldots, v - 2, r = q + 1, \ldots, v - 1$.

Using (11), $\det(X_{ij} G_{jj}^\dagger) > 0$ iff

$$\det([GG^\dagger]_{\{i,j\},\{i,j\}}) > 0$$  \hspace{1cm} (14)

for all $j = 2, \ldots, M, i = 1, \ldots, j - 1$. Next, assume that (14) holds. When $M \geq 3$, the ordered set $\{r, q\}$ is a subset of \{i, j\}. Then, $\det(X_{qr} G_{rr}^\dagger) > 0$ and using (12), $\det(Z_{eq} G_{v^e}^\dagger) > 0$, iff

$$\det([GG^\dagger]_{\{q,r,v\},\{q,r,v\}}) > 0$$  \hspace{1cm} (15)

for all $v = 3, \ldots, M, q = 1, \ldots, v - 2, r = q + 1, \ldots, v - 1$. Similarly, when $M = 4$, $\det(WG_{44}^\dagger) > 0$, iff

$$\det(\tilde{GG}^\dagger) > 0$$  \hspace{1cm} (16)

Now, the necessity and sufficiency of (10) follows by combining (14)-(16) and noting that $\psi = \{i, j\} \cup \{q, r, v\}$. □
As $M$ increases, the expressions presented by Güncel and Kabuli [12] become increasingly complex (cf. (6)-(8)). On the other hand, the extension to the general case is simple (by induction), when the conditions are compactly expressed in terms of NI.

When all the blocks of the partitioned system are square, the conditions of Theorem 2 can also be equivalently expressed in terms of BRG, as shown below.

**Definition 5** For $G(s)$ partitioned into square blocks, let $G_{ii} \in \mathbb{R}^{m_i \times m_i}$ be non-singular for all $i = 1, \ldots, M$ and let $j = \{1, \ldots, M\}/i$. The BRG of $G_{ii}$ in $G$ is defined as [15]

$$[A_B(G)]_{ii} = G_{ii} \left(G_{ii} - G_{jj}G_{jj}^{-1}G_{ji}\right)^{-1}$$

(17)

**Proposition 6** Let $G(s)$ be partitioned into square blocks such that $G_{ii} \in \mathbb{R}^{m_i \times m_i}$ is non-singular for all $i = 1, \ldots, M$. Then the conclusions of Theorem 2 or (10) hold iff,

$$\det([A_B(G_{ii})]_{kk}) > 0 \quad \forall \psi \in \Psi, k = 1, \ldots, |\psi|$$

(18)

where $|\psi|$ denotes the cardinality of the set $\psi$.

**PROOF.** For equivalence, we show that (18)$\Leftrightarrow$(10), which in turn implies that the conclusions of Theorem 2 hold iff (18) holds. Since $NI(G_{ii,j},(i,j)) = \det([A_B(G_{ii,j})]_{ii}), NI(G_{i,j},(i,j)) > 0$, iff

$$\det([A_B(G_{ii,j})]_{ii}) > 0$$

(19)

for all $i, j \leq M, i \neq j$. Next, assume that (19) holds. When, $M \geq 3$ [4],

$$NI(G_{i,j,k},(i,j,k)) = \frac{NI(G_{i,j,k},(i,j,k))}{\det([A_B(G_{i,j,k})]_{kk})}$$

(20)

for all $i, j, k \leq M, i \neq j \neq k$. Since $NI(G_{i,j,k},(i,j,k)) > 0$ for all $i, j \leq M, i \neq j$, $NI(G_{i,j,k},(i,j,k)) > 0$, iff

$$\det([A_B(G_{i,j,k})]_{ii}) > 0$$

(21)

for all $i, j, k \leq M, i \neq j \neq k$. When, $M = 4$, using (13) and similar arguments as above, $NI(G) > 0$ iff

$$\det([A_B(G)]_{ii}) > 0$$

(22)

for all $i, j, k \leq M, i \neq j \neq k, \ell = \{1, \ldots, M\}/(i,j,k)$. If all the terms on the right hand side of (22) are positive,

$$\det([A_B(G_{ii,j})]_{ii})$$

is always positive. The task of finding the set of $2^M - (M + 1)$ non-redundant BRGs requires some book-keeping. In this sense, the use of (10) is advantageous over the use of (18). The usefulness of the results presented in this section is demonstrated next by a numerical example.

**Example 7** Consider the following system adapted from Hovd and Skogestad [14],

$$G(s) = \frac{1}{(1 + 5s)^2} \begin{bmatrix} 1 & -4.19 & -25.96 \\ 6.19 & 1 & -25.96 \\ 1 & 1 & 1 \end{bmatrix}$$

(23)

The objective is to ascertain the integrity of system with pairings selected on the diagonal elements. The $NI(G)$ and $NI(G_{ii})$ are 26.9 for $\psi = (1,2), (1,3)$ and (2,3). Then, Proposition 4 guarantees that the system has integrity. This result is also confirmed using
Proposition 6, where \([\Lambda_{P}(G)]_{ii} = 1\) for \(i = 1, 2, 3\) and 
\([\Lambda_{P}(G_{\psi})]_{jj} = 0.037\) for \(\psi = (1, 2), (1, 3)\) and 
\((2, 3)\) for all \(j = 1, 2, 3\).

For fully decentralized control, satisfying (10) or (18) 
guarantees the existence of a pure integral action 
controller such that \(G(s)\) has integrity. We design a 
controller of the form \(\text{diag}(k_i/s)\) using the algorithm of 
Gündes and Kabuli [12], where \(k_i = 0.01, 0.002\) and 
\(10^{-5}\) for \(i = 1, 2, 3\), respectively. This controller maintains 
closed loop stability, when any combination of loops fail. 
Alternatively, we find using trial and error that the same 
objective is achieved by the controller \(k \cdot I, k = 0.001\).

4 Computational complexity

In this section, we present some results on computational 
complexity for establishing the existence of a block 
diagonal controller such that \(G(s)\) has integrity. It is 
shown that this problem is NP-hard, unless \(P=NP\) [8]. 
We introduce the useful notion of \(P\)-matrices, which 
form the basis of the proof for NP-hardness.

Definition 8 A matrix \(A \in \mathbb{R}^{n \times n}\) is called a \(P\)-matrix, 
if all the principal minors of \(A\) are positive [16].

In the subsequent discussion, we refer to the problem of 
establishing the existence of a block diagonal controller 
such that \(G(s)\) has integrity, simply as the integrity 
problem. Note that the integrity problem involves search 
over all possible partitions of \(G(s)\) and the controller 
structure is not specified \textit{a priori}.

Proposition 9 Let \(\text{rank}(G_{\psi}) = m\) for all \(i = 1, \ldots, M\) 
and \(G = \text{diag}(G_{\psi})\). If the controller \(K(s)\) is restricted 
to be strictly proper, the integrity problem is NP-hard, 
unless \(P=NP\).

\textbf{Proof.} For the NP-hardness of the integrity problem, it suffices to show that the integrity problem 
is NP-hard, when the individual blocks of controller are 
single input multi output (SIMO). Let the class of \(n \times n\) 
real matrices be classified as,

- Matrices with at least one negative or zero diagonal elements, \(A_1\)
- Matrices with all positive diagonal elements, \(A_2\)

It readily follows that for any \(A \in A_1\), the \(P\)-matrix 
problem can be solved in polynomial time through \(n\) 
evaluations. Coxson [5] has shown that verifying whether 
a given matrix is \(P\)-matrix is co-NP-complete. Then, the 
\(P\)-matrix problem must be co-NP-complete for the set 
\(A_2\), otherwise the results of Coxson [5] are contradicted. 
For any \(A\), since
\[
\det(A) = \det(A)\det(A^{-1}) \quad (23)
\]

any \(A \in A_2\) is a \(P\)-matrix, iff \(A A^{-1}\) is a \(P\)-matrix.

By reversing the proof of Proposition 4, it follows that 
\(A A^{-1}\) is a \(P\)-matrix, iff for every transfer matrix \(G(s)\) 
satisfying \(A A^{-1} = G G^T\), there exists a decentralized 
controller having SIMO blocks and integral action such 
that \(G(s)\) has integrity. Clearly, the transformations 
\(A A^{-1}\) and \(G G^T\) require finite number of operations 
and can be completed in polynomial time. Thus, the integrity 
problem is at least as hard as the \(P\)-matrix problem and 
is NP-hard, unless \(P=NP\). \(\square\)

Based on Proposition 9, it is possible to establish 
the computational complexity of some more general or 
special cases of integrity problem, as discussed below:

1. When the system is partitioned into MISO blocks, 
satisfying (6)-(8) guarantees the existence of a 

pure integral action controller such that \(G(s)\) has integrity. 
In this case, when the controllers are 

further restricted to have poles at origin only, the 

integrity problem is also NP-hard.

2. When the controller is block decentralized, one 

only needs to check the positiveness of the minors 
of the sub-matrices of \(G G^T\) that can be formed 

by combining elements of different blocks and the 

corresponding off-block diagonal elements. In this 

case, if \(\det((G G^T)_{\psi}) > 0\) for all \(\psi \in \Psi\), we call 

\(G G^T\) a block \(P\)-matrix in the spirit of \(P\)-matrices. 

The worst-case time complexity of an algorithm 

for the block \(P\)-matrix problem is approximately 

\(O(n^3 2^M)\). Then, for the special case, where the 

controller structure is specified \textit{a priori} with \(M\) 

being independent of the system dimensions, the 

integrity problem lies in class \(P\).

Though the integrity problem is NP-hard it may still 
be possible to solve the integrity problem in polynomial 
time for particular instances of the problem. The time 
complexity of an algorithm evaluating all the principal 
mins of the given real matrix is approximately 
\(O(n^3 2^n)\). Tsatsomeros and Li [17] presented a recursive 
algorithm that reduces the time complexity to \(O(2^n)\). 
This algorithm is based on Schur complement lemma 
and is easily extended for verifying block \(P\)-matrices.

Recently, Rump [16] presented an algorithm, whose time 
complexity is not necessarily exponential, but can be 
exponential in the worst case. Rump [16] has applied 
this algorithm to a test set of parameterized matrices, 
whose membership in the class of \(P\)-matrices is known 
beforehand for the given value of the parameter. It is 
shown that the algorithm can successfully verify whether 
these matrices having dimensions up to 100 \times 100 are 
\(P\)-matrices in polynomial time. Future work will focus 
on generalizing Rump’s algorithm [16] for verification of
The principal sub-matrices of a positive-definite matrix. We next present a sufficient condition for verifying whether $GG^\dagger$ is a $\mathcal{P}$- or block $\mathcal{P}$-matrix.

**Proposition 10** Let $\text{rank}(G_{ii}) = m_i$ for all $i = 1, \ldots, M$ and $G = \text{diag}(G_{ii})$. Define $E = (G - G)G^\dagger$. Then, $GG^\dagger$ is block $\mathcal{P}$-matrix with respect to the structure of $G$, if $\text{det}(I + 0.5E) \neq 0$ and

$$\mu_\Delta((I + 0.5E)^{-1}E) < 2$$

where $\mu$ is the structured singular value [6] and

$$\Delta = \{\text{diag}(\delta_i \cdot I_{m_i}), \delta_i \in \mathbb{C}, |\delta_i| \leq 1, i = 1, \ldots, M\}$$

PROOF. Define, $\Delta_1 = \{\text{diag}(\epsilon_i \cdot I_{m_i}), \epsilon_i = \{0, 1\}, i = 1, \ldots, M\}$. Then, $GG^\dagger = I + E$, is a block $\mathcal{P}$-matrix iff,

$$\text{det}(I + E\Delta_1) > 0 \quad \forall \Delta_1 \in \Delta_1$$

(26)

Further, defining $\Delta_2 = \{\text{diag}(\epsilon_i \cdot I_{m_i}), \epsilon_i \in \mathbb{C}, |\epsilon_i| \leq 1, i = 1, \ldots, M\}$ and noting that $\Delta_1 \subset \Delta_2$, (26) holds if $\text{det}(I + E\Delta_2) > 0$ for all $\Delta_2 \in \Delta_2$. Since the determinant is a continuous function over convex sets, if $\text{det}(I + E\Delta_2)$ changes sign over the set $\Delta_2$, there exists some $\Delta_2 \in \Delta_2$ such that $\text{det}(I + E\Delta_2) = 0$. Since, $\Delta_1 \subset \Delta_2$, (26) holds if,

$$\text{det}(I + E\Delta_2) \neq 0 \quad \forall \Delta_2 \in \Delta_2 \quad \Leftrightarrow \quad \mu_{\Delta_2}(E) < 1$$

(27)

The condition (27) is conservative as $I, -I \in \Delta_2$. To reduce conservatism [1], for every $\Delta \in \Delta$, $\Delta_2 \in \Delta_2$, define $\Delta_2 = 0.5(I + \Delta)$. Then,

$$\text{det}(I + E\Delta_2) = \text{det}(I + 0.5E + 0.5E\Delta)$$

$$= \text{det}(I + 0.5E)\text{det}(I + 0.5(I + 0.5E)^{-1}E\Delta)$$

When (24) holds, $\text{det}(I + 0.5(I + 0.5E)^{-1}E\Delta)$ does not change sign over the set $\Delta$ and $GG^\dagger$ is block $\mathcal{P}$-matrix with respect to the structure of $G$. □

A practical approach is to check if the upper bound on $\mu_\Delta((I + 0.5E)^{-1}E) < 2$ and if not, use the algorithms of Tsatsomeros and Li [17] for block decentralized control or Rump [16] for fully decentralized control.

**Example 11** To show the advantage of Proposition 10 over the sufficient condition $GG^\dagger > 0$, we consider $G(s)$ with

$$G = \begin{bmatrix} 1 - \frac{\alpha}{\gamma} \\ \alpha & 1 \end{bmatrix}; \quad \alpha, \gamma > 0$$

For fully decentralized control with pairing on diagonal elements, $\text{NI}(G) = 1 + \gamma$ and the system has integrity for all allowable values of $\alpha$ and $\gamma$. For $2 \times 2$ systems, we note that the upper bound on the structured singular value obtained using the $D$-scaling method is exact [6]. Using this method and some lengthy but straightforward algebraic manipulations, it can be shown that

$$\mu_\Delta((I + 0.5E)^{-1}E) = 2\sqrt{\frac{\gamma}{4 + \gamma}}$$

which satisfies (24) for all allowable finite values of $\alpha$ and $\gamma$. This example demonstrates that Proposition 10 is not always conservative. On the other hand, the eigenvalues of $(GG^\dagger + (GG^\dagger)^T)$ are $2 \pm (\alpha - \frac{\gamma}{2})$ and the sufficient condition $GG^\dagger > 0$ is satisfied only when $|(\alpha - \frac{\gamma}{2})| < 2$.

For example, when $\gamma = 1$, $GG^\dagger > 0$ only for $0.416 \leq \alpha \leq 2.416$. Clearly, this is highly conservative, as the integrity of the system is independent of $\alpha$.

5 Conclusions

When the controller is allowed to have unstable poles other than at origin, it is shown that the conditions for the integrity problem based on Niederlinski Index and Block relative gain, generally believed to be necessary, are both necessary and sufficient. It is also shown that solving the integrity problem, i.e. establishing the existence of a block diagonal controller with integral action such that the system has integrity, is NP-hard. This result implies that no computationally easy algorithm exists for solving the integrity problem and the engineer needs to be content with conditions that are easily computable but are either necessary or sufficient, but not both.

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