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Scaling of geometric phase versus band structure in cluster-Ising models

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We study the phase diagram of a class of models in which a generalized cluster interaction can be quenched by an Ising exchange interaction and external magnetic field. The various phases are studied through winding numbers. They may be ordinary phases with local order parameters or exotic ones, known as symmetry protected topologically ordered phases. Quantum phase transitions with dynamical critical exponents $z = 1$ or $z = 2$ are found. In particular, the criticality is analyzed through finite-size scaling of the geometric phase accumulated when the spins of the lattice perform an adiabatic precession. With this study, we quantify the scaling behavior of the geometric phase in relation to the topology and low-energy properties of the band structure of the system.

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I. INTRODUCTION

Gapped ground states define quantum phases of matter, yet they can be of a very different nature. Some of them exhibit approximate orders on a local scale and they can be characterized by their symmetries. Others possess subtler orders that can only be captured by highly nonlocal observables. One of the major challenging themes in modern condensed matter physics, with applications in quantum technology, is to devise a unified understanding of all the possible quantum phases of matter [1–5]. Indeed, the scientific community has been applying integrated methods by combining quantum information, foundational notions of quantum mechanics, and many-body physics to study the problem [6–9]. Recent outcomes in topological matter, e.g., topological insulators, Weyl semimetals [10–12], and superconductors, have demonstrated how the topology of the energy bands of the system can be useful in a novel way for analyzing the quantum phases of matter [12,13]. Here, we study a specific many-body system that can display exotic orders by exploiting the Berry phase and winding numbers [14–16].

We focus on a one-dimensional spin system whose ground state can be tuned to be an ordered state with a local order parameter or to be a state with an exotic order of a topological nature (a so-called symmetry protected topological order [17,18]). The different regimes that may be established in the system are separated by a quantum phase transition driven by certain control parameters [4,19]. There has been growing interest in the criticality between ground states with exotic order [20,21]. Interestingly, it was found that the Berry phase can be invoked to study quantum phase transitions [22–25]. A finite-size scaling analysis of the geometric phase to reveal order-disorder quantum phase transitions was studied in the XY spin chain [23].

In this Rapid Communication, we consider a set of localized spins in a one-dimensional lattice enjoying a specific higher-order (multispin) interaction; at the same time, such an interaction competes with an Ising exchange; finally, the chain is placed in a transverse magnetic field. Indeed, the systems under scrutiny are generalizations of the cluster-Ising model that was formulated in the cross-fertilization area between many-body physics, quantum correlations, and ultracold atoms [26–28]. The cluster-Ising model displays a second-order quantum phase transition between a phase with a local order parameter and a symmetry protected topological quantum phase [29,30]. Here, we study the phase diagrams of the generalized cluster-Ising models by looking at the winding numbers of the ground states (for a specific subclass of models studied here, winding numbers were recently studied in Ref. [31]). We explore the criticality of the system by studying the finite-size scaling of the geometric phase. In particular, we find that critical points with nonlinear low-energy dispersions are characterized by an anomalous logarithmic scaling of the geometric phase.

II. MODEL

We consider a class of models describing interactions between $l+1$ spins competing with exchange interactions, in an external field. The Hamiltonian reads [20,21,26–30]

$$H^{(l)} = \sum_{j=-M}^{M} -\lambda\sigma_j^x \mathcal{Z}_{j,l}\sigma_{j+l}^x + a\sigma_j^y\sigma_{j+1}^y + g\sigma_j^z, \qquad (1)$$

with $\mathcal{Z}_{j,l} = \sigma_{j+1}^z \cdots \sigma_{j+l-1}^z$ and $M = (L-1)/2$ for odd L. The operators σ_n^α ($\alpha = x,y,z$) are the Pauli matrices defining the spin state in the nth site of the one-dimensional lattice. Equation (1) can be mapped to a system of decoupled $l+1$

free fermions, $H^{(l)} = \sum_k \Psi_k^\dagger H_k^{(l)} \Psi_k$, where $\Psi_k^\dagger = (c_k^\dagger, c_{-k})$ and $H_k^{(l)} = \mathbf{d}^{(l)}(k) \cdot \boldsymbol{\sigma}$ lies in a pseudospin Hilbert space, with $\mathbf{d}^{(l)}(k) = h_y^{(l)} \hat{\mathbf{e}}_y + h_z^{(l)} \hat{\mathbf{e}}_z$, $h_y^{(l)} = \lambda \sin kl + a \sin k$, $h_z^{(l)} = -\lambda \cos kl + a \cos k - g$, with $\hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ the unit vectors in the directions y,z.

For $l = 1$, $Z_{j,l} = 1$, and therefore the Hamiltonian defines the transverse Ising model with the well-known antiferromagnet-paramagnet quantum phase transition in the Ising universality class. For $l = 2$, Eq. (1) defines the cluster-Ising model in an external magnetic field. Assuming periodic boundary conditions $\sigma_{L+1}^\alpha = \sigma_1^\alpha$, the ground state of Eq. (1) for $a = g = 0$ is a unique state known as a cluster state [32]. Such a state enjoys a nontrivial global symmetry of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ type. For open boundary conditions, the cluster state is fourfold degenerate. Such a degeneracy can be lifted only by resorting to operators in the Hamiltonians's symmetry algebra. In such a specific sense, the cluster ground state provides an example of a quantum phase of matter with symmetry protected topological order [17]. Remarkably, such a kind of order is preserved by the Ising interaction and the external field in Eq. (1) until quantum phase transitions occur in the system. The cluster-Ising models enjoy nontrivial duality properties [5,21,26,27,29,33,34]. In particular, our Hamiltonians Eq. (1) can be mapped to the class of models considered in Ref. [21] by $\sigma_j^z = \tau_j^y \tau_{j+1}^y$, $\sigma_j^y \sigma_{j+1}^y = -\tau_j^y \tau_{j+1}^z \tau_{j+2}^y$, and $\sigma_{j-1}^x \sigma_j^z \sigma_{j+1}^z \sigma_{j+2}^x = -\tau_j^x \tau_{j+1}^z \tau_{j+2}^x$. Since the construction of the phase diagram of the systems relies on energy properties, the phase diagrams are unaltered by duality.

III. PHASE DIAGRAM

Different ground states can be characterized by order parameters. In the cluster phases, however, such order parameters need to be highly nonlocal (the string order parameters) [21,29]. Winding numbers provide an alternative description of the different phases, bypassing the notion of an order parameter [15]. Indeed, winding numbers have integer values and they cannot change without closing the spectral gap. Remarkably, these numbers correspond to the number of zero modes appearing at the edge of the system (when open boundary conditions are imposed).

The winding number counts the times that a closed curve encircles the origin in the pseudospin Hilbert space $[h_y^{(l)}(k), h_z^{(l)}(k)]$,

$$W = \frac{1}{2\pi} \int_{\text{BZ}} d\theta_k^{(l)}, \tag{2}$$

with $\theta_k^{(l)} = \mathbf{d}^{(l)}(k)/|\mathbf{d}^{(l)}(k)|$. The winding numbers in the different ground states of the system are summarized in Table I.

TABLE I. Phase and winding number for specific part in the cluster-Ising models.

Interaction	Phase	Winding number
$\pm\sum_j \sigma_j^z$	P	0
$\pm\sum_j \sigma_j^y \sigma_{j+1}^y$	AFM$^{(Y)}$,FM$^{(Y)}$	+1
$\pm\sum_j \sigma_j^x \sigma_{j+1}^x$	AFM$^{(X)}$,FM$^{(X)}$	−1
$\pm\sum_j \sigma_j^x \mathcal{Z}_{j,l} \sigma_{j+l}^x$	C_l^*, C_l	$-l$

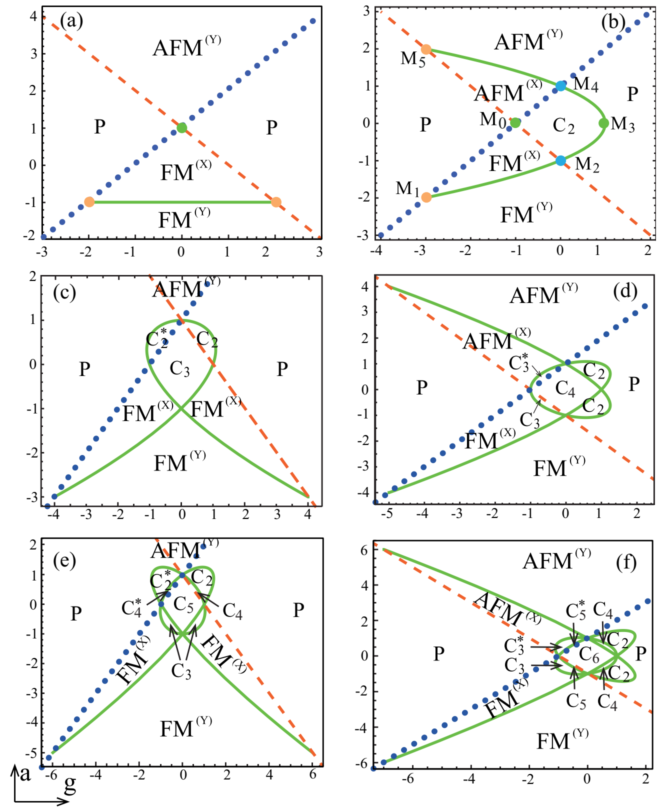

FIG. 1. Phase diagrams of cluster-Ising models. (a)–(f) are phase diagrams of the system for l from 1 to 6. We choose $\lambda = 1$. The horizontal and vertical axes are represented by g and a, respectively. The abbreviations mean different phases: paramagnetic (P), ferromagnetic (FM), antiferromagnetic (AFM), cluster (C); the superscripts X,Y specify the directions of the orders. C_l represent the cluster phase ($l = 2$) or generalized cluster phases ($l > 2$). C_l^* mean dual (generalized) cluster phases (see the text for the definition).

FM$^\alpha$ (AFM$^\alpha$) denotes ferromagnetic (antiferromagnetic) order along the spin direction α.

The cluster order can be quenched by the local field g and the nonlocal exchange interaction a [35,36]. In Figs. 1(a)–1(f) we show the phase diagrams with l from 1 to 6. Figures 1(a) and 1(b) show the detailed phase diagrams for $l = 1$ and 2 (see Refs. [21,31]). The cases $l > 2$ were recently studied by Lahtinen and Ardonne [20].

We observe that the generalized cluster states with winding number $-l$ are "broken" into phases characterized by winding numbers $[-(l-1), \ldots, -1]$. The structure of the phase diagrams is related to the symmetry of $[h_y(k), h_z(k)]$. Such symmetry implies the parity of the number of l. For even l [Figs. 1(b), 1(d), and 1(f)], the Zeeman field is the control parameter. When $g > 0$, phases with even integer winding numbers are generated. The Ising interaction a tunes the ferromagnetic or antiferromagnetic phase. The roles of g and a are exchanged for odd l [Figs. 1(a), 1(c), and 1(e)]. The C_2 and C_2^* (the so-called dual cluster phase) display a string order of the cluster state type with two Majorana modes at each edge of the system; the two such phases are characterized by string order parameters with different spin polarizations

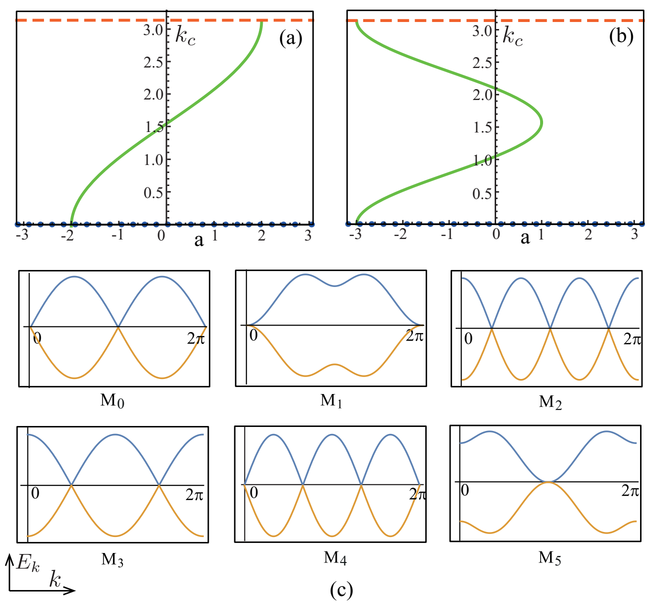

FIG. 2. (a) and (b) are the critical momenta corresponding to $l = 2$ and 3, respectively. (c) Energy band structures of critical points M_0–M_5 for $l = 2$ in Fig. 1(b). M_1 and M_5 are non-Lorentz-invariant critical points, with nonlinear low-energy modes. At M_0 and M_3, band structures share the same topology. Similarly for multicritical points M_2 and M_4.

at the edges [21]. Similarly, C_3 phases are cluster phases with three Majorana modes at each edge of the system. C_3 and C_3^* phases in Fig. 1(d) are distinguished from each other by the negative and positive Ising interaction a. The C_m and C_m^* phases with $m > 2$ in Figs. 1(c)–1(f) are defined with a similar logic. The different phases C_m with fixed l in the different panels of Fig. 1 can be connected adiabatically: A fixed phase C_m of a given Hamiltonian $H^{(l)}$ evolves into C_m of $H^{(l+1)}$ under $H^{(l,l+1)} = (1-t)H^{(l)} + tH^{(l+1)}$, $t \in [0,1]$. We start with $t = 0$ and the ground state of $H^{(l)}$ is also the ground state of $H^{(l,l+1)}$. Then we increase t until $t = 1$. If the system does not go through a degeneracy in the energy band, we know that the ground states of $H^{(l)}$ and $H^{(l+1)}$ are in the same phase.

Phase boundaries are obtained as the combination of the Hamiltonian parameters for which a specific low-energy mode emerges (for a specific value of critical momentum k_c) in the band structure. The critical momenta of the phase boundaries Figs. 1(b) and 1(c) are shown in Figs. 2(a) and 2(b), respectively. The green solid lines in Fig. 1 are in the XY universality class. Along there, the critical momentum depends on the parameters a and g. For the blue-dotted (red-dashed) straight lines indicating Ising phase transitions in Fig. 1, there is one Dirac point at $k_c = 0$ (π). The XY and Ising transitions have a topological difference. The two phases separated by the XY line have a winding number difference equal to 2. However, for ground states separated by the Ising type transition, the winding number difference is 1. Elaborating on the findings for $l = 2$ [29], Lahtinen and Ardonne demonstrated that the multicritical points of the system may be indeed characterized by the so$(N)_1$ conformal field theory. For M_0 with two Ising criticalities, the XY gets to the so$(2)_1$ universality class. If a cluster type of order is involved, more branches ($\geqslant 3$) with linear dispersions show up at the criticality [Fig. 2(c)]. The multicritical points are combined in the so$(l)_1$ and so$(l+1)_1$ universality classes [20]. Specifically, in M_3 there are two degenerate points in the Brillouin zone. As for M_4, there are three degenerate points in the band structure. Therefore, M_4 enjoys a so$(3)_1$ criticality rather than an XY one. It is interesting to note that there is quadratic band touching at M_1 and M_5. Indeed, quadratic band touching may lead to interesting non-Fermi-liquid interaction effects [37,38].

IV. SCALING OF GEOMETRIC PHASE

A Berry phase arises when the spin variables localized in the lattice points along the chain are rotated adiabatically [22]. The rotating system can be described by $H_{\mathcal{R}}^{(l)} = \mathcal{R}^\dagger H^{(l)} \mathcal{R}$, with $\mathcal{R}^\dagger = e^{i\sum_{j=-M}^{M}\phi\sigma_j^z/2}$. For our model Eq. (1), the ground state of $H_{\mathcal{R}}^{(l)}$ is the vacuum of free fermionic modes, $|gs\rangle^{(l)} = \prod_{k=1}^{M} |gs\rangle_k^{(l)}$ with $|gs\rangle_k^{(l)} = (\cos\frac{\theta_k^{(l)}}{2}|0\rangle_k|0\rangle_{-k} - ie^{-i2\phi}\sin\frac{\theta_k^{(l)}}{2}|1\rangle_k|1\rangle_{-k})$, where $|0\rangle_k,|1\rangle_k$ are the vacuum and single excitation of the kth mode, c_k, respectively. Adiabatically varying the angle ϕ from 0 to π, the geometric phase of $|gs\rangle$ results in [22,23]

$$\varphi_{gs}^{(l)} = \frac{i}{M}\int_0^\pi {}^{(l)}\langle gs|\partial_\phi|gs\rangle^{(l)} d\phi, \\ = \frac{\pi}{M}\sum_k \left(1 - \cos\theta_k^{(l)}\right). \quad (3)$$

In Fig. 1(a) the horizontal line (green solid) at $a = -1$, $-2 < g < 2$, defines a XY critical state with quasi-long-range order. In such a state, the Berry phase is identically vanishing. If a nontrivial cluster state order is involved, the universality class of the transition changes: Multicritical points M_2 and M_4 appear as shown in Fig. 1(b). The Berry phase near the criticality (green solid) is nonvanishing. We explore the criticality via $d\varphi/dg$.

(I) Scaling close to the quantum phase transitions with the critical exponent $z = 1$: The scaling *Ansatz* for (the derivative of) the geometric phase is [23]

$$\left.\frac{d\varphi_{gs}^{(l)}}{dg}\right|_{g_m} \simeq \kappa_1 \ln N + \text{const}, \quad (4)$$

$$\frac{d\varphi_{gs}^{(l)}}{dg} \simeq \kappa_2 \ln|g - g_c| + \text{const}, \quad (5)$$

where g_c is the critical value of g for infinite long spin chain, and g_m marks the anomaly for the finite-size system. According to the scaling *Ansatz*, in the case of logarithmic singularities, the ratio $|\kappa_2/\kappa_1|$ is the exponent ν that governs the divergence of correlation length. We note that the scaling behavior is related to the band structure at low energy.

As for topological quantum phase transitions, we first consider $l = 2$. The critical properties are found symmetric about $a = 0$, as shown in Fig. 1(b). We discuss the phase boundaries with $0 \leqslant a \leqslant 2$. For $a = 0$, phase transitions occur at $|g| = 1$ which separate a paramagnetic and a cluster phase. As expected by looking at the dispersion curves, M_0 and M_3 share the same criticality. Similarly, the quantum multicritical

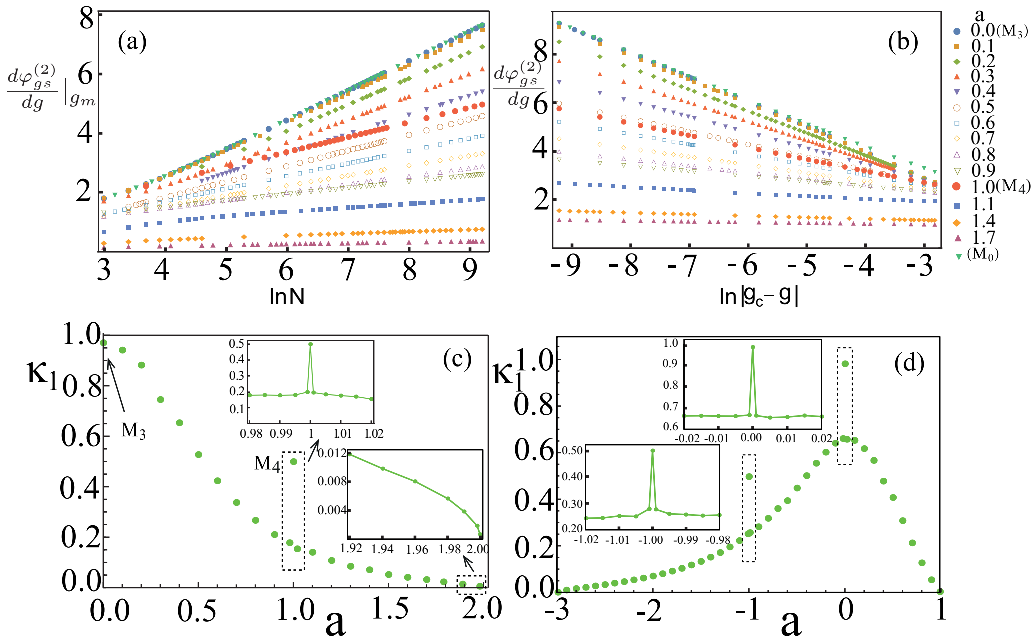

FIG. 3. (a), (b) The scaling behavior close to the XY and so(3)$_1$ (M_4) criticalities in Fig. 1(b) with various value of a [indicated on the right-hand side of (b)]. (c) displays the scaling parameter κ_1 shown in (a) with $a \in [0,2)$. (d) denotes κ_1 of XY (the part with k_c changing from 0 to $\pi/2$), so(3)$_1$, and so(4)$_1$ criticalities for $l = 3$ [Fig. 1(c)] with $a \in (-3,1]$.

points M_2, M_4 involving the cluster phase enjoy the same scaling behavior. In Figs. 3(a) and 3(b) we present the scaling behaviors characterized by Eqs. (4) and (5). For a critical regime with a critical exponent $z = 1$ and linear low-energy dispersions, the ratio $|\kappa_2/\kappa_1| \sim 1$. The scaling parameter κ_1 for $l = 2$ is represented in Fig. 3(c). We find that for multicritical points with multiple degeneracies in the energy bands, the scaling coefficients are discontinuously connected to the neighboring critical points which share the same topologies of the band structures. The discontinuity (sudden change) of the scaling parameter κ_1 renders the topological change of the band structure. The smooth variation of κ_1 in the XY criticality arises from the fact that slopes of linear dispersions change depending on the Ising exchange interaction and transverse magnetic field. In Fig. 3(d), we show κ_1 for $l = 3$ and also observe the sudden change at the multicritical points. Similar behaviors also exist for κ_2.

(II) Scaling close to quantum phase transitions with $z = 2$: Quantum phase transitions implied in Eq. (1) may be characterized by a low-energy dispersions $\sim k^2$ (e.g., M_1 and M_5 for $l = 2$). Consistently with the scaling theory, the dynamical critical index for such a phase transition is $z = 2$.

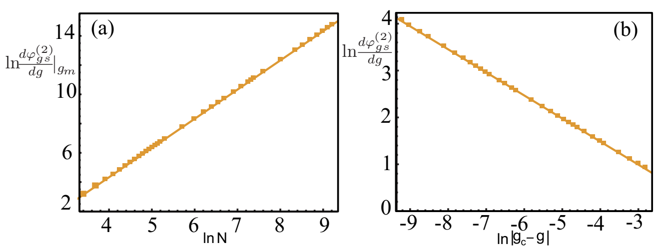

FIG. 4. (a) and (b) display scaling behavior at the point M_5.

The scaling behaviors of this band topology at a critical regime are shown in Fig. 4. For M_1 and M_5, we found that the scaling *Ansätze* Eqs. (4) and (5) should be modified to

$$\ln \left.\frac{d\varphi_{gs}^{(l)}}{dg}\right|_{g_m} \simeq \tilde{\kappa}_1 \ln N + \text{const} \tag{6}$$

and

$$\ln \frac{d\varphi_{gs}^{(l)}}{dg} \simeq \tilde{\kappa}_2 \ln |g - g_c| + \text{const.} \tag{7}$$

$\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ in Figs. 4(a) and 4(b) are found to be 1.999 and -0.492, respectively. Close to the critical points with quadratic dispersions for $l > 2$, we find a similar log scaling behavior.

V. CONCLUSION AND OUTLOOK

We have studied the criticality of generalized cluster-Ising models through the scaling properties of the geometric phase. The criticality with a parameter-dependent critical momentum is generically found for the XY type. At the multicritical points with linear gapless modes the quantum phase transitions are in the so$(N)_1$ universality classes. We have found that the critical points with linear and quadratic low-energy dispersions obey different scaling *Ansätze*. Specifically, the critical point with critical exponent $z = 2$ shows anomalous logarithmic scaling behavior which is markedly different from the one with $z = 1$, with linear dispersions. We also employed scaling parameters to study the band topology in critical regimes. We have also found that the scaling parameters change smoothly along the phase boundary with $z = 1$. In contrast, the scaling parameters are found to be very sensitive to topological changes [Figs. 3(c) and 3(d)]. In this Rapid Communication, we observed that there is a close connection between topological phase transitions, quantum criticalities, energy band structures, and geometric phases.

Our approach may be generalized to other spin chains with multispin interactions, such as the Wen-plaquette model, simulated in nuclear magnetic resonance systems [39], or Baxter-Wu models [40,41]. Recently, multispin interactions were demonstrated to arise in Floquet driven lattices [42,43]. We finally observe that the multicritical points in generalized cluster-Ising models can be used to investigate nonequilibrium dynamics in many-body physics [44,45]. It will be interesting to study the nonadiabatic driving scheme across the multicritical points in cluster-Ising spin chains and the interplay between geometric phases and dynamics in the near future.

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