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RATE OF CONVERGENCE OF SOME SPACE DECOMPOSITION METHODS FOR LINEAR AND NONLINEAR PROBLEMS*

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Abstract. Convergence of a space decomposition method is proved for a class of convex programming problems. A space decomposition refers to a method that decomposes a space into a sum of subspaces, which could be a domain decomposition or a multilevel method when applied to partial differential equations. Two algorithms are proposed. Both can be used for linear as well as nonlinear elliptic problems, and they reduce to the standard additive and multiplicative Schwarz methods for linear elliptic problems.

Key words. parallel, domain decomposition, nonlinear, elliptic equation, space decomposition

AMS subject classifications. 65J10, 65M55, 65Y05

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1. Introduction.

We use space decomposition methods to solve a convex programming problem. When the minimization space is suitably decomposed into subspaces, two algorithms are proposed to solve the minimization problem. The first algorithm solves a minimization subproblem in parallel in each of the decomposed subspaces. The second algorithm solves a minimization problem sequentially in each of the subspaces. When the algorithms are used for linear partial differential equations with domain decomposition methods, the first algorithm reduces to the standard additive Schwarz method and the second algorithm reduces to the standard multiplicative Schwarz method. For different kinds of domain decomposition methods and multigrid methods for linear partial differential equations, see the books [6] and [5].

An error estimate is presented in sections 3 and 4. The rate of convergence of the proposed algorithms only depends on some parameters. For domain decomposition methods and multigrid methods, the estimation of these parameters is already known for a class of linear and nonlinear problems.

The algorithms are proposed for minimization problems. They can be used for a wide class of problems, for example, eigenvalue problems, optimal control problems related to partial differential equations, and least-squares methods associated with linear and nonlinear equations.

The two algorithms given in this work were first proposed in [7], see also [8], [10], and [11], where the convergence of the algorithms was proved, but the rate of convergence was not given. In the present work, we modify the assumptions concerning the decomposed spaces. The convergence proof of this work can be applied to domain decomposition methods as well as multigrid methods; see [13].

2. Statement of the problem and the algorithms.

Consider the nonlinear problem

$$\min_{v\in V} F(v)\ . \tag{2.1}$$

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The functional F is differentiable and convex, and the space V is a reflexive Banach space. Partial differential equations of the type

$$-\sum D_i(a_{ij}D_ju) + bu = f \text{ in } \Omega ,$$

and

$$-\nabla \cdot (\rho(|\nabla u|)\nabla u) = f \text{ in } \Omega ,$$

with a suitably given ρ, can be formulated as (2.1) by defining the functional F and space V properly.

We shall use space decomposition methods to solve (2.1). A space decomposition method refers to a method that decomposes the space V into a sum of subspaces; i.e., there are spaces $V_i,\ i = 1, 2, \dots, m$, such that

$$V = V_1 + V_2 + \cdots + V_m . \tag{2.2}$$

The meaning of the above decomposition is that for any $v \in V$ there exists $v_i \in V_i$ such that $v = \sum_{i=1}^m v_i$. At the same time, if $v_i \in V_i$, then $\sum_{i=1}^m v_i \in V$. If the space can be decomposed as in (2.2), then the following algorithms can be used to solve (2.1).

Algorithm 2.1 (an additive space decomposition method).

Step 1. *Choose initial values* $u_i^0 \in V_i$ *and relaxation parameters* $\alpha_i > 0$ *such that* $\sum_{i=1}^m \alpha_i \le 1$.

Step 2. *For* $n \ge 0$, *let* $\hat{u}_i^{n+\frac{1}{2}} \in V_i$, $i = 1, 2, \dots, m$, *satisfy*

$$F\left(\sum_{k=1,k\ne i}^m u_k^n + \hat{u}_i^{n+\frac{1}{2}}\right) \le F\left(\sum_{k=1,k\ne i}^m u_k^n + v_i\right) \quad \forall v_i \in V_i . \tag{2.3}$$

Use an approximate solver to find $u_i^{n+\frac{1}{2}} \in V_i$ *in parallel for* $i = 1, 2, \dots, m$ *such that*

$$\|u_i^{n+\frac{1}{2}} - \hat{u}_i^{n+\frac{1}{2}}\|_V \le \epsilon_0 \|u_i^n - \hat{u}_i^{n+\frac{1}{2}}\|_V . \tag{2.4}$$

Step 3. *Set*

$$u_i^{n+1} = u_i^n + \alpha_i(u_i^{n+\frac{1}{2}} - u_i^n) , \tag{2.5}$$

and go to the next iteration.

Algorithm 2.2 (a multiplicative space decomposition method).

Step 1. *Choose initial values* $u_i^0 \in V_i$.

Step 2. *For* $n \ge 0$, *let* $\hat{u}_i^{n+1} \in V_i$, $i = 1, 2, \dots, m$, *satisfy*

$$\begin{aligned} &F\left(\sum_{1\le k<i} u_k^{n+1} + \hat{u}_i^{n+1} + \sum_{i<k\le m} u_k^n\right) \\ &\le F\left(\sum_{1\le k<i} u_k^{n+1} + v_i + \sum_{i<k\le m} u_k^n\right) \quad \forall v_i \in V_i . \end{aligned} \tag{2.6}$$

Use an approximate solver to find $u_i^{n+1} \in V_i$ sequentially for $i = 1, 2, \ldots, m$ such that

$$\|u_i^{n+1} - \hat{u}_i^{n+1}\|_V \le \epsilon_0 \|u_i^n - \hat{u}_i^{n+1}\|_V \ . \tag{2.7}$$

Step 3. *Go to the next iteration.*

In the following, the notation $\langle \cdot, \cdot \rangle$ is used to denote the duality pairing between V and V', where V' is the dual space of V. The functional F is assumed to be Gateaux differentiable (see [1]). We assume there are constants $K > 0$, $L < \infty$ such that

$$\begin{aligned} \langle F'(w) - F'(v), w - v \rangle \ge K\|w - v\|_V^2 \quad \forall w, v \in V \ , \\ \|F'(w) - F'(v)\|_{V'} \le L\|w - v\|_V \quad \forall w, v \in V \ . \end{aligned} \tag{2.8}$$

From (2.8) it is easy to deduce that

$$K\|w - v\|_V^2 \le \langle F'(w) - F'(v), w - v \rangle \le L\|w - v\|_V^2 \quad \forall w, v \in V \ . \tag{2.9}$$

Under assumption (2.8), problem (2.1) and subproblems (2.6) and (2.3) have unique solutions; see [4, p. 35].

For the decomposed spaces, we assume that there is a constant $C_1 > 0$ such that $\forall v \in V$ we can find $v_i \in V_i$ that satisfy

$$v = \sum_{i=1}^m v_i \quad \text{and} \quad \sum_{i=1}^m \|v_i\|_V^2 \le C_1^2 \|v\|_V^2 \ . \tag{2.10}$$

Moreover, assume that there is a $C_2 > 0$ such that the following holds:

$$\begin{aligned} \sum_{i=1}^m \sum_{j=1}^m \langle F''(w_{ij}) u_i, v_j \rangle \le C_2 \left(\sum_{i=1}^m \|u_i\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^m \|v_i\|_V^2 \right)^{\frac{1}{2}} \\ \forall w_{ij} \in V \quad \forall u_i \in V_i \quad \forall v_j \in V_j \ . \end{aligned} \tag{2.11}$$

Domain decomposition methods, multilevel methods and multigrid methods, can be viewed as different ways of decomposing finite element spaces into sums of subspaces. For the estimation of the constants C_1 and C_2 for different types of decomposition of finite element methods for linear problems, one can find the proofs or references in Xu [16]. A sharper estimate for two-level domain decomposition methods for linear problems can be found in [3] and [15].

The constant ϵ_0 controls how accurately we shall solve the subproblems. The analysis given later will show that if we take

$$(1 + \epsilon_0)\epsilon_0 \le \frac{K}{4L} \le \frac{1}{4} \ , \tag{2.12}$$

then both algorithms are convergent.

Later, the error reduction factor for the above two algorithms shall be estimated. In the following we shall use $e^n, n = 0, 1, 2, \ldots,$ which is defined as

$$e^n = |\langle F'(u^n) - F'(u), u^n - u \rangle|^{\frac{1}{2}}$$

as a measure of the error between u^n and u. Here and later, u stands for the unique solution of (2.1). For convenience, constants α_{min} and α_{max} are defined as

$\alpha_{min} = \min_{1\le i\le m} \alpha_i$, $\alpha_{max} = \max_{1\le i\le m} \alpha_i$, and α_i are the relaxation parameters in Algorithm 2.1. The constants C_p and C_s, which are

$$C_p = C_2 C_1(\alpha_{min}^{-\frac{1}{2}}(1+\epsilon_0) + \alpha_{max}^{\frac{1}{2}}), \qquad C_s = C_2 C_1 (2+\epsilon_0), \tag{2.13}$$

will play an important role in the analysis of the error reduction factor.

Remark 2.1.

(1) When F is differentiable and if we define

$$w_i^{n+\frac{1}{2}} = \sum_{k=1,k\neq i}^{m} u_k^n + u_i^{n+\frac{1}{2}} \ , \quad \hat{w}_i^{n+\frac{1}{2}} = \sum_{k=1,k\neq i}^{m} u_k^n + \hat{u}_i^{n+\frac{1}{2}} \ , \tag{2.14}$$

then (2.3) is equivalent to solving

$$\left\langle F'(\hat{w}_i^{n+\frac{1}{2}}), v_i \right\rangle = 0 \quad \forall v_i \in V_i \ . \tag{2.15}$$

(2) Let

$$u^{n+1} = \sum_{i=1}^{m} u_i^{n+1}, \ n = 0, 1, 2, \ldots, \tag{2.16}$$

and $w_i^{n+\frac{1}{2}}$ be defined as in (2.14); then

$$w_i^{n+\frac{1}{2}} = u^n + u_i^{n+\frac{1}{2}} - u_i^n$$

and the value of u^{n+1} corresponding to (2.5) can be obtained by

$$\begin{aligned} u^{n+1} &= \sum_{i=1}^{m} u_i^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n) \\ &= u^n + \sum_{i=1}^{m} \alpha_i (u_i^{n+\frac{1}{2}} - u_i^n) \\ &= \sum_{i=1}^{m} \alpha_i (u^n + u_i^{n+\frac{1}{2}} - u_i^n) + \left(1 - \sum_{i=1}^{m} \alpha_i\right) u^n \\ &= \sum_{i=1}^{m} \alpha_i w_i^{n+\frac{1}{2}} + \left(1 - \sum_{i=1}^{m} \alpha_i\right) u^n. \end{aligned} \tag{2.17}$$

When using the proposed algorithms with a two-level domain decomposition method, only the values of u^{n+1} and the coarse mesh problem are needed for the next iteration, and u^{n+1} is updated by the above formula after the computation of each of the subdomain problems.

(3) For Algorithm 2.2, if we define

$$\begin{aligned} w_i^{n+1} &= \sum_{1\le k<i} u_k^{n+1} + u_i^{n+1} + \sum_{i<k\le m} u_k^n \ , \\ \hat{w}_i^{n+1} &= \sum_{1\le k<i} u_k^{n+1} + \hat{u}_i^{n+1} + \sum_{i<k\le m} u_k^n \ , \end{aligned} \tag{2.18}$$

then $\hat{w}_i^{n+1}$ satisfies

$$\big\langle F'(\hat{w}_i^{n+1}), v_i \big\rangle = 0 \quad \forall v_i \in V_i \ . \tag{2.19}$$

and after solving w_i^{n+1} for each i, we only need to set $u^{n+1} = w_m^{n+1}$.

Remark 2.2. Algorithm 2.2 solves the minimization problems sequentially over each subspace. Algorithm 2.1 solves the minimizations in parallel over all the subspaces. In applications, by suitably decomposing the minimization space, the minimization problem over each subspace can be done in parallel, and hence both algorithms are suitable for parallel machines; see [9] and [13]. Moreover, with a suitable decomposition, the constants C_1 and C_2 can be made independent of the size of the problem, and thus the convergence of the above two algorithms does not depend on the size of the problem.

3. The convergence of the additive algorithm. We first give the rate of convergence for Algorithm 2.1.

THEOREM 3.1. *If the space decomposition satisfies* (2.10) *and* (2.11)*, the constant* ϵ_0 *satisfies* (2.12) *and the functional* F *satisfies* (2.8)*, then for Algorithm* 2.1 *we have the following estimates.*

(a) *If* F *is quadratic with respect to* v *and the norm of* V *is chosen as* $\|v\|_V = \langle F'(v), v\rangle$*, then*

$$|e^{n+1}|^2 \le \frac{2C_p^2}{1+2C_p^2}|e^n|^2 \quad \forall n \ge 0 \ . \tag{3.1}$$

(b) *If* F *is three times continuously differentiable, then*

$$|e^{n+1}| \to 0 \ \textit{as} \ n \to \infty \ , \quad \text{and} \quad |e^{n+1}|^2 \le \beta_n |e^n|^2 \quad \forall n \ge 0 \ .$$

Moreover,

$$\lim_{n\to\infty} \beta_n = \frac{C}{1+C} < 1 \quad \textit{and} \quad C = \frac{2{C_p}^2}{K^2} \ , \tag{3.2}$$

which means that the asymptotic convergence rate of Algorithm 2.1 *is* $\frac{C}{1+C}$.

Before we go to the proof of the theorem, we first present a lemma that is needed in the proof. The lemma can be proved in a similar way as in Proposition 5.4 in [4, p. 25], and the proof can be found in [7].

LEMMA 3.2. *If condition* (2.9) *is valid, then we have for the functional* F

$$F(w) - F(v) \ge \langle F'(v), w - v\rangle + \frac{K}{2}\|w - v\|_V^2 \quad \forall v, w \in V, \tag{3.3}$$

$$F(w) - F(v) \le \langle F'(v), w - v\rangle + \frac{L}{2}\|w - v\|_V^2 \quad \forall v, w \in V \ . \tag{3.4}$$

Proof of Theorem 3.1. We use the definitions of (2.16) and (2.14). As F is a convex functional, one obtains by using (2.5), (2.15), (2.17), and (3.3)

$$
\begin{aligned}
&F(u^n) - F(u^{n+1}) \\
&= F(u^n) - F\left(\sum_{i=1}^{m} \alpha_i w_i^{n+\frac{1}{2}} + \left(1 - \sum_{i=1}^{m} \alpha_i\right) u^n\right) \\
&\geq F(u^n) - \sum_{i=1}^{m} \alpha_i F\left(w_i^{n+\frac{1}{2}}\right) - \left(1 - \sum_{i=1}^{m} \alpha_i\right) F(u^n) \\
&= \sum_{i=1}^{m} \alpha_i F(u^n) - \sum_{i=1}^{m} \alpha_i F(w_i^{n+\frac{1}{2}}) \\
&\geq \sum_{i=1}^{m} \alpha_i \langle F'(w_i^{n+\frac{1}{2}}), u_i^n - u_i^{n+\frac{1}{2}} \rangle + \frac{K}{2} \sum_{i=1}^{m} \alpha_i \|u_i^n - u_i^{n+\frac{1}{2}}\|_V^2 \\
&\geq \sum_{i=1}^{m} \alpha_i \langle F'(w_i^{n+\frac{1}{2}}) - F'(\hat{w}_i^{n+\frac{1}{2}}), u_i^n - u_i^{n+\frac{1}{2}} \rangle \\
&\quad + \frac{K}{2} \sum_{i=1}^{m} \alpha_i \|u_i^n - u_i^{n+\frac{1}{2}}\|_V^2 \\
&\geq \frac{K}{2} \sum_{i=1}^{m} \alpha_i \|u_i^n - u_i^{n+\frac{1}{2}}\|_V^2 \\
&\quad - L \sum_{i=1}^{m} \alpha_i \|u_i^{n+\frac{1}{2}} - \hat{u}_i^{n+\frac{1}{2}}\|_V \|u_i^n - u_i^{n+\frac{1}{2}}\|_V \\
&\geq \frac{K}{2} \sum_{i=1}^{m} \alpha_i \|u_i^n - u_i^{n+\frac{1}{2}}\|_V^2 - (1+\epsilon_0)\epsilon_0 L \sum_{i=1}^{m} \alpha_i \|u_i^{n+\frac{1}{2}} - u_i^{n+\frac{1}{2}}\|_V^2 \\
&\geq \frac{K}{4} \sum_{i=1}^{m} \alpha_i \|u_i^n - u_i^{n+\frac{1}{2}}\|_V^2 \ .
\end{aligned} \tag{3.5}
$$

From (2.4) it follows that

$$
\begin{aligned}
&\|u_i^n - \hat{u}_i^{n+\frac{1}{2}}\|_V \\
&\leq \|u_i^n - u_i^{n+\frac{1}{2}}\|_V + \|u_i^{n+\frac{1}{2}} - \hat{u}_i^{n+\frac{1}{2}}\|_V \\
&\leq \|u_i^n - u_i^{n+\frac{1}{2}}\|_V + \epsilon_0 \|u_i^{n+\frac{1}{2}} - u_i^n\|_V \\
&\leq (1+\epsilon_0)\|u_i^{n+\frac{1}{2}} - u_i^n\|_V \ .
\end{aligned} \tag{3.6}
$$

As u is the solution of (2.1), it satisfies $\langle F'(u), v\rangle = 0\ \forall v \in V$. For any $v_i \in V_i,\ i = 1, 2, \ldots, m$, such that $\sum v_i = u$, we shall use (2.15), (3.6), and (2.8) to estimate

$$
\begin{aligned}
&\left\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\right\rangle \\
&= \langle F'(u^{n+1}), u^{n+1} - u\rangle = \sum_{i=1}^{m} \left\langle F'(u^{n+1}), u_i^{n+1} - v_i\right\rangle \\
&= \sum_{i=1}^{m} \left\langle F'(u^{n+1}) - F'(u^n + \hat{u}_i^{n+\frac{1}{2}} - u_i^n), u_i^{n+1} - v_i\right\rangle \\
&= \sum_{i=1}^{m} \left\langle F''(\theta_i^{n+1})(u^{n+1} - u^n), u_i^{n+1} - v_i\right\rangle \\
&\quad - \sum_{i=1}^{m} \left\langle F''(\theta_i^{n+1})(\hat{u}_i^{n+\frac{1}{2}} - u_i^n), u_i^{n+1} - v_i\right\rangle \\
&\quad (\ \theta_i^{n+1} = \theta u^{n+1} + (1-\theta) w_i^{n+1},\ \theta \in [0,1]\ ) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \left\langle F''(\theta_i^{n+1})(u_j^{n+1} - u_j^n), u_i^{n+1} - v_i\right\rangle \qquad (3.7)\\
&\quad - \sum_{i=1}^{m} \left\langle F''(\theta_i^{n+1})(\hat{u}_i^{n+\frac{1}{2}} - u_i^n), u_i^{n+1} - v_i\right\rangle \\
&\leq C_2 \left(\sum_{i=1}^{m} \|u_i^{n+1} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 \right)^{\frac{1}{2}} \\
&\quad + C_2 \sum_{i=1}^{m} \|u_i^n - \hat{u}_i^{n+\frac{1}{2}}\|_V \|u_i^{n+1} - v_i\|_V \\
&\leq C_2 \left(\sum_{i=1}^{m} \alpha_i^2 \|u_i^{n+\frac{1}{2}} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 \right)^{\frac{1}{2}} \\
&\quad + \frac{C_2(1+\epsilon_0)}{\min \sqrt{\alpha_i}} \left(\sum_{i=1}^{m} \alpha_i \|u_i^n - u_i^{n+\frac{1}{2}}\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 \right)^{\frac{1}{2}} .
\end{aligned}
$$

From the property of the space decomposition (2.10), there exists $\phi_i \in V_i$ such that $u^{n+1} - u = \sum_{i=1}^m \phi_i$, and $\sum_{i=1}^m \|\phi_i\|_V^2 \leq C_1^2 \|u^{n+1} - u\|_V^2$. We take $v_i = u_i^{n+1} - \phi_i$ and see that

$$
\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 = \sum_{i=1}^{m} \|\phi_i\|_V^2 \leq C_1^2 \|u^{n+1} - u\|_V^2 \ . \qquad (3.8)
$$

By combining (3.5)–(3.8), we see that

$$
\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\
&\le C_2\sqrt{\max\alpha_i}\left(\sum_{i=1}^{m}\alpha_i\|u_i^{n+\frac{1}{2}} - u_i^n\|_V^2\right)^{\frac{1}{2}}\cdot C_1\|u^{n+1}-u\|_V \\
&\qquad + C_2(1+\epsilon_0)\alpha_{min}^{-\frac{1}{2}}\left(\sum_{i=1}^{m}\alpha_i\|u_i^{n+\frac{1}{2}} - u_i^n\|_V^2\right)^{\frac{1}{2}}\cdot C_1\ \|u^{n+1}-u\|_V \\
&\le C_1\left(\alpha_{max}^{\frac{1}{2}}C_2 + \alpha_{min}^{-\frac{1}{2}}C_2(1+\epsilon_0)\right)\left[\frac{4}{K}\left(F(u^n) - F(u^{n+1})\right)\right]^{\frac{1}{2}}\cdot\|u^{n+1}-u\|_V\ .
\end{aligned} \tag{3.9}
$$

Let us note that

$$
\|u^{n+1}-u\|_V^2 \le K^{-1}\langle F'(u^{n+1}) - F'(u), u^{n+1}-u\rangle\ .
$$

Therefore, it follows from (3.9) that

$$
\begin{aligned}
\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle &\le C_p\left[\frac{4}{K}\left(F(u^n) - F(u^{n+1})\right)\right]^{\frac{1}{2}} \\
&\cdot K^{-1/2}\sqrt{\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle}\ ,
\end{aligned}
$$

and so

$$
\begin{aligned}
&K^2\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\rangle \\
&\le 4C_p^2[F(u^n) - F(u^{n+1})] \\
&= 4C_p^2\left[F(u^n) - F(u) + F(u) - F(u^{n+1})\right]\ .
\end{aligned} \tag{3.10}
$$

Summing (3.10) for $n = 0, 1, 2, \ldots, N$, we find that

$$
\sum_{n=0}^{N}|e^{n+1}|^2 \le 4{C_p}^2/K^2[F(u^0) - F(u^{N+1})] \le 4{C_p}^2/K^2[F(u^0) - F(u)]\ ,
$$

and so

$$
|e^{n+1}| \to 0 \text{ as } n \to \infty\ . \tag{3.11}
$$

We shall first prove (a) and then (b). Relations (3.3)–(3.4) give

$$
\begin{aligned}
F(u^n) - F(u) &\le \langle F'(u), u^n - u\rangle + \frac{L}{2}\|u^n - u\|_V^2 \\
&= \frac{L}{2}\|u^n - u\|_V^2
\end{aligned} \tag{3.12}
$$

and

$$
\begin{aligned}
F(u) - F(u^{n+1}) &\le -\langle F'(u), u^{n+1} - u\rangle - \frac{K}{2}\|u^{n+1} - u\|_V^2 \\
&= -\frac{K}{2}\|u^{n+1} - u\|_V^2\ .
\end{aligned} \tag{3.13}
$$

Substituting (3.12) and (3.13) into (3.10) and using (2.9) gives

$$\begin{aligned} & K^2|e^{n+1}|^2 \\ & \leq 4{C_p}^2\left(\frac{L}{2}\|u^n-u\|_V^2-\frac{K}{2}\|u^{n+1}-u\|_V^2\right) \\ & \leq 4{C_p}^2\left(\frac{L}{2K}|e^n|^2-\frac{K}{2L}|e^{n+1}|^2\right) \ , \end{aligned}$$

which shows that

$$|e^{n+1}|^2 \leq \frac{LK^{-1}2{C_p}^2}{K^2+KL^{-1}2{C_p}^2}|e^n|^2 \ . \tag{3.14}$$

As F is quadratic and satisfies (2.9), $\sqrt{\langle F'(v),v\rangle}$ defines a norm for V, and $F'(v)$ is linear with respect to v. In the proof given above, if we choose the norm of V to be

$$\|v\|_V=\sqrt{\langle F'(v),v\rangle} \quad \forall v \in V \ ,$$

then we have $K=L=1$ in (2.9); moreover,

$$|e^n|^2=|\langle F'(u^n)-F'(u),u^n-u\rangle|=|\langle F'(u^n-u),u^n-u\rangle|=\|e^n\|_V^2 \ ,$$

and so (3.14) implies (3.1).

Next we prove (b). First, we note that $|e^n| \to 0$ as $n\to\infty$; hence there exists a ball $B(u,\delta)$ which is centered at u, with radius δ such that $u^n \in B(u,\delta)$ $\forall n$. As F is three times continuously differentiable, there is a constant $C(u)$ such that

$$|F'''(\xi)\cdot(v,v,v)| \leq C(u)\|v\|_V^3 \quad \forall \xi \in B(u,\delta) \quad \forall v \in V \ .$$

We use Taylor's formula (see Cea [1, Chap. 2]) to get

$$\begin{aligned} F(u^n)-F(u) &= \langle F'(u),u^n-u\rangle+\frac{1}{2}F''(u)\cdot(u^n-u)^2 \\ & +\frac{1}{6}F'''(u+\theta^n(u^n-u))\cdot(u^n-u)^3 \ , \end{aligned} \tag{3.15}$$

$$\begin{aligned} F(u)-F(u^{n+1}) &= -\langle F'(u),u^{n+1}-u\rangle-\frac{1}{2}F''(u)\cdot(u^{n+1}-u)^2 \\ & -\frac{1}{6}F'''(u+\theta^{n+1}(u^{n+1}-u))\cdot(u^{n+1}-u)^3 \ . \end{aligned} \tag{3.16}$$

Above, $\theta^n,\theta^{n+1}\in[0,1]$. Taking the sum of (3.15) and (3.16) and using (2.9) and the property that $\langle F'(u),v\rangle=0$ $\forall v \in V$, it follows that

$$F(u^n)-F(u^{n+1})=\frac{1}{2}|e^n|^2-\frac{1}{2}|e^{n+1}|^2+I_1+I_2+I_3+I_4 \ , \tag{3.17}$$

where

$$
\begin{aligned}
I_1 &= \frac{1}{2}F''(u)\cdot(u^n-u)^2 - \frac{1}{2}\langle F'(u^n)-F'(u), u^n-u\rangle \\
&\le C(u)\|u^n-u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^n|^3 \ , \\
I_2 &= -\frac{1}{2}F''(u)\cdot(u^{n+1}-u)^2 + \frac{1}{2}\langle F'(u^{n+1})-F'(u), u^{n+1}-u\rangle \\
&\le C(u)\|u^{n+1}-u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^{n+1}|^3 \ , \\
I_3 &= \frac{1}{6}F'''(u+\theta^n(u^n-u))\cdot(u^n-u)^3 \\
&\le C(u)\|u^n-u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^n|^3 \ , \\
I_4 &= -\frac{1}{6}F'''(u+\theta^{n+1}(u^{n+1}-u))\cdot(u^{n+1}-u)^3 \\
&\le C(u)\|u^{n+1}-u\|_V^3 \le \frac{C(u)}{K^{\frac{3}{2}}}|e^{n+1}|^3 \ .
\end{aligned}
\tag{3.18}
$$

Let $C^* = \frac{8C_p^2 C(u)}{K^{\frac{3}{2}}}$. From relations (3.10) and (3.17)–(3.18), it follows that

$$
\begin{aligned}
&\left(K^2 + 2{C_p}^2\right)|e^{n+1}|^2 \\
&\le 2{C_p}^2|e^n|^2 + C^*|e^n|^3 + C^*|e^{n+1}|^3
\end{aligned}
$$

and so

$$
|e^{n+1}|^2 \le \frac{2C_p^2 + C^*|e^n|}{K^2 + 2C_p^2 - C^*|e^{n+1}|}|e^n|^2 \ .
$$

From (3.11), we see that $|e^n| \to 0$ as $n \to \infty$. So, if n is large enough we have

$$
|e^{n+1}| \le \frac{K^2}{2C^*}, \quad |e^n| \le \frac{K^2}{2C^*} \ , \tag{3.19}
$$

and then

$$
\beta_n = \frac{2C_p^2 + C^*|e^n|}{K^2 + 2C_p^2 - C^*|e^{n+1}|} < 1 \ .
$$

Moreover,

$$
\lim_{n\to\infty}\beta_n = \frac{C}{1+C} < 1 \text{ and } C = \frac{2C_p^2}{K^2} \ . \qquad \square
$$

4. The convergence of the multiplicative algorithm. The convergence of Algorithm 2.2 is similar to Algorithm 2.1.

THEOREM 4.1. *Under the same conditions as in Theorem* 3.1, *we have the following estimates for Algorithm* 2.2.

(a) *If F is quadratic with respect to v and the norm of V is taken as $\|v\|_V = \langle F'(v), v\rangle$, then*

$$|e^{n+1}|^2 \le \frac{2{C_s}^2}{1+2{C_s}^2}|e^n|^2 \quad \forall n \ge 0\ . \tag{4.1}$$

(b) *If F is three times continuously differentiable, then*

$$|e^{n+1}| \to 0 \text{ as } n \to \infty\ , \qquad \text{and} \qquad |e^{n+1}|^2 \le \beta_n |e^n|^2 \quad \forall n \ge 0\ .$$

Moreover,

$$\lim_{n\to\infty} \beta_n = \frac{C}{1+C} < 1 \quad \text{and} \quad C = \frac{2{C_s}^2}{K^2}\ , \tag{4.2}$$

which means that the asymptotic convergence rate of Algorithm 2.2 *is $\frac{C}{1+C}$.*

Proof of Theorem 4.1. Let u^{n+1} and w_i^{n+1} be defined as in (2.16) and (2.18). We see that $u^{n+1} = w_m^{n+1}$. If we also define $w_0^{n+1} = u^n$, we observe that

$$\begin{aligned} F(u^n) - F(u^{n+1}) &= \sum_{i=1}^m \left(F(w_{i-1}^{n+1}) - F(w_i^{n+1})\right) \\ &\ge \sum_{i=1}^m \langle F'(w_i^{n+1}), u_i^n - u_i^{n+1}\rangle + \frac{K}{2}\sum_{i=1}^m \|u_i^n - u_i^{n+1}\|_V^2 \\ &= \sum_{i=1}^m \langle F'(w_i^{n+1}) - F'(\hat{w}_i^{n+1}), u_i^n - u_i^{n+1}\rangle + \frac{K}{2}\sum_{i=1}^m \|u_i^n - u_i^{n+1}\|_V^2 \\ &= \frac{K}{2}\sum_{i=1}^m \|u_i^n - u_i^{n+1}\|_V^2 - L\sum_{i=1}^m \|u_i^{n+1} - \hat{u}_i^{n+1}\|_V \|u_i^n - u_i^{n+1}\|_V \\ &= \frac{K}{4}\sum_{i=1}^m \|u_i^n - u_i^{n+1}\|_V^2\ . \end{aligned} \tag{4.3}$$

Similar to the proof of (3.7), there exist $v_i \in V_i,\ i = 1, 2, \ldots, m$ such that $\sum v_i = u$. We use (2.19) and (2.9) to get the following estimate

$$\begin{aligned} &\left\langle F'(u^{n+1}) - F'(u), u^{n+1} - u\right\rangle \\ &= \langle F'(u^{n+1}), u^{n+1} - u\rangle = \sum_{i=1}^m \left\langle F'(u^{n+1}), u_i^{n+1} - v_i\right\rangle \\ &= \sum_{i=1}^m \left\langle F'(u^{n+1}) - F'(\hat{w}_i^{n+1}), u_i^{n+1} - v_i\right\rangle \\ &= \sum_{i=1}^m \left\langle F''(\theta_i^{n+1})(u^{n+1} - \hat{w}_i^{n+1}), u_i^{n+1} - v_i\right\rangle \end{aligned} \tag{4.4}$$

$$(\ \theta_i^{n+1} = \theta u^{n+1} + (1-\theta) w_i^{n+1}, \ \theta \in [0,1] \ )$$

$$\begin{aligned}
&= \sum_{i=1}^{m} \Bigg[\sum_{j>i+1} \Big\langle F''(\theta_i^{n+1})(u_j^{n+1} - u_j^n), u_i^{n+1} - v_i \Big\rangle \\
&\quad + \Big\langle F''(\theta_i^{n+1}(\hat{u}_i^{n+1} - u_i^n), u_i^{n+1} - v_i) \Big\rangle \Bigg] \\
&\le C_2 \left(\sum_{i=1}^{m} \|u_i^{n+1} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 \right)^{\frac{1}{2}} \\
&\quad + C_2 \left(\sum_{i=1}^{m} \|\hat{u}_i^{n+1} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 \right)^{\frac{1}{2}} \\
&\le C_2 (2 + \epsilon_0) \left(\sum_{i=1}^{m} \|u_i^{n+1} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} \|u_i^{n+1} - v_i\|_V^2 \right)^{\frac{1}{2}} .
\end{aligned}$$

Take v_i such that (3.8) is valid. By combining (4.3)–(4.4), we get

$$\begin{aligned}
&\langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle \\
&\le C_2 (2 + \epsilon_0) \left(\sum_{i=1}^{m} \|u_i^{n+1} - u_i^n\|_V^2 \right)^{\frac{1}{2}} \cdot C_1 \|u^{n+1} - u\|_V \\
&\le C_1 C_2 (2 + \epsilon_0) \left[\frac{4}{K} \left(F(u^n) - F(u^{n+1}) \right) \right]^{\frac{1}{2}} \cdot \|u^{n+1} - u\|_V .
\end{aligned} \tag{4.5}$$

Similar to the reasonings for (3.10), one deduces from (4.5)

$$\begin{aligned}
&K^2 \langle F'(u^{n+1}) - F'(u), u^{n+1} - u \rangle \\
&\le 4 {C_s}^2 [F(u^n) - F(u^{n+1})] \\
&= 4 {C_s}^2 \left[F(u^n) - F(u) + F(u) - F(u^{n+1}) \right] .
\end{aligned}$$

The rest of the proof is the same as for Theorem 3.1. ∎

5. Conclusion. Using suitable coloring procedures, both domain decomposition methods and multigrid methods can be regarded as space decomposition techniques; see Tai [12] and Tai and Espedal [13], [14]. The proposed algorithms have been tested both for domain decomposition methods and multigrid methods. In report [13], some numerical experiments for linear elliptic and interface problems are given for a two-level domain decomposition method. Some preliminary experiments for nonlinear elliptic problems are also carried out in report [13] by a two-level domain decomposition method. Numerical experiments concerning the use of multigrid methods for nonlinear problems will be given in later reports.

Special note. In report [2] by Dryja and Hackbusch, an elegant convergence proof was given for the additive algorithm under weaker conditions. In [2], no assumptions on the subspaces are required and it was shown that the asymptotic convergence rate of the additive nonlinear space decomposition algorithm is the same as the rate of the additive linear algorithm for the corresponding linearized problem. However, for certain nonlinear problems, the linearized problem does not satisfy the ellipticity condition and the convergence rate for the corresponding linearized problem may not be obtained by existing theories.

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REFERENCES

[1] J. Cea, *Optimisation – théorie et algorithmes*, Dunod, Paris, 1971.

[2] M. Dryja and W. Hackbusch, *On the Nonlinear Domain Decomposition Method*, Report 95-5, Christian-Albrechts-Universität zu Kiel, November 1995.

[3] M. Dryja and O. B. Widlund, *Domain decomposition algorithms with small overlap*, SIAM J. Sci. Comput., 15 (1994), pp. 604–620.

[4] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North–Holland, Amsterdam, 1976.

[5] W. Hackbusch, *Iterative Methods for Large Sparse Linear Systems*, Springer, Heidelberg, 1993.

[6] B. F. Smith, P. E. Bjørstad, and W. D. Gropp, *Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations*, Cambridge University Press, Cambridge, 1996.

[7] X.-C. Tai, *Parallel Function Decomposition and Space Decomposition Methods with Applications to Optimisation, Splitting and Domain Decomposition*, Preprint 231-1992, Institut für Mathematik, Technische Universität Graz, 1992; see also http://www.mi.uib.no/˜tai.

[8] X.-C. Tai, *Parallel function and space decomposition methods*, in Finite Element Methods, Fifty Years of the Courant Element, Lecture Notes in Pure and Appl. Math. 164, P. Neittaanmäki, ed., Marcel Dekker, New York, 1994, pp. 421–432.

[9] X.-C. Tai, *Domain decomposition for linear and nonlinear elliptic problems via function or space decomposition*, in Domain Decomposition Methods in Scientific and Engineering Computing, Proc. of the 7th International Conference on Domain Decomposition, Pennsylvania State University, 1993, D. Keyes and J. Xu, eds., American Mathematical Society, Providence, RI, 1995, pp. 355–360.

[10] X.-C. Tai, *Parallel function and space decomposition methods—Part* I. *Function decomposition*, Beijing Math., 1 (1995), pp. 104–134.

[11] X.-C. Tai, *Parallel function and space decomposition methods—Part* II. *Space decomposition*, Beijing Math., 1 (1995), pp. 135–152.

[12] X.-C. Tai, *A space decomposition method for parabolic problems*, Numer. Methods Partial Differential Equations, 14 (1998), pp. 27–46.

[13] X.-C. Tai and M. Espedal, *Rate of Convergence of a Space Decomposition Method and Applications to Linear and Nonlinear Elliptic Problems*, Report 103, Mathematics Department, University of Bergen, 1996; see also http://www.mi.uib.no/˜tai.

[14] X.-C. Tai and M. Espedal, *A space decomposition method for minimization problems*, in Proc. of the 9th Domain Decomposition Conference, June 3–8, 1996, Ullensvang, Norway, P. Bjørstad, M. Espedal, and D. Keyes, eds., John Wiley and Sons, to appear; see also http:// www.mi.uib.no /˜tai.

[15] O. Widlund, *Some Schwartz methods for symmetric and nonsymmetric elliptic problems*, in Proc. 5th International Symposium on Domain Decomposition Methods for Partial Differential Equations, Norfolk, VA, May 1991, SIAM, Philadelphia, 1992.

[16] J. C. Xu, *Iteration methods by space decomposition and subspace correction*, SIAM Rev., 34 (1992), pp. 581–613.