

This document is downloaded from DR-NTU, Nanyang Technological University Library, Singapore.

Title	Unification of nonclassicality measures in interferometry
Author(s)	Yuan, Xiao; Zhou, Hongyi; Gu, Mile; Ma, Xiongfeng
Citation	Yuan, X., Zhou, H., Gu, M., & Ma, X. (2018). Unification of nonclassicality measures in interferometry. <i>Physical Review A</i> , 97(1), 012331-.
Date	2018
URL	<a href="http://hdl.handle.net/10220/45159">http://hdl.handle.net/10220/45159</a>
Rights	© 2018 American Physical Society. This paper was published in <i>Physical Review A</i> and is made available as an electronic reprint (preprint) with permission of American Physical Society. The published version is available at: [ <a href="http://dx.doi.org/10.1103/PhysRevA.97.012331">http://dx.doi.org/10.1103/PhysRevA.97.012331</a> ]. One print or electronic copy may be made for personal use only. Systematic or multiple reproduction, distribution to multiple locations via electronic or other means, duplication of any material in this paper for a fee or for commercial purposes, or modification of the content of the paper is prohibited and is subject to penalties under law.

## Unification of nonclassicality measures in interferometry

Xiao Yuan,<sup>1</sup> Hongyi Zhou,<sup>1</sup> Mile Gu,<sup>2,3,4</sup> and Xiongfeng Ma<sup>1,\*</sup>

<sup>1</sup>Center for Quantum Information, Institute for Interdisciplinary Information Sciences, Tsinghua University, Beijing 100084, China

<sup>2</sup>School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 639673, Republic of Singapore

<sup>3</sup>Complexity Institute, Nanyang Technological University, Singapore 639673, Republic of Singapore

<sup>4</sup>Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117543, Republic of Singapore



(Received 3 August 2017; revised manuscript received 7 December 2017; published 24 January 2018)

From an operational perspective, nonclassicality characterizes the exotic behavior in a physical process which cannot be explained with Newtonian physics. There are several widely used measures of nonclassicality, including coherence, discord, and entanglement, each proven to be essential resources in particular situations. There exists evidence of fundamental connections among the three measures. However, the sources of nonclassicality are still regarded differently and such connections are yet to be elucidated. Here, we introduce a general framework of defining a unified nonclassicality with an operational motivation founded on the capability of interferometry. Nonclassicality appears differently as coherence, discord, and entanglement in different scenarios with local measurement, weak basis-independent measurement, and strong basis-independent measurement, respectively. Our results elaborate how these three measures are related and how they can be transformed from each other. Experimental schemes are proposed to test the results.

DOI: [10.1103/PhysRevA.97.012331](https://doi.org/10.1103/PhysRevA.97.012331)

### I. INTRODUCTION

Nonclassicality may come in various shapes in different scenarios. One of the earliest forms of nonclassicality is the coherent superposition of a single quantum system. Quoting Einstein, “*God does not play dice with the world*” [1], it is nowadays widely believed that the intrinsic randomness in a quantum measurement is a key feature that distinguishes the quantum theory from classical ones. From the perspective of quantum resource, intrinsic randomness comes from breaking the coherence of quantum states [2,3]. The superposition or interference of distinguishable states—coherence—thus can be regarded as a mark for single-partite nonclassicality [4,5].

In the past few decades or so, quantum information theory has been well developed. In quantum information processing, the existence of nonclassicality has been witnessed by specific tasks that can be fulfilled by quantum processes, but not by any classical process. For instance, Bell inequalities, satisfied by the classical theory, can be violated with certain quantum settings [6,7]; quantum correlations enable extending secret keys between two remotely separated users [8,9], which is impossible with classical processes; and quantum computing can tackle classically intractable problems [10]. In these and many other tasks, entanglement, which measures a special form of correlation of multipartite quantum systems, has been recognized as the central element that is responsible for the advantage of the quantum process [11]. Entanglement has become the most widely used measure of nonclassicality [12,13].

Besides entanglement, another important nonclassicality measure in quantum information processing is discord. As a

general measure for multipartite quantum correlation, discord plays an important role in remote state preparation [14] and quantum metrology [15], and reveals quantum advantages in other tasks [16,17]. In those tasks, entanglement might not be detected and the quantum advantage might be associated to the existence of discord.

In this work, we focus on the nonclassicality theories of coherence, discord, and entanglement [11,18,19]. There are many examples indicating fundamental connections between the three nonclassicality measures. For quantum correlation, including entanglement and discord, it is shown that all nonclassical correlations can be activated into distillable entanglement [20,21]. Recently, enormous efforts have been devoted to investigate the relation between coherence and quantum correlation. For instance, the trade-off between coherence and correlation measures has been analyzed under different scenarios [22–27]. Also, considering incoherent operations, it is shown that coherence can be converted into quantum correlation [28,29]. In addition, coherence and quantum correlation are shown to play important roles in several information tasks, such as frozen quantumness [30–32] and quantum state merging [33–35].

From these observations, we see that coherence, discord, and entanglement are deeply connected concepts. This hints that the three nonclassicality measures may play similar roles in quantum information tasks. Our work supports this intuition by unifying nonclassicality measure in a simple information task, i.e., the three measures play a similar role in the same task only with different conditions. A challenge of this unification lies in the fact that coherence characterizes nonclassicality of a single system, while entanglement and discord characterize multipartite quantum correlations. In addition, difficulties also stem from the fact that coherence is defined on a specific measurement basis, while entanglement and discord are

\*xma@tsinghua.edu.cn

TABLE I. Properties of a generalized nonclassicality quantifier.

(Q1)	Vanishes for classical state: $Q(\sigma) = 0, \forall \sigma \in \mathcal{C}$ . Stronger condition: (Q1') $Q(\sigma) = 0$ iff $\sigma \in \mathcal{C}$ .
(Q2)	<i>Monotonicity</i> : classical operation cannot increase nonclassicality, (Q2a) $Q(\rho) \geq Q(\Phi^C(\rho))$ , (Q2b) $Q(\rho) \geq \sum_n p_n Q(\rho_n)$ .

independent of the (local) basis. Furthermore, although discord and entanglement both describe multipartite correlation, their similarities and differences are not fully understood.

Tracing back the origin of the mystery of quantum mechanics, it is the wave-particle duality that first confused physicists, including Einstein. From today's viewpoint, Einstein's quote indicates that he disagreed with Born's probability interpretation of wave functions [2]. It has long been well known, before the birth of quantum mechanics, that the wave property of a physical subject can be demonstrated by interference. In fact, it is Young's double-slit experiment that confirms the wave property of light. We follow this track to unify various notions of nonclassicality. In our work, we operationally identify nonclassicality measures based on the capability of interferometry. Under different scenarios, we show that the associated nonclassicality measures are coherence, discord, and entanglement.

## II. BACKGROUND AND DEFINITION

### A. Preliminaries

We first review a general nonclassicality framework, which consists of the definitions of classical or nonclassical states, classical operations, and nonclassicality quantifiers [36,37]. Focusing on an operational task, the corresponding nonclassicality is witnessed when quantum behavior that cannot be explained classically is observed. A state  $\sigma$  is called *classical* when it exhibits no quantum behavior. Denote the set of classical states by  $\mathcal{C} = \{\sigma\}$ ; then a state  $\rho$  that does not belong to  $\mathcal{C}$  is called nonclassical or *quantum*. Besides the definition of classical states, a nonclassicality framework is completed by defining classical operations and quantifiers of the nonclassicality. Based on the definition of classical state, classical operation  $\Phi^C$  should be physically realizable, i.e., it is a completely positive trace preserving (CPTP) map, and, in the meantime, cannot generate nonclassicality from classical states, i.e.,  $\Phi^C(\sigma) \in \mathcal{C}, \forall \sigma \in \mathcal{C}$ . In Kraus representation, classical operation is defined by  $\Phi^C(\sigma) = \sum_n \hat{K}_n \sigma \hat{K}_n^\dagger \in \mathcal{C}, \forall \sigma \in \mathcal{C}$ , where  $\{\hat{K}_n\}$  is a series of Kraus operators satisfying  $\sum_n \hat{K}_n^\dagger \hat{K}_n = I$ . Extra constraints can be added to the definition of classical operations. For instance, we can further require that classical operation cannot generate nonclassicality even under postselection,  $\hat{K}_n \sigma \hat{K}_n^\dagger / \text{Tr}[\hat{K}_n \sigma \hat{K}_n^\dagger] \in \mathcal{C}, \forall \sigma \in \mathcal{C}$ . With classical states and operations, the amount of nonclassicality can be given by a real-valued function of states,  $Q(\rho)$ . Generally, a nonclassicality quantifier should satisfy the properties in Table I. Extra requirements such as convexity or other physical conditions can be added.

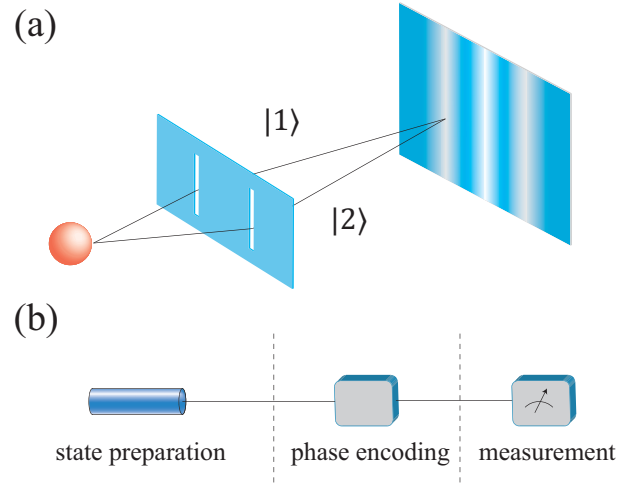


FIG. 1. Double-slit experiment of an electron. (a) Interference pattern is observed when the electron is in a superposition of the path basis,  $J = \{|1\rangle, |2\rangle\}$ . (b) A general interferometry process that consists of state preparation, phase encoding, and measurement. The nonclassicality of the prepared state is witnessed when an interference pattern, i.e., nonzero phase information, is observed from the measurement outcome.

From the previous paragraph, we can see that a key step in the construction of a general nonclassicality framework lies in distinguishing classical and nonclassical states. In this work, we concentrate on identifying the classical states of four different nonclassicality measures, i.e., coherence [4,5], basis-dependent (BD) discord [38], discord [38,39], and entanglement [12], of which BD discord plays as a bridge that links coherence and quantum correlation. We will show a unified framework based on interferometry, in which these sources of nonclassicality naturally arise.

### B. Double-slit experiment

As an illustrative example, we consider the double-slit experiment of an electron, shown in Fig. 1(a), as our first operational task. Classically, the electron will go through either path |1> or |2> and display no interference pattern, while when the electron is in a superposition of the two paths, the quantum behavior of interference can be observed. Schematically [see Fig. 1(b)], the double-slit experiment can be regarded as an interferometry process that probes the phase difference between different paths. Considering this superposition as a quantum feature, while considering a mixture of two paths as classical states, the interferometry capability is thus a traditional signature of nonclassicality.

### C. Interferometry capability and nonclassicality

The double-slit experiment can be generalized to observe the phase behavior on  $d_A$  paths (labeled by  $A$ ) and, in the meantime, the particle potentially has  $d_B$  internal degrees of freedom (labeled by  $B$ ). In this case, we focus on a  $d_A \times d_B$ -dimensional Hilbert space  $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$  with bases  $J_A = \{|j_A\rangle\}_{j_A=1,2,\dots,d_A}$  and  $J_B = \{|j_B\rangle\}_{j_B=1,2,\dots,d_B}$  for path and

internal degrees of freedom, respectively. In principle, systems  $A$  and  $B$  can also be regarded as two individual subsystems.

An interferometry process generally consists of three stages: state preparation, phase encoding, and measurement. After state preparation, an initial state  $\rho_{AB}$  from  $\mathcal{H}_{AB}$  is prepared. By denoting  $\phi_{j_A}$  as the accumulated phase for path  $|j_A\rangle$ , the phase-encoding process can be described by a unitary operation,

$$U_{J_A, \phi_A} = \sum_{j_A=1}^{d_A} e^{-i\phi_{j_A}} |j_A\rangle \langle j_A| \otimes I_B, \quad (1)$$

with  $\phi_A = (\phi_1, \phi_2, \dots, \phi_{d_A})$  and  $I_B$  being the identity matrix of the internal degrees of freedom. After phase encoding, the state evolves to  $\rho'_{AB} = U_{J_A, \phi_A} \rho_{AB} U_{J_A, \phi_A}^\dagger$ . In the measurement phase, we consider a general positive-operator valued measure (POVM)  $\{M_1, M_2, \dots, M_D | \sum_i M_i = I, M_j \geq 0\}$  on  $\rho'_{AB}$ , where  $D$  is the number of POVM elements. The measurement outcome is denoted as a random variable  $X_M$ .

Under the generalized interferometry process, the nonclassical or quantum behavior is defined by the *interferometry capability*, i.e., the ability of probing the phase information  $\phi_A$ . A state  $\rho_{AB}$  is called classical when the measurement outcome  $X_M$  is independent of the phase information  $\phi_A$ , i.e.,  $I(X_M, \phi_A) = 0$ , for any possible measurement. Here, we consider  $\phi_A$  as a random variable and  $I(X, Y)$  is the mutual information of two random variables. On the other hand, a state is considered nonclassical or quantum if one can acquire nonzero information of the phase with a proper measurement. We will show in the following that the interferometry capability is powered by coherence and BD discord. That is, one can acquire nonzero information of the phase if and only if the input state has coherence or BD discord.

#### D. Adversarial scenario

While discord and entanglement are basis-independent nonclassicality measures, we have to consider interferometry without a fixed basis. In practice, this corresponds to the case where the phase-encoding basis varies with time or is even unknown [15]. Such a practical issue is equivalent to the worst-case scenario where an adversary, Eve, controls the phase-encoding basis according to her local information  $\rho_E$  to minimize the phase information that can be learned from the measurement result. It is thus also interesting to investigate the interferometry capability, i.e., nonclassicality, of quantum states under the adversary's control. In such an adversarial scenario, a state  $\rho_{AB}$  is called quantum only when the measurement outcome  $X_M$  has nonzero phase information, i.e.,  $\min_E I(X_M, \vec{\phi}_A) > 0$ , where the minimization is over all possible manipulations by the adversary, who may share entanglement with  $\rho_{AB}$  and control the phase-encoding basis as described below.

We consider that the adversary first measures her local system  $\rho_E$  to generate a basis choice  $e$ ; then she rotates the phase-encoding basis to  $J_A^e = \{|j_A^e\rangle = U_e^\dagger |j_A\rangle\}$  by applying the rotation  $U_e^\dagger$  based on  $e$ . We assume that the measurement outcome  $e$  is revealed to the interferometry measurement. Otherwise, it is not hard to see that the adversary can always destroy the interferometry capability for any input state  $\rho_{AB}$ .

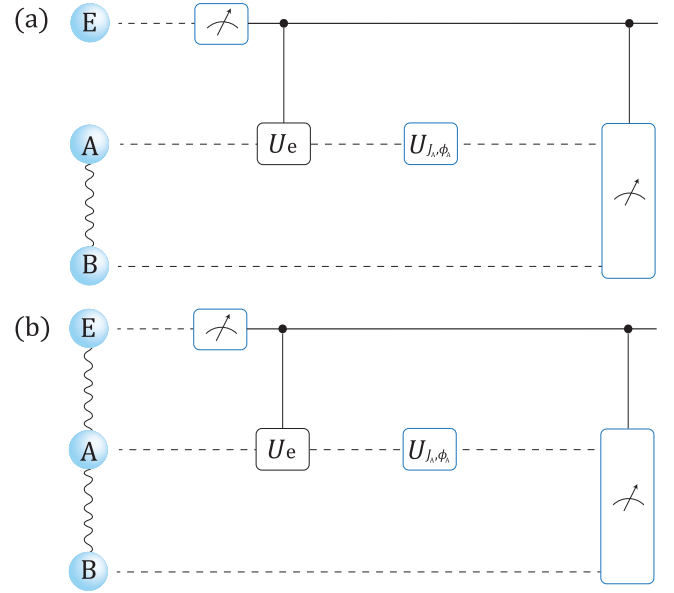


FIG. 2. Interferometry process under adversarial scenarios. (a) The weak adversary is not entangled with system  $AB$ , while (b) the strong adversary shares maximal entanglement with system  $AB$ . The adversary measures her local system to generate a basis choice  $e$ , based on which she rotates the phase-encoding basis  $J_A$  to an arbitrary one,  $J_A^e = \{|j_A^e\rangle = U_e^\dagger |j_A\rangle\}$ , by acting a unitary operation  $U_e = \sum_{j_A} |j_A\rangle \langle j_A^e|$  on system  $A$ . That is,  $\rho'_{AB} = U_{J_A, \phi_A} U_e \rho_{AB} U_e^\dagger U_{J_A, \phi_A}^\dagger = U_e U_{J_A^e, \phi_A} \rho_{AB} U_{J_A^e, \phi_A}^\dagger U_e^\dagger$  with  $U_{J_A^e, \phi_A}^\dagger = U_e^\dagger U_{J_A, \phi_A} U_e$ . The solid and dashed lines represent classical and quantum states, respectively. Note that a strong adversary is strictly more powerful than a weak adversary.

In this work, we consider two different ways that the adversary generate the basis choices, as shown in Fig. 2. The adversary is called *weak* when her local system  $E$  is not entangled with system  $AB$  and *strong* when system  $ABE$  is maximally entangled. As the phase-encoding basis is unknown for each basis choice, we identify such interferometry capability by weak and strong basis-independent nonclassicality measures for weak and strong adversaries, respectively. We will show in the following that the nonclassicality measures for weak and strong adversaries are discord and entanglement, respectively.

#### E. Four types of classical states

Now, we briefly summarize the definitions of the four nonclassicality measures. Coherence is defined for a single quantum system on a specific measurement basis [4,5]. Considering the space of the path and the phase-encoding basis  $J_A$ , a state  $\sigma_A^{J_A}$  is called incoherent state when

$$\sigma_A^{J_A} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A|, \quad (2)$$

and is called a coherent state otherwise.

For BD discord [38], we consider the joint state  $\rho_{AB}$  of path and internal degrees of freedom. A state has no BD discord on

$J_A$  iff

$$\sigma_{AB}^{J_A} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \rho_B^{j_A}, \quad (3)$$

where  $p_{j_A} \geq 0$ ,  $\sum_{j_A} p_{j_A} = 1$ , and  $\rho_B^{j_A}$  is an arbitrary state from  $\mathcal{H}_B$ .

Quantum correlation also defines the nonclassicality measures of a joint system. The set of states that has zero discord [38,39] is defined by the union of the zero BD-discord state of all local bases,

$$\mathcal{C} = \bigcup_{J_A} \{\sigma_{AB}^{J_A}\}, \quad (4)$$

where  $\sigma_{AB}^{J_A}$  is defined in Eq. (A7) for basis  $J_A$ .

A state that has no entanglement is called a separable state, which is given by

$$\sigma_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j. \quad (5)$$

### III. INTERFEROMETRY CAPABILITY REVEALS NON-CLASSICALITY

In this section, we investigate the relation between the existence of nonclassicality and interferometry capability. We show that coherence, BD discord, discord, and entanglement are necessary and sufficient resources for demonstrating the interferometry capability in different scenarios. Here we only discuss the necessary argument and present the rigorous derivations in the Appendix.

For quantum coherence, we consider the interferometry process without the presence of an adversary. When the Hilbert spaces of path and internal degrees of freedom are uncorrelated, the input state can be expressed as  $\rho_{AB} = \rho_A \otimes \rho_B$  and the interferometry ability is independent of the internal degrees of freedom. Focusing on the path state, i.e., system  $A$ , we prove that coherence is the resource for demonstrating interferometry capability.

*Result 1 (coherence).* State  $\sigma_A$  displays no interferometry capability [i.e.,  $I(X_M, \phi_A) = 0, \forall M$ ] iff  $\sigma_A$  is an incoherent state on basis  $J_A$ , i.e.,  $\sigma_A = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A|$ .

This result is more rigorously expressed in Theorem 1 and the proof can be found in the Appendix. The intuition is that when the input state  $\sigma_A$  is incoherent on the phase-encoding  $J_A$ , the state after phase encoding, i.e.,  $\sigma'_A = U_{J_A, \phi_A} \sigma_A U_{J_A, \phi_A}^\dagger$ , is identical to the input state  $\sigma_A$  and hence is independent of the phase information. On the other hand, as long as the input state has nonzero coherence, phase information can be encoded and read out by a proper measurement. The result holds for general interferometry tasks. As an example, consider a special case where only a finite number of different phases is chosen. The probability of guessing the phase information correctly is quantitatively characterized by the robustness of coherence. We refer to Ref. [40] for details.

In general, when the input state has correlation between the path and internal degrees of freedom, we can similarly prove the following statement. The intuition follows the argument of coherence similarly. That is, the state displays no interferometry capability if it can be represented as a convex mixture

of states that is a product of an incoherent state in system  $A$  and any state  $B$ . We refer to Theorem 2 in the Appendix for a detailed discussion.

*Result 2 (BD discord).* State  $\sigma_{AB}^{J_A}$  displays no interferometry capability iff  $\sigma_{AB}^{J_A}$  has zero BD discord on basis  $J_A$ , i.e.,  $\sigma_{AB}^{J_A} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \rho_B^{j_A}$ .

In the presence of an adversary, correlation between the path and internal degrees of freedom is necessary for displaying quantum behavior. Under a weak adversary, the prepared state is independent of the adversary's intervention, i.e., the prepared state will not be affected by the adversary's measurement. Therefore, different measurements of the adversary only correspond to the same case where there is one, but unknown, phase-encoding basis  $J'_A = \{|j'_A\rangle\}$ . In this case, an uncorrelated state  $\sigma_{AB} = \rho_A \otimes \rho_B$  becomes classical when  $J'_A = \{|j'_A\rangle\}$  is chosen in which  $\rho_A$  has a spectral decomposition  $\rho_A = \sum_{j'_A} \lambda_{j'_A} |j'_A\rangle \langle j'_A|$ . In general, states with zero discord, as defined in Eq. (4), display zero interferometry capability under a weak adversary. This is because the adversary can always choose a phase-encoding basis  $J'_A$  in which a zero-discord state also has zero BD discord. We refer to Theorem 3 for details of the proof.

*Result 3 (discord).* State  $\sigma_{AB}$  displays no interferometry capability under a weak adversary iff  $\sigma_{AB}$  has zero discord, i.e., it is a zero BD-discord state of any local basis.

Consider a strong adversary, who holds a purification of  $|\phi\rangle_{ABE}$  with  $\rho_{AB} = \text{tr}[|\phi\rangle_{ABE} \langle \phi|_{ABE}]$ . She can rotate the phase-encoding basis according to the measurement result on her local quantum system  $E$ . Since the local measurement on  $E$  will effectively collapse the remaining system to a decomposition of  $\rho_{AB} = \sum_e p_e |\psi_{AB}\rangle_e \langle \psi_{AB}|_e$ , she can thus rotate the measurement basis to  $J_A^e$  individually for each measurement outcome  $e$ . Therefore, as long as  $\rho_{AB}$  can be decomposed into a convex combination of BD-discord states of all measurement bases, i.e., a separable state defined in Eq. (5), it cannot be used for interferometry under a strong adversary. As rigorously expressed in Theorem 4, we can also relate interferometry capability under strong adversary to entanglement.

*Result 4 (entanglement).* State  $\sigma_{AB}$  displays no interferometry capability under a strong adversary iff  $\sigma_{AB}$  has zero entanglement, i.e.,  $\sigma_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j$ .

### IV. EXAMPLES: PHOTONIC SETUP

Here, we present a photonic setup for demonstrating the relation between nonclassicality and interferometry capability. Focusing on Fig. 3(a), we can test that the coherence on two paths  $\{|0\rangle, |1\rangle\}$  is necessary for probing the phase  $\phi$ . When the beam splitter is replaced by a random switch, which selects the path according to a random bit, the prepared state will be in a mixture of the two paths and hence display no quantum effects. In Fig. 3(b), we consider that the phase-encoding basis can be controlled by a weak adversary. Under this scenario, the phase information cannot be obtained when the adversary selects an appropriate basis. In Fig. 3(c), we consider interferometry with internal degrees of freedom, i.e., polarization. Even though the local state of the path contains no coherence, the correlation between path and polarization can

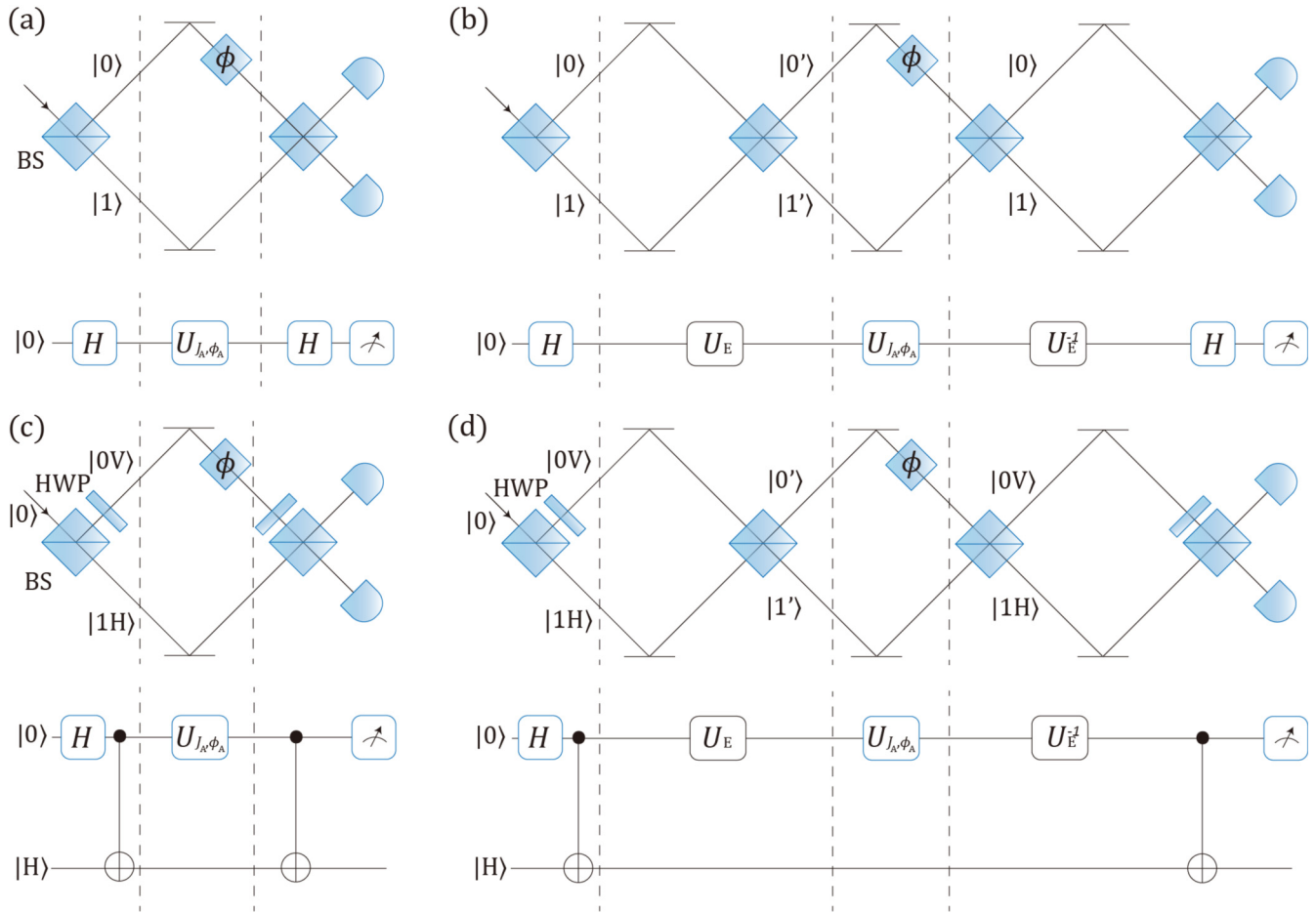


FIG. 3. Photonic setups of four interferometry processes: (a) assisted with coherence of paths, (b) under adversary's attacks, (c) with correlation between path and polarization, and (d) with correlation and under adversary's attack. BS: beam splitter;  $|0\rangle$ ,  $|1\rangle$ : two paths; HWP: half wave plate;  $|H\rangle$ : horizontal polarization;  $|V\rangle$ : vertical polarization; H: Hadamard gate.

still be used for probing the phase information. In Fig. 3(d), we consider an adversarial scenario. In this picture, we can see that the quantum correlation of the prepared state guarantees the interferometry capability.

## V. DISCUSSION

As summarized in Table II, we derive a unified framework for coherence, BD discord, discord, and entanglement and show them as the resource for demonstrating the interferometry capability in different scenarios. Our work supplies a general framework for unifying nonclassicality measures via the interferometry process. Although we only focus on coherence, discord, and entanglement of discrete-variable quantum states, we expect that the result can be extended

to other nonclassicality measures such as negativity of the Wigner function [41] and contextuality [42,43]. Besides, we can also consider general continuous-variable systems instead of discrete-variable systems and the nonclassicality measures. Furthermore, combining our results with the channel-state duality [44], the framework of defining the nonclassicality measures of quantum processes might also be a potential direction for future work.

In Ref. [45], the authors relate the coherence quantifier to the visibility of the interference pattern. As an example, the  $l_1$  norm coherence quantifier is related to interferometry visibility [45], which is also formulated to define the set of discorded states and construct an entanglement monotone [46]. Based on our unified framework of nonclassicality, we will consider in a future project the quantifiers for different nonclassicality

TABLE II. Different measures for nonclassicality based on interferometry.

Nonclassicality	System	Basis	Adversary	Classical states	Example
Coherence	$A$	$J_A$ basis	No	Eq. (2)	Fig. 3(a)
BD-discord	$AB$	$J_A$ basis	No	Eq. (3)	Fig. 3(c)
Discord	$AB$	Basis independent	Weak	Eq. (4)	Fig. 3(d)
Entanglement	$AB$	Basis independent	Strong	Eq. (5)	Fig. 3(d)

measures. We will show that the BD-discord, discord, and entanglement quantifiers can be constructed from an arbitrary coherence quantifier.

Finally, quantum coherence has been shown as the resource for randomness generation [3,47]. Based on the connection between coherence, discord, and entanglement, another research direction is to investigate the role of discord and entanglement in randomness generation.

### ACKNOWLEDGMENTS

We acknowledge H.-K. Lo and J. Ma for the insightful discussions. This work was supported by the National Key R&D Program of China (2017YFA0303900, 2017YFA0304004), the National Natural Science Foundation of China Grant No. 11674193, the National Research Foundation (NRF), NRF-Fellowship (Reference No. NRF-NRFF2016-02), the John Templeton Foundation (Grant No. 54914), and the Foundational Questions Institute.

X.Y. and H.Z. contributed equally to this work.

### APPENDIX: NONCLASSICALITY FOR INTERFEROMETRY FRAMEWORK

In this Appendix, we will discuss the nonclassicality that is defined by the power of interferometry that consists of state preparation, phase encoding, and measurement.

#### 1. Coherence

First, we focus on a general  $d$ -dimensional Hilbert space and a classical computational basis  $J = \{|j\rangle\}_{j=1,2,\dots,d}$ . As we only focus on system  $A$ , we abbreviate the subscript  $A$  in this part.

The phase-encoding process is defined by a unitary operator,

$$U_{J,\phi} = \sum_{j=1}^d e^{-i\phi_j} |j\rangle \langle j|, \quad (\text{A1})$$

where  $\phi_j$  is an additional phase for basis  $|j\rangle$ , and  $\phi = (\phi_1, \phi_2, \dots, \phi_d)$ . For any input state  $\rho$ , it evolves to  $\rho' = U_{J,\phi} \rho U_{J,\phi}^\dagger$  after the information-encoding process. Then a measurement is performed on state  $\rho'$  to extract the phase information  $\phi$ . For a given unitary operator  $U_{J,\phi}$ , we can define that the measurement exhibits an interference pattern as follows:

*Definition 1.* A quantum state  $\rho$  can probe the phase information of process  $U_{J,\phi}$  when there exists a general positive-operator valued measure  $M = \{M_1, M_2, \dots, M_D\}$  that can be performed to extract the information of  $\phi$ . Here,  $D$  denotes the number of POVM elements.

*Definition 2.* Denote the measurement outcome of a measurement by a random variable  $X_M$ ; then the measurement have no information of the phase  $\phi$  and is defined by  $I(X_M, \phi) = 0$ .

For a state that cannot probe phase information, we define it to be classical.

*Definition 3.* A state  $\sigma_J$  is called classical on the classical computational basis  $J$ , if and only if it cannot probe the phase information for any information-encoding process defined in Eq. (A1).

Otherwise, we call the state a quantum state, which has the ability to probe the phase information. For the classical

computational basis  $J$ , we can identify all classical states, which is summarized in the following theorem.

*Theorem 1.* A quantum state  $\sigma_J$  is classical on the classical computational basis  $J$  if and only if

$$\sigma_J = \sum_{j=1}^d p_j |j\rangle \langle j|, \quad (\text{A2})$$

where  $p_j$  is non-negative and  $\sum_{j=1}^d p_j = 1$ .

To prove Theorem 1, we first prove the following lemma:

*Lemma 1.* A quantum state  $\sigma_J$  is classical if and only if

$$U_{J,\phi} \sigma_J U_{J,\phi}^\dagger = \sigma_J. \quad (\text{A3})$$

*Proof.* First we prove the “if” part. We notice that after an arbitrary phase encoding, the output state stays invariant since the probability distribution of a measurement outcome only depends on  $\sigma_J$ , that is,  $\{X_M(\sigma_J)\}$ . Consider that  $\sigma_J$  is independent with the phase-encoding method  $\phi$ ; hence  $\{X_M(\sigma_J)\}$  is also independent with  $\phi$ , that is,  $I(X_M, \phi) = 0$ .

To prove the “only if” part, we look at the converse negative proposition and focus on a unitary  $U_{J,\phi}$  that has different  $\phi_j$  for different  $j$ . Here, we prove that if  $U_{J,\phi} \sigma_J U_{J,\phi}^\dagger \neq \sigma_J$ , then  $I(X_M, \phi) \neq 0$ , that is, there exists a measurement  $M$  to learn information of  $\phi$ . When focusing on the density matrix of  $\sigma_J = \sum_{j,k} p_{j,k} |j\rangle \langle k|$  and  $U_{J,\phi} \sigma_J U_{J,\phi}^\dagger = \sum_{j,k} p'_{j,k} |j\rangle \langle k|$ ,  $U_{J,\phi} \sigma_J U_{J,\phi}^\dagger \neq \sigma_J$  implies that there exists  $j_0, k_0$  such that  $p_{j_0, k_0} \neq p'_{j_0, k_0} = e^{-i(\phi_{j_0} - \phi_{k_0})} p_{j_0, k_0}$ . In this case, one can perform a POVM  $\{|j_0\rangle + |k_0\rangle, |j_0\rangle - |k_0\rangle\}$ , and the corresponding probability distribution  $X_M$  is

$$\begin{aligned} p_0 &= p'_{j_0, j_0} + p'_{k_0, k_0} + p'_{j_0, k_0} + p'_{k_0, j_0}, \\ p_1 &= 1 - p_0, \end{aligned} \quad (\text{A4})$$

where  $p_0$  is a function of  $\phi_{j_0}$  and  $\phi_{k_0}$ . For different  $\phi_{j_0}$  and  $\phi_{k_0}$ ,  $X_M$  will also be different. Therefore,  $X_M$  depends on  $\phi$  and  $I(X_M, \phi) \neq 0$ . ■

With Lemma 1, we can now prove Theorem 1.

*Proof.* First, we prove the “if” part. After the unitary operation  $U_{J,\phi}$ , it is easy to verify that the state  $\sigma_J$  in Eq. (A2) is unchanged. By Lemma 1, we thus prove that  $\sigma_J$  is classical.

To prove the “only if” part, we can first make use of Lemma 1. Thus we only need to show that any state satisfying  $U_{J,\phi} \sigma_J U_{J,\phi}^\dagger = \sigma_J$  should be represented as in Eq. (A2). This is true because

$$\begin{aligned} U_{J,\phi} \sigma_J U_{J,\phi}^\dagger &= \sum_{j,k=1}^d p_{j,k} U_{J,\phi} |j\rangle \langle k| U_{J,\phi}^\dagger \\ &= \sum_{j,k=1}^d e^{-i(\phi_j - \phi_k)} p_{j,k} |j\rangle \langle k|. \end{aligned} \quad (\text{A5})$$

To have  $U_{J,\phi} \sigma_J U_{J,\phi}^\dagger = \sigma_J$ , we thus require  $e^{-i(\phi_j - \phi_k)} p_{j,k} = p_{j,k}$ . That is, we have  $e^{-i(\phi_j - \phi_k)} = 1$ , or  $p_{j,k} = 0$ . As these two equations should be fulfilled for all  $\phi_j$  and  $\phi_k$ , we thus have  $p_{j,k} = 0$  for  $j \neq k$ . Relabelling  $p_{j,j}$  by  $p_{j,A}$ , we thus proved our result. ■

It is easy to check that the classical state defined in Eq. (A2) is exactly the incoherent state defined in Eq. (2). Therefore,

the nonclassicality of a single quantum system based on interferometry power is equivalent to the nonclassicality of quantum coherence.

## 2. Basis-dependent discord

In this part, we focus on a general bipartite  $d_A \otimes d_B$ -dimensional Hilbert space and a classical computational basis  $J_A = \{|j_A\rangle\}_{j=1,2,\dots,d}$  on system  $A$ . The phase is only locally encoded on system  $A$  on the  $J_A$  basis, which can be described by

$$U_{J_A, \phi_A} = \sum_{j_A=1}^{d_A} e^{-i\phi_{j_A}} |j_A\rangle \langle j_A| \otimes I_B, \quad (\text{A6})$$

where  $I^B$  is the identity matrix of system  $B$ .

The unitary operation defined in Eq. (A6) can be understood as a natural generalization of the one defined in Eq. (A1), where the only difference is that we consider the other system  $B$ . In this case, although the information is only encoded in system  $A$ , we consider general measurement on joint systems  $A$  and  $B$ .

*Definition 4.* A quantum state  $\rho_{AB}$  can probe the phase information of process  $U_{J_A, \phi_A}$  when a quantum measurement  $M_{AB} = \{M_1, M_2, \dots, M_D\} \sum_j M_j = I, M_j \geq 0\}$  on the joint system  $AB$  can be performed to extract the information of  $\phi_A$ .

Compared to Definition 1, the only difference is that we allow joint measurement on both systems  $A$  and  $B$  to extract the phase information. In this case, a locally classical state may also exhibit a global interference pattern if other systems are also considered.

For a quantum state  $\rho_{AB}^{J_A}$  that cannot probe the phase information in  $U_{J_A, \phi_A}^A$ , we define it to be locally classical on system  $A$ .

*Definition 5.* A quantum state  $\sigma_{AB}^{J_A}$  is called locally (system  $A$ ) classical on the classical computational basis  $J$  if and only if it cannot probe the phase information  $\phi_A$  in the phase-encoding process  $U_{J_A, \phi_A}$  with any measurement  $M_{AB}$ , that is,  $I(X_{M, AB}, \phi_A) = 0$ .

In the following, if not specified, by saying locally classical, we mean it is local on system  $A$ . Similar to the single-system case, we can also identify all locally classical states as the state has zero basis-dependent discord.

*Theorem 2.* A quantum state  $\sigma_{AB}^{J_A}$  is locally classical on the classical computational basis  $J_A$  if and only if

$$\sigma_{AB}^{J_A} = \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \rho_B^{j_A}, \quad (\text{A7})$$

where  $p_{j_A} \geq 0$ ,  $\sum p_{j_A} = 1$ ,  $\rho_B^{j_A}$  is a density matrix on system  $B$ .

To prove Theorem 2, we need a Lemma similar to Lemma 1

*Lemma 2.* A quantum state  $\sigma_{AB}^{J_A}$  is locally classical if and only if

$$U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger = \sigma_{AB}^{J_A}. \quad (\text{A8})$$

*Proof.* The proof is also very similar to the proof of Lemma 1. The “if” part follows directly. To prove the “only if” part, we also look at the converse negative proposition, where we prove that if  $U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger \neq \sigma_{AB}^{J_A}$ , then

$I(X_{M, AB}, \phi_A) \neq 0$ . Focusing on the density matrix of  $\sigma_{AB}^{J_A} = \sum_{j,k,m,n} p_{j,k,m,n} |j\rangle \langle k| \otimes |m\rangle \langle n|$  and  $U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger = \sum_{j,k,m,n} e^{-i(\phi_j - \phi_k)} p_{j,k,m,n} |j\rangle \langle k| \otimes |m\rangle \langle n|$ ,  $U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger \neq \sigma_{AB}^{J_A}$  implies that there exists  $j_0, k_0, m_0, n_0$  such that  $p_{j_0, k_0, m_0, n_0} \neq p'_{j_0, k_0, m_0, n_0} = e^{-i(\phi_{j_0} - \phi_{k_0})} p_{j_0, k_0, m_0, n_0}$ . In this case, one can perform a POVM,  $\{|j_0\rangle + |k_0\rangle\rangle \langle j_0| + \langle k_0| \otimes (|m_0\rangle + |n_0\rangle) \langle m_0| + \langle n_0|, I - (|j_0\rangle + |k_0\rangle) \langle j_0| + \langle k_0| \otimes (|m_0\rangle + |n_0\rangle) \langle m_0| + \langle n_0|\}$ , and the corresponding probability distribution  $X_{M, AB}$  is

$$p_0 = \sum_{m,n} (p'_{j_0, j_0, m, n} + p'_{k_0, k_0, m, n} + p'_{j_0, k_0, m, n} + p'_{k_0, j_0, m, n}) + \sum_{j,k} (p'_{j,k, m_0, m_0} + p'_{j,k, n_0, n_0} + p'_{j,k, m_0, n_0} + p'_{j,k, n_0, m_0}),$$

$$p_1 = 1 - p_0, \quad (\text{A9})$$

where  $p_0$  is a function of  $\phi_{j_0}$  and  $\phi_{k_0}$  for arbitrary  $m$  and  $n$ . Therefore,  $I(X_{M, AB}, \phi_A) \neq 0$ . ■

With Lemma 2, we can now prove Theorem 2.

*Proof.* The proof of Theorem 2 is very similar to the proof of Theorem 1. The “if” part follows directly. To prove the “only if” part, we can first make use of Lemma 2. Thus we only need to show that any state satisfying  $U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger = \sigma_{AB}^{J_A}$  should be represented as in Eq. (A7). This is true because

$$U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger = \sum_{j,k,m,n} p_{j,k,m,n} U_{J_A, \phi_A} |j\rangle \langle k| \otimes |m\rangle \langle n| U_{J_A, \phi_A}^\dagger = \sum_{j,k,m,n} e^{-i(\phi_j - \phi_k)} p_{j,k,m,n} |j\rangle \langle k| \otimes |m\rangle \langle n|. \quad (\text{A10})$$

To have  $U_{J_A, \phi_A} \sigma_{AB}^{J_A} U_{J_A, \phi_A}^\dagger = \sigma_{AB}^{J_A}$ , we thus require  $e^{-i(\phi_j - \phi_k)} p_{j,k,m,n} = p_{j,k,m,n}$ . That is, we have  $e^{-i(\phi_j - \phi_k)} = 1$ , or  $p_{j,k,m,n} = 0$ . As these two equations should be fulfilled for all  $\phi_j$  and  $\phi_k$ , we thus have  $p_{j,k,m,n} = 0$  for  $j \neq k$ . Relabelling  $p_{j,k,m,n}$  by  $p_{j_A}$ , we thus have

$$\sigma_{AB}^{J_A} = \sum_{j,m,n} p_{j,m,n} |j_A\rangle \langle j_A| \otimes |m\rangle \langle n|. \quad (\text{A11})$$

Suppose we perform a local (on system  $A$ ) projective measurement on basis  $J$ ; then a measurement outcome  $j$  indicates a postselected state  $\rho_B^j$  of system  $B$ ,

$$\rho_B^j = \frac{1}{p_{j_A}} \sum_{m,n} p_{j,m,n} |m\rangle \langle n|, \quad (\text{A12})$$

where  $p_{j_A} = \sum_{m,n} p_{j,m,n}$ . Then, it is easy to verify that

$$\sigma_{AB}^{J_A} = \sum_{j_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \rho_B^{j_A}, \quad (\text{A13})$$

which proves Eq. (A7). ■

## 3. Discord

In this part and the next part, we introduce an adversary who has the ability of manipulating the phase-encoding basis. In this



case, we will also find the corresponding classical states within the interferometry framework. A weak adversary, as mentioned in the main text, measures her local system  $\rho_E$  which is not entangled with  $\rho_{AB}$  and gets an outcome  $e$ . She then applies a rotation  $U_e$  according to  $e$  on the phase-encoding basis  $|j_A\rangle$ , i.e.,  $J_A^e = \{|j_A^e = U_e^\dagger |j_A\rangle\}$ . Since  $\rho_E$  is not entangled with  $\rho_{AB}$ , her different measurement results will not change  $\rho_{AB}$ . In the following analysis, we consider a fixed  $U_e$  since the adversary reveals  $U_e$  to the interferometry measurement in each round. The phase-encoding process is the same as that in Eq. (A6).

**Definition 6.** A state  $\sigma_{AB}^w$  is called a classical state against a weak adversary if and only if the phase information  $\phi_A$  cannot be probed with any measurement  $M_{AB}$  in the presence of a weak adversary, that is,  $I(X_{M,AB}, \phi_A) = 0$ .

**Theorem 3.** A quantum state  $\sigma_{AB}^w$  is a classical state against a weak adversary if and only if

$$\sigma_{AB}^w = \bigcup_{J_A} \sigma_{AB}^{J_A}, \quad (\text{A14})$$

where  $\sigma_{AB}^{J_A}$  is defined by Eq. (A7).  $\sigma_{AB}^{J_A}$  is a union of the zero BD-discord state of all local bases.

*Proof.* First we prove the “if” part. For a state  $\sigma_{AB}^w$  and an arbitrary fixed  $U_e E$ , the output state will be  $\sigma_{AB}^{J_A} = \sum_{j'_A} p_{j'_A} |j'_A\rangle \langle j'_A| \otimes \rho_B^{j'_A}$ . Using Theorem 2, there is no interferometry power for  $\sigma_{AB}^{J_A}$ .

To prove the “only if” part, for an arbitrary  $U_e$  and the output state  $\sigma_{AB}^{J_A}$ , if the interferometry power is zero, i.e.,  $I(X_{M,AB}, \phi_A) = 0$ , we can use Theorem 2 again and  $\sigma_{AB}^{J_A}$  should have the form  $\sigma_{AB}^{J_A} = \sum_{j'_A} p_{j'_A} |j'_A\rangle \langle j'_A| \otimes \rho_B^{j'_A} \in \sigma_{AB}^w$ . ■

#### 4. Entanglement

In this part, we consider a strong adversary who holds a purification  $|\phi\rangle_{ABE}$  of system  $AB$ . Similar to the weak version, she performs a unitary operation  $U_e$  on the phase-encoding basis  $|j_A\rangle$  according to her local measurement outcome. The phase-encoding process is the same as that in Eq. (A6).

**Definition 7.** A state  $\sigma_{AB}^s$  is called a classical state against a strong adversary if and only if the phase information  $\phi_A$  cannot be probed with any measurement  $M_{AB}$  in the presence of a strong adversary, that is,  $I(X_{M,AB}, \phi_A) = 0$ .

**Lemma 3.** A separable state is a convex combination of zero BD-discord states of different measurement basis.

*Proof.* For an arbitrary separable state given in Eq. (A15), we show that it can be represented as a convex combination

of zero BD-discord states of different measurement basis. We only need to make use of the spectral decomposition of  $\rho_j^A = \sum_k \lambda_{n_{jk}} |n_{jk}\rangle \langle n_{jk}|$ , where  $\{|n_{jk}\rangle\}$  forms an orthogonal basis. Therefore,

$$\begin{aligned} \sigma_{AB} &= \sum_{j=1}^d p_j \rho_A^j \otimes \rho_B^j \\ &= \sum_{j=1}^d p_j \sum_{kj} \lambda_{n_{jk}} |n_{jk}\rangle \langle n_{jk}| \otimes \rho_B^j \\ &= \sum_k \lambda_k \left( \sum_{j=1}^d p_j |n_{jk}\rangle \langle n_{jk}| \otimes \rho_B^j \right). \end{aligned} \quad (\text{A15})$$

Next, we need to prove that any convex combination of zero BD-discord states of different measurement basis is a separable state or, equivalently, is not entangled. This is true because zero BD-discord states are separable states and any convex combination of separable states is still separable. ■

**Theorem 4.** A quantum state  $\sigma_{AB}^s$  is a classical state against a strong adversary if and only if

$$\sigma_{AB}^s = \sum_j p_j \rho_A^j \otimes \rho_B^j, \quad (\text{A16})$$

which is a separable state.

*Proof.* To prove the “if” part, using Lemma 3, we rewrite the separable state  $\sigma_{AB}^s = \sum_j p_j \rho_A^j \otimes \rho_B^j$  as  $\sigma_{AB}^s = \sum_e p_e \sum_{j'_A=1}^{d_A} p_{j'_A} |j'_A\rangle \langle j'_A| \otimes \sigma_B^{j'_A}$ . Eve can always let her ancilla be correlated with  $\sigma_{AB}^s$  to form a three-party state  $\rho_{ABE} = \sum_e p_e |e\rangle \langle e| \otimes (\sum_{j'_A=1}^{d_A} p_{j'_A} |j'_A\rangle \langle j'_A| \otimes \sigma_B^{j'_A})$ . After a measurement on her ancilla, the state of party  $AB$  becomes  $\sum_{j'_A=1}^{d_A} p_{j'_A} |j'_A\rangle \langle j'_A| \otimes \sigma_B^{j'_A}$ , which is a zero BD-discord state. Using Theorem 2, there is no interferometry power, i.e.,  $I(X_{M,AB}, \phi_A) = 0$ .

Then we prove the “only if” part. If  $I(X_{M,AB}, \phi_A) = 0$ , the state of party  $AB$  is  $\sum_{j'_A=1}^{d_A} p_{j'_A} |j'_A\rangle \langle j'_A| \otimes \sigma_B^{j'_A}$  according to Theorem 2. Recall that the adversary chooses  $U_e$  according to her measurement outcome  $e$ . The state of party  $AB$  before  $U_e$  will also be a zero BD-discord state,  $\sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \sigma_B^{j_A}$ . Ignoring the measurement outcome  $e$ , the reduced state of party  $AB$  is a mixed zero BD-discord state,  $\sum_e p_e \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \sigma_B^{j_A}$ . And the state of party  $AB$  before Eve’s measurement will also be  $\sum_e p_e \sum_{j_A=1}^{d_A} p_{j_A} |j_A\rangle \langle j_A| \otimes \sigma_B^{j_A}$ , which is a separable state. ■

- [1] M. Born, H. Born, I. Born, and A. Einstein, *The Born-Einstein Letters: Correspondence Between Albert Einstein and Max and Hedwig Born from 1916 to 1955* (Walker, London, 1971).
- [2] M. Born, *Z. Phys.* **37**, 863 (1926).
- [3] X. Yuan, H. Zhou, Z. Cao, and X. Ma, *Phys. Rev. A* **92**, 022124 (2015).
- [4] J. Aberg, [arXiv:quant-ph/0612146](https://arxiv.org/abs/quant-ph/0612146).

- [5] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [6] J. S. Bell, *Physics* **1**, 195 (1964).
- [7] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Phys. Rev. Lett.* **23**, 880 (1969).
- [8] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers*,

- Systems and Signal Processing* (IEEE, New York, 1984), pp. 175–179.
- [9] A. K. Ekert, *Phys. Rev. Lett.* **67**, 661 (1991).
- [10] P. W. Shor, *SIAM J. Comput.* **26**, 1484 (1997).
- [11] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [12] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
- [13] V. Vedral and M. B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).
- [14] B. Dakić, Y. O. Lipp, X. Ma, M. Ringbauer, S. Kropatschek, S. Barz, T. Paterek, V. Vedral, A. Zeilinger, Č. Brukner *et al.*, *Nat. Phys.* **8**, 666 (2012).
- [15] D. Girolami, T. Tufarelli, and G. Adesso, *Phys. Rev. Lett.* **110**, 240402 (2013).
- [16] A. Datta, A. Shaji, and C. M. Caves, *Phys. Rev. Lett.* **100**, 050502 (2008).
- [17] M. Gu, H. M. Chrzanowski, S. M. Assad, T. Symul, K. Modi, T. C. Ralph, V. Vedral, and P. K. Lam, *Nat. Phys.* **8**, 671 (2012).
- [18] A. Streltsov, G. Adesso, and M. B. Plenio, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [19] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, *Rev. Mod. Phys.* **84**, 1655 (2012).
- [20] M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, and A. Winter, *Phys. Rev. Lett.* **106**, 220403 (2011).
- [21] A. Streltsov, H. Kampermann, and D. Bruß, *Phys. Rev. Lett.* **106**, 160401 (2011).
- [22] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, *Phys. Rev. A* **92**, 012118 (2015).
- [23] J. Svozilík, A. Vallés, J. Peřina, Jr., and J. P. Torres, *Phys. Rev. Lett.* **115**, 220501 (2015).
- [24] Y. Yao, X. Xiao, L. Ge, and C. P. Sun, *Phys. Rev. A* **92**, 022112 (2015).
- [25] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, *Phys. Rev. Lett.* **116**, 160406 (2016).
- [26] C. Radhakrishnan, M. Parthasarathy, S. Jambulingam, and T. Byrnes, *Phys. Rev. Lett.* **116**, 150504 (2016).
- [27] A. Streltsov, S. Rana, M. N. Bera, and M. Lewenstein, *Phys. Rev. X* **7**, 011024 (2017).
- [28] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, and G. Adesso, *Phys. Rev. Lett.* **115**, 020403 (2015).
- [29] J. Ma, B. Yadin, D. Girolami, V. Vedral, and M. Gu, *Phys. Rev. Lett.* **116**, 160407 (2016).
- [30] J. Maziero, L. C. Céleri, R. M. Serra, and V. Vedral, *Phys. Rev. A* **80**, 044102 (2009).
- [31] T. R. Bromley, M. Cianciaruso, and G. Adesso, *Phys. Rev. Lett.* **114**, 210401 (2015).
- [32] X.-D. Yu, D.-J. Zhang, C. L. Liu, and D. M. Tong, *Phys. Rev. A* **93**, 060303 (2016).
- [33] M. Horodecki, J. Oppenheim, and A. Winter, *Nature (London)* **436**, 673 (2005).
- [34] M. Horodecki, J. Oppenheim, and A. Winter, *Commun. Math. Phys.* **269**, 107 (2007).
- [35] A. Streltsov, E. Chitambar, S. Rana, M. N. Bera, A. Winter, and M. Lewenstein, *Phys. Rev. Lett.* **116**, 240405 (2016).
- [36] M. B. Plenio and S. Virmani, *Quantum Inf. Comput.* **7**, 1 (2007).
- [37] F. G. S. L. Brandão and G. Gour, *Phys. Rev. Lett.* **115**, 070503 (2015).
- [38] H. Ollivier and W. H. Zurek, *Phys. Rev. Lett.* **88**, 017901 (2001).
- [39] L. Henderson and V. Vedral, *J. Phys. A* **34**, 6899 (2001).
- [40] C. Napoli, T. R. Bromley, M. Cianciaruso, M. Piani, N. Johnston, and G. Adesso, *Phys. Rev. Lett.* **116**, 150502 (2016).
- [41] A. Kenfack and K. Życzkowski, *J. Opt. B: Quantum Semiclassical Opt.* **6**, 396 (2004).
- [42] A. Cabello, S. Severini, and A. Winter, *Phys. Rev. Lett.* **112**, 040401 (2014).
- [43] M. Howard, J. Wallman, V. Veitch, and J. Emerson, *Nature* **510**, 351 (2014).
- [44] M.-D. Choi, *Lin. Algebra Applic.* **10**, 285 (1975).
- [45] T. Biswas, M. G. Díaz, and A. Winter, *Proc. R. Soc. London, Ser. A* **473**, 20170170 (2017).
- [46] K. C. Tan, H. Kwon, C.-Y. Park, and H. Jeong, *Phys. Rev. A* **94**, 022329 (2016).
- [47] X. Yuan, Q. Zhao, D. Girolami, and X. Ma, [arXiv:1605.07818](https://arxiv.org/abs/1605.07818).