<table>
<thead>
<tr>
<th>Title</th>
<th>A simplified model of catenary action in reinforced concrete frames under axially restrained conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Pham, Anh Tuan; Tan, Kang Hai</td>
</tr>
<tr>
<td>Citation</td>
<td>Pham, A. T., &amp; Tan, K. H. (2017). A simplified model of catenary action in reinforced concrete frames under axially restrained conditions. Magazine of Concrete Research, 69(21), 1115-1134. doi:10.1680/jmacr.17.00009</td>
</tr>
<tr>
<td>Date</td>
<td>2017</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10220/46148">http://hdl.handle.net/10220/46148</a></td>
</tr>
<tr>
<td>Rights</td>
<td>© 2017 ICE Publishing. This paper was published in Magazine of Concrete Research and is made available as an electronic reprint (preprint) with permission of ICE Publishing. The published version is available at: [<a href="http://dx.doi.org/10.1680/jmacr.17.00009">http://dx.doi.org/10.1680/jmacr.17.00009</a>]. One print or electronic copy may be made for personal use only. Systematic or multiple reproduction, distribution to multiple locations via electronic or other means, duplication of any material in this paper for a fee or for commercial purposes, or modification of the content of the paper is prohibited and is subject to penalties under law.</td>
</tr>
</tbody>
</table>
A simplified model of catenary action in reinforced concrete frames under axially restrained conditions

Anh Tuan Pham
PhD Student, School of Civil and Environmental Engineering, Nanyang Technological University, Singapore (corresponding author: atpham@ntu.edu.sg)

Kang Hai Tan
Professor, School of Civil and Environmental Engineering, Nanyang Technological University, Singapore

From previous experimental studies on missing column scenarios, catenary action could provide beneficial effects on structures during load redistribution after a target column is suddenly removed under an alternative load path procedure to assess structural resistance to progressive collapse. However, full-scale structural tests are costly and time consuming. Equally, finite-element models require in-depth expertise and large computing resources. Practising engineers are not amenable to this approach for preliminary sizing of members or for a quick assessment of structural resilience. To address this need, this paper provides a simplified semi-analytical model for predicting structural response under a column loss scenario. The model is constructed based on finite-element simulations and is verified by test results. The simplified model idealises the structural response of reinforced concrete beam–column sub-assemblages under column loss scenarios as a piecewise multi-linear curve. It shows good agreement with published test results and validated finite-element simulations.

Notation

- **A**: area of beam section, \( A = A_s + A_k \)
- **A_s**: sectional area of concrete
- **A_k**: sectional area of total longitudinal rebars at the end joint
- **A_{top}**: sectional area of top longitudinal rebars in the beam near the end joint
- **c_1, c_2**: locations of curtailment region
- **d**: distance from extreme compression fibre of concrete to tension reinforcement
- **d_{A_k}, d_{B_k}, d_{C_k}, d_{D_k}, d_{E_k}, d_{F_k}, d_{G_k}**: displacements at points A, B, C, D, E, F, G
- **d_{beam}**, **d_{mid}, d_{end}**: concrete beam depth and end and mid joints
- **d'**: distance from extreme compression fibre of concrete to compression reinforcement
- **E_c**: elastic modulus of concrete, \( E_c = 4.7(f_{ck})^{1/2} \) GPa
- **E_{eq}**: equivalent elastic modulus of beam section, \( E_{eq} = (E_c A_c + E_s A_s)(A_c + A_s) \)
- **E_1**: the flexural stiffness of the beam of the removed column when the sub-assemblage is non-symmetrically designed
- **E_{top}**: elastic modulus of longitudinal rebars
- **F_{top}**: ultimate tensile force of the top reinforcement, \( F_{top} = f_u A_{top} \)
- **F'_{top}**: yield force of the top reinforcement, \( F'_{top} = f_y A_{top} \)
- **f_{ck}**: concrete compressive strength
- **f_u**: ultimate strength of longitudinal rebars
- **f_y**: yield strength of longitudinal rebars
- **H, \( H_{max} \)**: horizontal reaction and maximum horizontal reaction of the sub-assemblage
- **I_g**, **K_k**, **K_s**, **K_{cex}**, **K_{bot}**, **K_{top}**, **L_{n}**, **L_{n\text{mid}}**, **L_{n\text{end}}**, **L_{top}**, **L_{op}**, **L_{end}**, **L_{span}**: moment of inertia of the gross section area, horizontal spring of the end-joint restraint, rotational spring of the end-joint restraint, relative bending stiffness, normalised horizontal stiffness, normalised rotational stiffness, span–depth ratio, steel–to–concrete stiffness ratio, clear span of the single beam, plastic hinge length at the end joint (for calculation of \( d_{B_k} \)), plastic hinge lengths at the middle and end joints (for calculation of \( d_{D_k} \)), sagging moment capacities of beam section near the middle joint and hogging moment capacity of beam section near the end joint, applied loads at points A, B, C, D, E, F, G
- **P_{left}, P_{right}, P_{total}**: simplified responses of the ‘left’, ‘right’ and total systems
- **R_s**, **a_1, a_2, \( \beta_1, \beta_2 \)**: capacity of the horizontal restraint, variables of the proposed model for sub-assemblages
Assemblages, catenary action was observed to start kicking in – in the previous quasi-static tests on beam mechanisms to prevent the structure from collapse. While compressive arch action (at small deflections) is an efficient means to enhance the maximum flexural capacity of the beam section, catenary action requires contributions of beam deflection within segments XY and YZ, and arch action and subsequent catenary action, besides flexural contributions of upper-bound mechanisms, such as compressive reinforcement. The use of sophisticated numerical simulations, either with physics-based solid elements or with simplified fibre elements incorporating a component-based joint model (Bao et al., 2014; Pham et al., 2016; Yu and Tan, 2013a), can provide reasonable predictions compared to actual response. Nonetheless, users are required to possess a high level of modelling skill, knowledge of the finite-element method (FEM) and vast computing resources when resorting to numerical simulations. In the conceptual design stage where engineers need to explore various options to arrive at reasonable and economical solutions, a well-validated and simple approach is preferred instead of detailed finite-element simulations. In this regard, Li et al. (2014) proposed equations for a straight type of catenary mechanism under point load application at the middle joint, defining the relationship between applied loads and corresponding deformations. However, Li’s model assumes perfectly rigid horizontal restraints at both sides of the two-span beam, which is difficult to achieve in practice. Other analytical methods have been proposed (Park and Gamble, 2000; Valipour et al., 2013; Yu and Tan, 2014) to predict the maximum structural capacity of beams or one-way slabs under axial compression condition, but they are not applicable for the catenary action phase. In addition, as progressive collapse is a dynamic phenomenon, strain energy from structural response, rather than the peak static resistance, is more important in assessing the maximum dynamic load and corresponding deformation. To study the behaviour of RC substructures under the single column removal scenario, a physics-based model applying the FEM from a related study (Pham et al., 2016) is employed to investigate important variables affecting catenary action. Based on actual tests and numerical predictions, a semi-analytical

\[ \Delta_c \] beam deformation caused by constant yield curvature at curtailment region
\[ \Delta_u \] beam deformation caused by ultimate curvature at plastic hinge
\[ \Delta_{u\text{mid}} \] beam deformation caused by ultimate curvature at plastic hinge (segment XY)
\[ \Delta_{XY}, \Delta_{YZ} \] contributions of beam deflection within segments XY and YZ
\[ \Delta_y \] beam deformation caused by linear distribution of curvature along the beam
\[ \Delta_{y\text{mid}} \] beam deformation caused by linear distribution of curvature along segment XY
\[ \delta \] translational deformation at the end joint
\[ \varepsilon_u \] ultimate strain of longitudinal reinforcement
\[ \varepsilon_y \] yield strain of longitudinal reinforcement
\[ \theta, \psi \] rotational deformation at the end joint and rotational angle
\[ \phi, \phi_u \] yield curvature of concrete beam section at the curtailment region under hogging moment
\[ \phi_y, \phi_u \] yield and ultimate curvatures of concrete beam section at the end joint under hogging moment
\[ \phi'_y, \phi'_u \] yield and ultimate curvatures of concrete beam section at the middle joint under sagging moment

**Introduction**

Although progressive collapse of buildings has been identified as a very low-probability event, its catastrophic consequence through the loss of human lives has raised significant research interest during the past two decades. In terms of design guides for progressive collapse mitigation (DOD, 2013; GSA, 2003), the assumption of a single column being suddenly removed is generally accepted in the engineering community. The surviving structure will be analysed for double-span bridging effect and alternative paths for possible load redistribution. Recent experimental work (Lim et al., 2015; Lu et al., 2016; Ren et al., 2016; Saddek et al., 2011; Sasani and Kropelnicki, 2008; Su et al., 2009; Valipour et al., 2013; Yi et al., 2008; Yu and Tan, 2013a, 2013b) and numerical studies (Bao et al., 2014; Pham et al., 2016; Qian et al., 2015; Valipour and Foster, 2010) on reinforced concrete (RC) beam–column structures under the single column removal scenario have confirmed the contribution of upper-bound mechanisms, such as compressive arch action and subsequent catenary action, besides flexural mechanisms. While compressive arch action (at small deformations) is an efficient means to enhance the maximum flexural capacity of the beam section, catenary action requires large deformations and is considered as the last line of defence mechanism to prevent the structure from collapse.

In the previous quasi-static tests on beam–column sub-assemblages, catenary action was observed to start kicking in when the middle-joint displacement of a two-span beam had reached around one beam depth. Subsequently, as the deflection kept increasing, beam bottom rebars near the middle joint started fracturing, leading to a sudden drop in load capacity. The displacement corresponding to this failure was about 1/8 to 1/11 of the single clear span in the tests of Yu and Tan (2013a, 2013b) and Lim et al. (2015); whereas in Yi et al. (2008) and Saddek et al. (2011), that ratio was 1/6 and 1/5, respectively. Suffice to say, the fracture of bottom rebars is significantly influenced by span–depth ratio, rebar arrangement and ultimate strain of longitudinal reinforcement. If the downward movement were to be continued after the fracture of bottom rebars (Lim et al., 2015; Sasani and Kropelnicki, 2008; Yu and Tan, 2013a, 2013b), the load capacity would increase again and the final load resistance might be greater than both the peak strength provided by compressive arch action and the catenary action stage before the fracture of bottom rebars.

The behaviour of RC beam–column structures against the column loss scenario involves various non-linear phenomena such as changing of geometry under large displacements, crushing of concrete at large strain, yielding and post-yielding behaviour of reinforcement. The use of sophisticated numerical simulations, either with physics-based solid elements or with simplified fibre elements incorporating a component-based joint model (Bao et al., 2014; Pham et al., 2016; Yu and Tan, 2013a), can provide reasonable predictions compared to actual response. Nonetheless, users are required to possess a high level of modelling skill, knowledge of the finite-element method (FEM) and vast computing resources when resorting to numerical simulations. In the conceptual design stage where engineers need to explore various options to arrive at reasonable and economical solutions, a well-validated and simple approach is preferred instead of detailed finite-element simulations. In this regard, Li et al. (2014) proposed equations for a straight type of catenary mechanism under point load application at the middle joint, defining the relationship between applied loads and corresponding deformations. However, Li’s model assumes perfectly rigid horizontal restraints at both sides of the two-span beam, which is difficult to achieve in practice. Other analytical methods have been proposed (Park and Gamble, 2000; Valipour et al., 2013; Yu and Tan, 2014) to predict the maximum structural capacity of beams or one-way slabs under axial compression condition, but they are not applicable for the catenary action phase. In addition, as progressive collapse is a dynamic phenomenon, strain energy from structural response, rather than the peak static resistance, is more important in assessing the maximum dynamic load and corresponding deformation.

To study the behaviour of RC substructures under the single column removal scenario, a physics-based model applying the FEM from a related study (Pham et al., 2016) is employed to investigate important variables affecting catenary action. Based on actual tests and numerical predictions, a semi-analytical...
A simplified model of catenary action in reinforced concrete frames under axially restrained conditions
Pham and Tan

model is proposed to provide a practical and yet reliable procedure to predict the overall response of two-dimensional RC beam–column structures. This simplified approach takes into account the effect of partial restraint, as well as structural response before and after the fracture of bottom rebars in the double-span beam. Equivalent dynamic capacity is also considered in the proposed model by using a simplified energy-based method (Izzuddin and Nethercot, 2009; Izzuddin et al., 2008). Published test results (Lim et al., 2015; Yu and Tan, 2013a, 2013b) are used to validate the proposed model and to assess its conservatism for design purposes.

Research significance

The paper presents a semi-analytical approach for RC beam–column substructures under progressive collapse, taking into account the effect of catenary action generated from horizontal restraint conditions. The purpose of this method is to provide a quick and reliable tool to practising engineers for preliminary sizing of structural members in a RC building subjected to the threat of single column removal.

Structural investigations on catenary action

Proposed simplified structural response under supporting column removal

From the test results (Lim et al., 2015; Sasani and Kropelnicki, 2008; Yu and Tan, 2013a, 2013b), a typical structural response with adequate horizontal restraints can be presented as in Figure 1(a) by the curve OABCDEFG. It should be noted that curve OA represents the initial stage in which steel reinforcement behaves elastically and there is little concrete damage. At point A, longitudinal rebars in the beam–column joints start yielding and cracks start developing within the concrete tensile regions if the section is under-reinforced. The first peak at point B denotes the maximum flexural capacity considering compressive arch action enhancement. Thereafter, vertical load gradually decreases due to crushing of concrete at the compressive zones, as well as a reduction of compressive arch action. Point C in Figure 1(a) marks the onset of catenary action and usually coincides with a vertical deflection proportional to one beam depth. Within the ascending phase CD, mobilisation of catenary action is identified by a change of axial force in beams from compression to tension. The sudden drop DE marks the fracture of the bottom longitudinal rebars in the beams at the middle joint region. For symmetrical structures, it is conservatively assumed that the bottom rebars from both sides of the middle joint fracture simultaneously. In actual tests, bottom rebars from two sides of the joint could fracture at different times due to imperfections in the symmetric condition. After which, if the middle-joint displacement continues to increase then the top reinforcing bars will contribute to the residual capacity of catenary mechanism until they also fracture.

To provide a simplified response representing the structural capacity of beam-column components, the typical response OABCDEFG is represented by a polyline with controlling points from A to G (Figure 1(b)). For parts BC and CD, additional intermediate points C1 and D1 are proposed to make these curves fit better with the actual response in Figure 1(a). For horizontal lines C1D1 and EF, it is assumed that the structural capacity remains unchanged while the displacement keeps increasing. Point C1 is defined so that C is the mid-point of C1D1. A, B, C, D, E and G are considered as the primary points of the proposed curve, which can be directly obtained from structural analysis. However, C1, D1 and F are secondary points, which are dependent on the locations of the primary points.

Proposed normalised factors for sub-assemblage behaviour

A series of normalised parameters is used to investigate the effects of important geometrical, material and external restraint factors on the behaviour of beam–column structures. To make the problem tractable, it is assumed that there is symmetry in loading, material and boundary conditions; thus,

Figure 1. Simplified response of beam–column structures: (a) actual response; (b) simplified piecewise response

Downloaded by [ Nanyang Technological University] on [11/09/18]. Copyright © ICE Publishing, all rights reserved.
only one half of the double-span beam is considered (Figure 2). The middle joint has vertical deflection as the only degree of freedom, while its horizontal and rotational degrees of freedom are restrained owing to symmetry at the centre. For the end joint, a pin-on-roller support with an axial spring ($K_A$) and a rotational spring ($K_R$) are considered as boundary conditions. The horizontal force from the axial spring is denoted as $H$. Such an arrangement allows the consideration of boundary effects (or imperfections) incorporating both horizontal and rotational restraints at the end joint. In Figure 2, $E_{eq}$ is the equivalent elastic modulus of the beam section; $\delta$ and $\theta$ are the translational and rotational deformations at the end joint, respectively. A semi-analytical model based on the simplified response shown in Figure 1(b) is developed employing the normalised factors, as presented in the following.

The first normalised parameter is the span–depth ratio $k_{span}$ defined by Equation 1. This factor describes the governing behaviour of the structure, which is either pure flexure or flexure with shear.

$$1. \quad k_{span} = \frac{L_n}{d_{beam}}$$

where $L_n$ is the single clear span before the column is removed and $d_{beam}$ is the overall beam depth.

The normalised horizontal stiffness, $k_{hor}$, which describes the axial stiffness of the beam relative to adjacent lateral restraints, is defined by Equation 2. The value of $k_{hor}$ depends on the axial stiffness of adjacent members to withstand a horizontal force, which is sensitive to the number of bays next to the removed column. A small value of $k_{hor}$ denotes a weak boundary restraint compared to the beam axial stiffness. A previous study by Pham et al. (2016) shows that if $k_{hor}$ is less than 0.011, catenary action in beams is unlikely to be mobilised.

$$2. \quad k_{hor} = \frac{K_AL_n}{E_{eq}(A_c + A_s)}$$

3. $E_{eq} = (E_cA_c + E_sA_s)/(A_c + A_s)$

where $E_c$, $A_c$ are, respectively, the elastic modulus and the total area of uncracked concrete; $E_s$, $A_s$ are, respectively, the elastic modulus and the total area of steel reinforcement at the end joint; and $E_{eq}$ is the equivalent elastic modulus of the beam section.

The normalised bending stiffness $k_{flex}$, which represents the flexural capacity of the beam compared to its flexural stiffness, is given by Equation 4. This factor relates the bending moment resistance of the section to the beam elastic stiffness. For example, a high value of $k_{flex}$ denotes that the beam is highly reinforced.

$$4. \quad k_{flex} = \frac{M_{mid} + M_{end}}{E_{eq}I_g/L_n}$$

where $I_g$ is the moment of inertia of the uncracked concrete cross-section; $M_{mid}$ and $M_{end}$ are, respectively, the sagging and
A simplified model of catenary action in reinforced concrete frames under axially restrained conditions
Pham and Tan

the hogging plastic moment capacities (yielding state) of the beam section at the middle joint and the end joint. $M_{\text{mid}}$ and $M_{\text{end}}$ can be obtained from the simplified equation proposed by Paulay and Priestley (1992)

\[ M = (d - d')A_s^t \sigma_f \]

where $d$ and $d'$ are the distances from the extreme compression fibre of concrete to the centre of tension and compression reinforcement, respectively; $A_s^t$ and $\sigma_f$ are the cross-sectional area and yield strength of tension longitudinal rebars, respectively. Equation 5 is applicable for under-reinforced sections in the absence of beam axial forces.

The steel-to-concrete stiffness ratio, $k_{\text{top bar}}$, relates the axial stiffness of the top reinforcement to that of the concrete material in the beam section at the end joint (Equation 6). A small reinforcement ratio yields a small $k_{\text{top bar}}$ or vice versa.

\[ k_{\text{top bar}} = \frac{E_s A_s^{\text{top}}}{E_c A_s} \]

where $A_s^{\text{top}}$ is the area of top reinforcement in the beam near the end joint.

Finally, Equation 7 specifies the normalised rotational stiffness $k_{\text{rot}}$ of the end joint. It is a ratio of the rotational stiffness $K_R$ of the boundary condition to the beam elastic stiffness. If the end restraint has low rotational stiffness, it will limit the capacity of compressive arch action.

\[ k_{\text{rot}} = \frac{K_R}{E_{\text{eq}}I_g / L_{\text{eq}}} \]

Investigations on structural behaviour
To investigate the effects of the five normalised variables on the overall response of the sub-assemblage structures, a physics-based FEM model developed in a related study (Pham et al., 2016), which has been validated by quasi-static tests (Yu and Tan, 2013a, 2013b), was employed. This model was constructed using an explicit finite-element software LS-Dyna (Hallquist, 2007). It employed eight-node solid elements with reduced integration scheme to simulate concrete materials, and two-node Hughes–Liu beam elements with 2 x 2 Gauss quadrature integration to simulate longitudinal reinforcement. A continuous surface cap model Mat_159 was employed to simulate the behaviour of the concrete material. It was shown from previous studies (Bao et al., 2014) that this material model can simulate actual responses and failure modes of RC structures against progressive collapse under quasi-static condition. An isotropic elasto-plastic material model ‘mat piecewise linear plasticity’ (Mat_024) was used for the steel reinforcement. Mesh sizes of 12.5 x 12.5 x 12.5 mm$^3$ (joint regions) and 12.5 x 12.5 x 25 mm$^3$ (other regions) were used for solid elements, whereas a mesh size of 25 mm was used for the beam elements. Although element erosion is not a physical phenomenon for concrete material, this attribute allows modelling of spalling and separation of concrete under extremely high tensile force. In the FEM model, the criterion for elements to be eroded was based on the maximum principal strain value. A value of 0.1 was chosen for the maximum principal strain as the erosion criterion. Element erosion was also applied for reinforcement material as it reaches the maximum tensile strain to represent fracture of rebars. Bond-slip behaviour between concrete and reinforcement was modelled applying a one-dimensional contact function (Contact_1D) between concrete solid elements and rebar beam elements (Bao et al., 2014; Shi et al., 2009). Figure 3 illustrates the one-half FEM model; Figure 4 shows some validation results of the model with experimental data from Yu and Tan (2013b).

Based on the validated model, parametric studies were conducted for the five factors ranging from $k_{\text{span}}$ to $k_{\text{rot}}$ and concrete strength; the results are shown in Figure 5. The purpose of the comparison is to find critical parameters that should be incorporated into the simplified model for catenary action. While direct comparisons are made for $k_{\text{span}}$, $k_{\text{hor}}$ and $k_{\text{rot}}$, the
Influences of $k_{\text{flex}}$ and $k_{\text{topbar}}$ are indirectly illustrated by comparing different ratios of top and bottom rebars at the end joint. From both numerical simulations (Pham et al., 2016) presented in Figure 5(a) and test results from Yu and Tan (2013b), it is shown that, if $k_{\text{span}}$ is smaller than 7, then after the first peak, the structural response will decrease quickly and there is no ascending phase; that is, flexural and compressive arch mechanisms will govern the response and shear dominates the failure instead of catenary action.

In terms of axial restraint, five values of $k_{\text{hor}}$ are compared, ranging from 0.011 to 2.271 (Pham et al., 2016). The results presented in Figure 5(b) clearly show that $k_{\text{hor}}$ has a significant effect on structural response, as well as on the development of the catenary mechanism in the beams. In this comparison, the value of $k_{\text{hor}} \geq 2$ can be considered as the perfectly rigid condition. However, if the horizontal restraint is too weak, say $k_{\text{hor}} < 0.01$ (for $k_{\text{span}} = 11$), the catenary mechanism cannot be efficiently mobilised. This observation agrees with Yu and Tan
Regarding the effect of longitudinal reinforcement on catenary action, Figures 5(c) and 5(d) show that, although the top rebars significantly affect the structural response before and after the fracture of the bottom rebars, the bottom rebars only affect the behaviour prior to their fracture at the middle joint (from point A to D). After the fracture, the load-carrying capacity drops to point E and the bottom rebars can no longer contribute to catenary action.

The normalised rotational stiffness, \( k_{\text{rot}} \), is studied in the range of 0–0.4 to the perfectly rigid condition. Results show that rotational restraint stiffness has less effect on structural response after the first peak (Figure 5(e)). To gain a thorough understanding of the effect of \( k_{\text{rot}} \) on catenary action, a more extensive investigation is made in which the span–depth ratio \( k_{\text{span}} \) is also considered. Five values of \( k_{\text{span}} \) (7, 9, 11, 12.8 and 15) and five values of \( k_{\text{rot}} \) (0.04, 0.41, 0.85, 4.36 and 8.9) are taken into account simultaneously. It is found that the effect of rotational stiffness on the curve from C1 to G (Figure 1(b)) is almost constant if both \( k_{\text{span}} \) and \( k_{\text{rot}} \) satisfy Equation 8. Moreover, the case of beam–column sub-assemblies with very little rotational restraint at the end supports (\( k_{\text{rot}} < 0.2 \)) is not realistic for concrete structures. Hence, it is concluded that the development of catenary action does not depend on \( k_{\text{rot}} \). Considering concrete strength, it is observed that \( f_{\text{ck}} \) greatly affects the peak compressive arch action (point B) but has limited effect on the remaining points from C to G (Figure 5(f)). In this phase, catenary action governs the response and is only dependent on \( k_{\text{span}}, k_{\text{hor}} \) and \( k_{\text{topbar}} \). Connection gaps, which exist in actual tests, can remarkably affect the mobilisation of compressive arch action (Yu and Tan, 2014). Nevertheless, it has very little influence on residual response after all the bottom rebars have fractured (part EG). Moreover, this boundary imperfection only occurs in the laboratory tests and has little physical meaning for cast in situ concrete structures. In summary, all the four important factors \( (k_{\text{span}}, k_{\text{hor}}, k_{\text{flex}}, k_{\text{topbar}}) \) have been incorporated into the proposed model, while the three other less important ones \( (k_{\text{rot}}, \text{concrete strength and connection gaps}) \) can be ignored.

### Simplified semi-analytical model on catenary action

Based on the comparison in the previous section, a comprehensive numerical parametric study on catenary action is employed to obtain a semi-analytical approach for predicting structural response. All the investigated parameters are summarised in Table 1. A total number of 120 case studies are considered, including four types of rebar arrangement, five different spans and six values of horizontal stiffness. The reinforcement contents are chosen within the practical range of beam design. Constant parameters include the beam cross-sectional dimensions, concrete grade and reinforcement strength. Perfect rotational restraint and zero connection gap are assumed, as they have little effect on catenary action. From numerical predictions, the values of controlling points (from A to G of Figure 1(b)) can be defined and semi-analytical equations are proposed as follows.

#### Point A

Assuming yielding of both the top rebars near the end joint and the bottom rebars near the middle joint, plastic hinge theory is employed to predict the capacity at point A (Figure 1(b)) as shown in Equation 9. Based on numerical results of displacement when structural capacity reaches the value of \( P_A \), the displacement at point A \( (d_A) \) is determined by Equation 10. In this equation, \( \alpha \) is a coefficient that considers the effect of horizontal restraint stiffness \( (k_{\text{hor}}) \) defined by Equation 11. Nonetheless, since the proposed model assumes rigid condition for rotational restraint, it tends to overestimate the stiffness at point A and yields a smaller value of \( \phi \) compared to test results.

### Table 1. Numerical parametric studies

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concrete</td>
<td>( f_{\text{ck}} = 30 \text{ MPa} )</td>
</tr>
<tr>
<td>Beam dimension</td>
<td>150 mm wide by 250 mm depth</td>
</tr>
<tr>
<td>Reinforcement</td>
<td>( f_y = 520 \text{ MPa}, f_y = 620 \text{ MPa}, \varepsilon_y = 0.027%, \varepsilon_u = 12% )</td>
</tr>
<tr>
<td>Top rebars</td>
<td>3H8 3H10 3H13 3H16</td>
</tr>
<tr>
<td>Top rebars mm²/%)</td>
<td>151 (0.45%) 236 (0.7%) 398 (1.2%) 603 (1.8%)</td>
</tr>
<tr>
<td>Bottom rebars</td>
<td>2T7 2T10 2T10</td>
</tr>
<tr>
<td>Bottom rebars mm²/%)</td>
<td>101 (0.3%) 157 (0.47%) 157 (0.47%) 157 (0.47%)</td>
</tr>
<tr>
<td>Span: mm</td>
<td>1750/2250/2750/3200/3750</td>
</tr>
<tr>
<td>Horizontal stiffness: kN/m</td>
<td>( 5 \times 10^3 ) ( 10^4 ) ( 2 \times 10^4 ) ( 4 \times 10^4 ) ( 10^5 ) ( 10^6 )</td>
</tr>
</tbody>
</table>
where $M_{\text{mid}}$ and $M_{\text{end}}$ are plastic moment capacities of beams at the middle and the end joints, respectively, computed following Equation 5.

10. $d_a = rac{a_1(M_{\text{mid}} + M_{\text{end}})I_n^2}{3EcI_g}$

11. $a_1 = \frac{1}{(1.25 + k_{\text{hor}})} + 0.56$

Point B
In this investigation, the analytical model proposed by Yu and Tan (2014) is employed to define the load-carrying capacity and the corresponding deformation at B. This approach considers imperfections in boundary conditions, such as connection gaps at restraints, and stiffness of both horizontal and rotational restraints.

Point C
At point C (Figure 1(b)), the effect of compressive arch action almost ceases and catenary action is about to be mobilised. The applied load at C, $P_c$, is assumed to be equal to $P_A$ (flexural strength). The vertical deflection at C, $d_c$, is assumed to be proportional to $d_{\text{beam}}$, and depends on $k_{\text{span}}$ and $k_{\text{hor}}$. A large value of $k_{\text{span}}$ and a small value of $k_{\text{hor}}$ lead to a large value of $d_c$. On the other hand, a small $d_c$ denotes the dominant influence of arching action. Consequently, Equations 12 and 13 are used together to define $d_c$.

12. $d_c = \alpha_2 d_{\text{beam}}$

where $\alpha_2$ is defined by Equation 13

13. $\alpha_2 = (0.4 + 0.065k_{\text{span}} - 12k_{\text{-flex}}) \left[ \frac{1}{(1.07 + k_{\text{hor}})^{0.8}} + 1 \right]$

Line FG
From the results of parametric studies, it is shown that the gradient of line FG greatly depends on the yield force of the top reinforcement $F_{\text{top}}$ at the end joint and the span-depth ratio $k_{\text{span}}$, while it is less influenced by axial restraint ratio $k_{\text{hor}}$. Nonetheless, the position of FG is dominated by the stiffness of axial restraint. Therefore, a linear function for line FG is proposed herein

14. $P_{FG}(d) = \beta_1 F_{\text{top}}^{\text{y}} \left( d - \beta_2 d_{\text{beam}} \right)$

$\beta_1$ and $\beta_2$ are specified by Equations 15 and 16, respectively.

15. $\beta_1 = (1.2 + 0.24k_{\text{span}} - 9k_{\text{topbar}})$

16. $\beta_2 = (-0.45 + 0.125k_{\text{span}} - 3.6k_{\text{topbar}}) \times \left[ \frac{1}{(1.05 + k_{\text{hor}})^{0.8}} + 1 \right]$

Line $D_1D$
Observations from both the test and numerical results show that line $D_1D$ tends to be parallel to FG (Figure 1(b)). Hence, for simplicity it is assumed that they both share the same gradient, which mostly depends on the yield strength $F_{\text{top}}$ of the top reinforcement (Equation 14). The reduction in terms of load-carrying capacity between $D_1D$ and FG is due to positive yield moment at the middle joint $M_{\text{mid}}$, which terminates after all the bottom rebars have fractured at D. Equation 17 shows the load capacity of the structure within $D_1D$.

17. $P_{D_1D}(d) = P_{FG}(d) + \frac{2M_{\text{mid}}}{I_n}$

= $\beta_1 F_{\text{top}}^{\text{y}} \left( d - \beta_2 d_{\text{beam}} \right) + \frac{2M_{\text{mid}}}{I_n}$

After obtaining the separate functions of FG and $D_1D$, the load-carrying capacities at points D and G can be defined by specifying corresponding displacements of these points. In the next section, an analytical approach based on plastic hinge length will be used to determine the corresponding deflections $d_D$ and $d_G$.

Displacements at G and D
The deformation at point D ($d_D$) is calculated based on the deflection caused by segments $XY$ ($\Delta_{XY}$) and $YZ$ ($\Delta_{YZ}$) shown in Figure 6(a), in which Y is the contra-flexural point. In segment $XY$, a plastic hinge is assumed to have fully developed in the beam section near the middle joint with a plastic hinge length $L_{\text{sp}}$ (Figure 6(b)). Beam curvature within the plastic hinge length is assumed to reach the ultimate value $\phi_u$, whereas the beam curvature along $Y$ is uniformly distributed from $\phi'_u$ (yield value) to zero. Hence, $\Delta_{XY} = \Delta_{u}^{\text{mid}} + \Delta_{u}^{\text{mid}}$, in which $\Delta_{u}^{\text{mid}}$ is due to ultimate curvature $\phi_u$ within $L_{\text{sp}}$, and $\Delta_{u}^{\text{mid}}$ comes from linear distribution of curvature along $XY$.

In segment $YZ$, the plastic hinge is partially developed within a plastic hinge length $L_{\text{sp}}$ (Figure 6(b)). Along $YZ$, beam curvature is assumed to be linear from $\phi_u$ (yield value) to zero. Since the plastic hinge is not yet fully developed, not all of the beam curvature within $L_{\text{sp}}$ reaches ultimate value $\phi_u$ (Figure 6(b)). Therefore, $\Delta_{YZ}$ is defined based on $\Delta_{XY}$, noting that both segments have the same rotational angle $\theta_D$ (Figure 6(a)).

When the structure reaches the maximum capacity at G and finally collapses at the two ends, a plastic hinge is assumed to have fully developed at the end joint. Near the middle joint,
however, concrete spalling penetrates through most of the beam sections, except at the region surrounding the top rebars. Thus, the middle joint can be considered as a pin (Figure 7(a)).

Within the region of top rebar cut-off points (curtailment regions), several flexural cracks are observed owing to negative bending moment reaching the yield capacity of the beam section. Total displacement $\Delta_d$ is divided into three components, that is $\Delta_d = \Delta_y + \Delta_u + \Delta_s$. The first component $\Delta_y$ is due to linear distribution of curvature along the beam: from zero at the middle joint to $\phi_y$ at the end joint. The second component $\Delta_u$ is
the displacement caused by ultimate curvature $\phi_u$ within the ideal plastic hinge length $L_{sp}$ near the end joint. The last term $\Delta_e$ comes from curvature distribution along the curtailment region $L_e$ where some top rebars are cut off. Based on test results and numerical studies, it is assumed that the remaining top rebars along the curtailment region attain the yield strength. Also the curvature reaches the yield value $\phi^\ast$ and is constant in this part (Figure 7(b)).

To calculate the displacements $d_1$ and $d_2$, a method proposed by Paulay and Priestley (1992) for cantilever columns under horizontal displacements is applied. The detailed derivations of $d_2$ and $d_1$ are presented in Appendix 1. Afterwards, the load capacities at $G$ and $D$ are obtained from Equations 14 and 17, respectively.

**Point E**

After the fracture of bottom rebars at both sides of the middle joint, the vertical load capacity has a sudden drop from $D$ to $E$. Results obtained from tests and FEM analyses show that the applied loads at $D$ and $E$ are proportional to the ratio of plastic moment resistance at the end joint and total plastic moment resistance from the middle and the end joints. Knowing the value of $P_D$, it is possible to compute $P_E$ from Equation 18.

$$18. \quad P_E = \frac{P_D}{M_{end}} = \frac{M_{end}}{M_{mid} + M_{end}}$$

**Points $C_1$, $D_1$, $F$**

After all the primary points $A$, $B$, $C$, $D$, $E$, $G$ and the equations of straight lines $D_3D$ and $FG$ have been determined, the secondary points $C_1$, $D_1$, $F$ can then be defined based on these data. Point $D_1$ is the intersection of $D_1D$ and $C_1C$, that is $P_{D_1} = P_C$. Similarly, point $F$ belongs to the line $FG$ and $P_F = P_E$. Lastly, $C_1$ is determined from $C_1D_1$ in which $C$ is the mid-point. However, if the results show that $P_D$ is smaller than $P_C$ then point $D_1$ does not exist. In this case, $P_D$ will be equal to $P_C$ while $d_1$ remains unchanged and $C_1$ is chosen so that $C_1C = CD$ (Figure 8(a)). This scenario represents the case when bottom rebars fracture early before catenary action is efficiently mobilised. On the other hand, if $E$ lies below line $FG$ then $F$ is determined as the intersection of lines $DE$ and $FG$ and point $E$ is no longer needed (Figure 8(b)). It demonstrates the response of a sub-assemblage with relatively large horizontal restraint stiffness.

**Minimum value for horizontal stiffness ratio $k_{hor}$**

Observations from different span–depth ratios show that, after the fracture of bottom rebars at the middle joint, catenary action can be mobilised efficiently at the latter part (FG) only if $k_{hor}$ is not less than a certain value defined by Equation 19. Therefore, in the proposed response, polyline $EFG$ will be considered as a horizontal line $EG$ if $k_{hor}$ does not satisfy the stiffness requirement in Equation 19. In this case, $P_E$ is equal to $P_G$ and point $F$ is no longer needed (Figure 8(c)).

$$19. \quad k_{hor} \geq 0.019 \quad \text{if} \quad 11 \geq k_{span} \geq 9$$

$$19. \quad k_{hor} \geq 0.029 \quad \text{if} \quad 9 \geq k_{span} \geq 7$$

**Minimum strength of axial restraint, $R_A$**

In the numerical simulations, the horizontal restraint stiffness $K_h$ is assumed to behave elastically. Nonetheless, in actual buildings, boundary restraints may respond beyond the elastic range and may be subject to large inward deflections if ductile failure governs, or may suddenly fail in a brittle manner. In this approach, the horizontal restraint is only allowed to behave elastically with the maximum elastic strength, $R_A$. If the horizontal reaction force exceeds $R_A$, catenary action cannot be developed in the FG part. To fully mobilise catenary action in the beam–column structure, the capacity of horizontal restraint $R_A$ must be greater than the maximum horizontal reaction $H_{max}$, which can be fully developed in the sub-assemblage before the top rebars fracture. At point $G$, it is

![Figure 8](image)

**Figure 8.** Special cases of the simplified model: (a) $P_D < P_C$; (b) point $E$ lies below $FG$; (c) $k_{hor}$ too small. $D_{new}$ is the new location of point $D$, in the case when $P_D$ is smaller than $P_C$. 

---

**A simplified model of catenary action in reinforced concrete frames under axially restrained conditions**

*Pham and Tan*
assumed that the beam top rebar near the end joint section reaches its ultimate strength, $f_u$. Through numerical predictions and neglecting the tensile force from the bottom rebars as well as the tensile strength of concrete, $H_{\text{max}}$ is related to the ultimate tensile force of the top reinforcement, $F_{\text{top}}$, by Equation 20.

$$R_A \geq H_{\text{max}} = F_{\text{top}} \left[1 - \frac{1}{(k_{\text{span}} - 4.5)^{0.6}}\right]$$

$$20. \quad \left[1 - \frac{1}{(k_{\text{hor}} + 1.05)^{0.3}}\right] \text{ for } k_{\text{span}} \geq 7$$

When both $k_{\text{span}}$ and $k_{\text{hor}}$ are large ($k_{\text{span}} \geq 11$ and $k_{\text{hor}} \geq 0.12$), $H_{\text{max}}$ is equal to $F_{\text{top}}$. But when $k_{\text{span}}$ is small or $k_{\text{hor}}$ is weak, $H_{\text{max}}$ can be reduced to 50% of $F_{\text{top}}$.

For the single column loss scenario, $K_A$ and $R_A$ will be calculated based on structural properties of adjacent column members. If $R_A \geq H_{\text{max}}$, then catenary action can be fully developed following all the equations of the proposed model. But if $R_A < H_{\text{max}}$, catenary action is neglected after all the bottom rebars have fractured and EFG will be replaced by a horizontal line EG, similar to the case of relatively small $k_{\text{hor}}$ presented in Figure 8(c).

Parameters of the primary controlling points (load and corresponding displacement, etc.) derived from the proposed model are compared to values obtained from finite-element parametric studies (Table 1). Results are presented in Appendix 2. Suffice to say, acceptable agreement is obtained between semi-analytical and finite-element predictions.

Dynamic assessment of sub-assemblies under missing column scenario

Once the static non-linear behaviour of the sub-assemblage is already defined, the dynamic effect can be easily achieved by employing the energy-based approach by Izzuddin et al. (2008). Based on energy conservation, this method transforms the non-linear static curve into a pseudo-static response which represents the maximum dynamic displacement for a certain value of dynamic load. The maximum value from the pseudo-static curve represents the ultimate progressive collapse load that the structure can sustain. Employing this approach to the proposed simplified static polylone, there are three possible peaks of dynamic response at displacements $d_{\text{b}}$, $d_{\text{d}}$, and $d_{\text{c}}$ (Figure 9). The equivalent dynamic loads at these deflections, which are $P_{\text{b}}^{\text{D}}$, $P_{\text{c}}^{\text{D}}$, and $P_{\text{d}}^{\text{D}}$, are derived by Equations 21–23. The largest value from these three dynamic loads is the ultimate dynamic capacity of the sub-assemblage.

21. $P_{\text{b}}^{\text{D}} = (P_A + P_B + P_C)(d_{\text{b}} - d_{\text{c}}) / 2d_{\text{b}}$

22. $P_{\text{c}}^{\text{D}} = (P_A + P_B + P_C)(d_{\text{b}} - d_{\text{c}}) + (P_C + P_B)(d_{\text{c}} - d_{\text{b}}) + 2P_C(d_{\text{d}} - d_{\text{c}}) + (P_D + P_{\text{D_1}})(d_{\text{d}} - d_{\text{b}}) / 2d_{\text{b}}$

23. $P_{\text{d}}^{\text{D}} = (P_A + P_B + P_C)(d_{\text{b}} - d_{\text{c}}) + (P_C + P_B)(d_{\text{c}} - d_{\text{b}}) + 2P_C(d_{\text{d}} - d_{\text{c}}) + (P_D + P_{\text{D_1}})(d_{\text{d}} - d_{\text{b}}) + (P_E + P_{\text{E_1}})(d_{\text{d}} - d_{\text{b}}) / 2d_{\text{b}}$

In summary, the solution procedure to obtain the simplified semi-analytical method is presented in Figure 10.

Test validations using the semi-analytical model

To verify the application of the proposed model for catenary action and structural response, experimental results include seven tests from Yu and Tan (2013a, 2013b), and two tests from Lim et al. (2015) are also used to construct the structural response using the equations and solution procedure described in the previous section. Other tests on two-dimensional

![Figure 9. Dynamic assessment for the simplified response: (a) dynamic load $P_{\text{b}}^{\text{D}}$ at $d_{\text{b}}$; (b) dynamic load $P_{\text{c}}^{\text{D}}$ at $d_{\text{c}}$; (c) dynamic load $P_{\text{d}}^{\text{D}}$ at $d_{\text{d}}$.](image)
beam–column structures (Sadek et al., 2011; Su et al., 2009; Yi et al., 2008) are not used for the validation, as the stiffness of horizontal restraints was not directly measured in these tests. Input information as well as all necessary variables for deriving the model are presented in Tables 2 and 3. Comparisons between semi-analytical and test results are presented in Figure 11. The proposed model provides reasonably good predictions of structural response, and yet requires much less effort compared to detailed FEM simulations. For specimen PR from Lim et al. (2015), because the test was terminated at a deflection of 396 mm before any fracture of top longitudinal rebars occurred, the response is defined up to this displacement only. In most of the tests, lateral boundary conditions for both sides were well restrained by either the strong wall or the A-frame. Hence, the ultimate strength $R_A$ of these restraints is considered to be relatively large. However, for specimen PR, one side of the frame was partially restrained while the other was fully restrained. The actual test results showed no enhancement from catenary action after the fracture of rebars at the middle joint. During the test, horizontal reaction of the partially restrained side gradually increased to 54 kN before the adjacent column was pulled inwards, indicating the yielding state of the restraint. Thus, in the simplified model, the strength of horizontal restraint $R_A$ is limited to 54 kN, which is smaller than $H_{max}$ calculated from Equation 20 (107·9 kN). As a result, the simplified response of specimen PR after the fracture of the bottom rebars is a horizontal line EG (Figure 8(c)), which shows good agreement with the actual test result.

By assuming perfectly rigid rotational restraint, the proposed model neglects rotational deformation at the end joint, and therefore tends to overestimate the stiffness of the substructure.
### Table 2. Test data and proposed model parameters

<table>
<thead>
<tr>
<th>Test</th>
<th>$F_{a,y}$: kN</th>
<th>$K_{A}$: KN/m</th>
<th>$k_{span}$</th>
<th>$k_{hor}$</th>
<th>$k_{flex}$</th>
<th>$k_{topbar}$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\phi_{\text{u}, A}$: m$^{-1}$</th>
<th>$\phi_{\text{c}, A}$: m$^{-1}$</th>
<th>$\phi_{\text{u}, E}$: m$^{-1}$</th>
<th>$\phi_{\text{c}, E}$: m$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yu and Tan (2013a, 2013b)</td>
<td>S1</td>
<td>148</td>
<td>90 000</td>
<td>11.0</td>
<td>0.237</td>
<td>0.0117</td>
<td>0.0600</td>
<td>0.76</td>
<td>0.99</td>
<td>3.30</td>
<td>0.77</td>
<td>0.64</td>
<td>0.0325</td>
<td>0.0325</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>120</td>
<td>140 000</td>
<td>11.0</td>
<td>0.372</td>
<td>0.0104</td>
<td>0.0488</td>
<td>0.70</td>
<td>0.99</td>
<td>3.40</td>
<td>0.77</td>
<td>0.64</td>
<td>0.0338</td>
<td>0.0338</td>
</tr>
<tr>
<td></td>
<td>S3</td>
<td>199</td>
<td>145 000</td>
<td>11.0</td>
<td>0.375</td>
<td>0.0140</td>
<td>0.0825</td>
<td>0.70</td>
<td>0.95</td>
<td>3.10</td>
<td>0.65</td>
<td>0.64</td>
<td>0.0357</td>
<td>0.0357</td>
</tr>
<tr>
<td></td>
<td>S4</td>
<td>199</td>
<td>145 000</td>
<td>11.0</td>
<td>0.369</td>
<td>0.0164</td>
<td>0.0825</td>
<td>0.71</td>
<td>0.92</td>
<td>3.10</td>
<td>0.65</td>
<td>0.64</td>
<td>0.0325</td>
<td>0.0325</td>
</tr>
<tr>
<td></td>
<td>S5</td>
<td>199</td>
<td>145 000</td>
<td>11.0</td>
<td>0.361</td>
<td>0.0193</td>
<td>0.0825</td>
<td>0.71</td>
<td>0.89</td>
<td>3.10</td>
<td>0.65</td>
<td>0.655</td>
<td>0.0325</td>
<td>0.0325</td>
</tr>
<tr>
<td></td>
<td>S6</td>
<td>309</td>
<td>145 000</td>
<td>11.0</td>
<td>0.357</td>
<td>0.0213</td>
<td>0.1249</td>
<td>0.74</td>
<td>0.86</td>
<td>2.72</td>
<td>0.49</td>
<td>0.64</td>
<td>0.0325</td>
<td>0.0325</td>
</tr>
<tr>
<td>Lim et al. (2015)</td>
<td>FR</td>
<td>118</td>
<td>20 000</td>
<td>12.3</td>
<td>0.083</td>
<td>0.0221</td>
<td>0.1016</td>
<td>0.88</td>
<td>1.05</td>
<td>3.25</td>
<td>0.93</td>
<td>0.774</td>
<td>0.0487</td>
<td>0.0487</td>
</tr>
<tr>
<td></td>
<td>PR</td>
<td>118</td>
<td>5000</td>
<td>12.3</td>
<td>0.021</td>
<td>0.0221</td>
<td>0.1016</td>
<td>0.94</td>
<td>1.19</td>
<td>3.25</td>
<td>1.09</td>
<td>0.774</td>
<td>0.0487</td>
<td>0.0487</td>
</tr>
<tr>
<td>PR</td>
<td>118</td>
<td>5000</td>
<td>12.3</td>
<td>0.021</td>
<td>0.0221</td>
<td>0.1016</td>
<td>0.94</td>
<td>1.19</td>
<td>3.25</td>
<td>1.09</td>
<td>0.774</td>
<td>0.0487</td>
<td>0.0487</td>
<td>0.0487</td>
</tr>
</tbody>
</table>

### Table 3. Controlling points from the proposed model

<table>
<thead>
<tr>
<th>Test</th>
<th>Point A $d_A$</th>
<th>Point B $d_B$</th>
<th>Point C$<em>1$ $d</em>{C1}$</th>
<th>Point C $d_C$</th>
<th>Point D$<em>1$ $d</em>{D1}$</th>
<th>Point D $d_D$</th>
<th>Point E $d_E$</th>
<th>Point F $d_F$</th>
<th>Point G $d_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yu and Tan (2013a, 2013b)</td>
<td>S1</td>
<td>17.7</td>
<td>33.6</td>
<td>60.0</td>
<td>43.7</td>
<td>181.9</td>
<td>33.6</td>
<td>314.2</td>
<td>33.6</td>
</tr>
<tr>
<td></td>
<td>S2</td>
<td>14.3</td>
<td>29.5</td>
<td>70.0</td>
<td>38.4</td>
<td>185.4</td>
<td>29.5</td>
<td>312.2</td>
<td>29.5</td>
</tr>
<tr>
<td></td>
<td>S3</td>
<td>19.8</td>
<td>40.9</td>
<td>57.5</td>
<td>53.7</td>
<td>195.6</td>
<td>40.9</td>
<td>280.0</td>
<td>40.9</td>
</tr>
<tr>
<td></td>
<td>S4</td>
<td>23.8</td>
<td>48.9</td>
<td>55.0</td>
<td>61.1</td>
<td>168.5</td>
<td>48.9</td>
<td>292.5</td>
<td>48.9</td>
</tr>
<tr>
<td></td>
<td>S5</td>
<td>28.6</td>
<td>58.6</td>
<td>55.0</td>
<td>69.0</td>
<td>151.1</td>
<td>58.6</td>
<td>292.8</td>
<td>58.6</td>
</tr>
<tr>
<td></td>
<td>S6</td>
<td>32.1</td>
<td>65.6</td>
<td>55.0</td>
<td>70.9</td>
<td>159.8</td>
<td>65.6</td>
<td>271.9</td>
<td>65.6</td>
</tr>
<tr>
<td>Lim et al. (2015)</td>
<td>FR</td>
<td>32.9</td>
<td>25.8</td>
<td>52.6</td>
<td>28.6</td>
<td>118.8</td>
<td>25.8</td>
<td>257.8</td>
<td>25.8</td>
</tr>
<tr>
<td></td>
<td>FR</td>
<td>35.4</td>
<td>25.8</td>
<td>54.0</td>
<td>27.0</td>
<td>142.1</td>
<td>25.8</td>
<td>286.4</td>
<td>25.8</td>
</tr>
</tbody>
</table>

Note: unit of displacement ($d_A$ to $d_G$) – mm; unit of applied load ($P_A$ to $P_G$) – kN
at the initial stage (part OA). As a result, the values of $d_A$ calculated from the model are smaller than those from actual tests. Nonetheless, this assumption has little bearing on the total strain energy of the static response, which is important for dynamic assessment of progressive collapse. In terms of maximum displacement and corresponding ultimate load capacity (point G), the simplified model provides relatively good agreement with all the test results (Table 4). A comparison study between the model predictions and test results on maximum reaction force $H_{\text{max}}$ at point G and maximum equivalent dynamic load using the Izzuddin method show very good agreement (Table 4).

**Discussion on the semi-analytical model for sub-assemblages**

**Limits of application**

Since the simplified model is developed based on FEM analyses, there are some assumptions that should be clearly stated when applying this approach. First of all, the model can only be used for structures with concrete cylinder strength ranging from 20 to 50 MPa. Structures using higher strength or fibre concrete are outside the scope of this research. Second, the longitudinal reinforcement consists of deformed bars with yield strength up to 600 MPa. It is assumed that, at the beam sections near the end joint and the middle joint, the ratio of top bars is larger than bottom bars, which is generally satisfied in normal RC gravity design. The anchorage length of reinforcement into the joint regions follows code provisions, for example Eurocode 2 (BS EN 1992-1-1:2004; BSI, 2004), to ensure good bonding between concrete and reinforcement. In terms of structural geometry, the model is only valid when $k_{\text{span}} \geq 7$ where flexure dominates behaviour instead of shear. With regard to catenary action developed in the double-span beam after the fracture of the bottom reinforcement, it is shown that catenary action is only significant when the requirements for horizontal stiffness (Equation 19) and strength (Equation 20) are satisfied. Concerning rotational stiffness, $k_{\text{rot}}$ has to satisfy Equation 8 to ensure rotational resistance at the end joint will not affect catenary action. This requirement was easily fulfilled in the tests for both sub-assemblages (Yu and Tan, 2013a, 2013b) and beam–column frames (Lim et al., 2015). The proposed model...
neglects connection gaps in the restraint boundaries. Such gaps have little effect on the final catenary action part (FG); they only affect the first peak of compressive arch action (point B).

Discussion on symmetric conditions

The simplified model is based on half-specimen simulations, assuming symmetry for geometry, material and boundary conditions. In practice, it is likely that the restraint stiffness values \((K_A \text{ and } K_B)\) from two sides of the two-span beam structure are not the same (e.g. the number of adjacent bays on each side is different). In this case, the smaller horizontal stiffness value should be applied.

In circumstances where geometry and material properties from two sides of the double-span beam are not identical, or the joint may not be located at the middle of the double span, then the total behaviour of the sub-assemblage \(P_{\text{total}}\) may be approximated as the average of the simplified responses from two symmetric systems which are similar to the single-span structures on the left and the right sides of the removed column (Figure 12(a)), the so-called ‘left’ and the ‘right’ systems, corresponding to the same middle-joint displacement, \(d\), as shown in Equation 24.

\[
24. \quad P_{\text{total}}(d) = \frac{P_{\text{left}}(d) + P_{\text{right}}(d)}{2}
\]

However, \(P_{\text{left}}(d)\) and \(P_{\text{right}}(d)\) are not totally independent. They are coupled with each other at specific displacements when rebars fracture. Owing to differences in structural response, the displacements corresponding to top and bottom rebar fractures of the ‘left’ and the ‘right’ systems are not the same. Figure 12(b) shows an example of the two simplified responses in which the bottom rebars of the ‘left’ system fracture earlier compared to those of the ‘right’ system \((d_{D,L} < d_{D,R})\), and vice versa regarding the fracture of the top rebars \((d_{G,L} > d_{G,R})\).

To simply combine the responses of the two systems into the approximate total response, \(P_{\text{total}}\), it is assumed that after the fracture of the bottom rebars from the ‘left’ system \(\text{at } d = d_{D,L}\), the bottom rebars from the ‘right’ system are also considered to have fractured. Therefore, the behaviour of the ‘right’ system will change from \(D_{1,R}D_{2,R}E_{R}F_{R}\) to \(D_{1,R}D_{2,R}E_{R}F_{R}\), as shown in Figure 12(b). Similarly, after the top rebars from the ‘right’ system have fractured (at \(d = d_{G,R}\)) causing a total failure to this structure, behaviour of the ‘left’ system is also considered to have terminated and segment \(F_{1}G_{L}\) is replaced by \(F_{1}G_{L}\), as represented in Figure 12(b).

To verify this coupling method for double-span beams with different geometry/material properties from the two sides, FEM analyses are conducted for an example shown in
Figure 13(a). Three simulations are performed representing responses from the ‘left’, the ‘right’ and the ‘total’ structures, and are compared with predictions from the simplifying approach using the coupling of rebar fracture from the ‘left’ and the ‘right’ systems (Figure 13(b)). It is shown that the simplified method gives a conservative result compared to FEM prediction.

Conclusions
The behaviour of beam–column structures under a single column removal scenario is greatly affected by the restraint stiffness of boundary conditions. In this paper, parametric studies based on numerical analyses are developed to gain a better understanding of the sub-assemblage structural resistance, especially the development and contribution of catenary action before and after the fracture of the bottom rebars at the middle joint. It is shown that, while rotational stiffness only affects the initial compressive arch action and has much less influence on the remaining response, horizontal stiffness has a greater effect on both compressive arch action and subsequent catenary action. Other factors such as concrete strength, connection gap or bottom reinforcement in beams has little effect on the subsequent catenary action phase. The assumption of elastic stiffness for horizontal restraint simplifies the analysis but is difficult to achieve in practice. Instead, if the strength of the horizontal restraint is smaller than the maximum horizontal reaction required by the sub-assemblage, then catenary action will not be fully mobilised after the fracture of the bottom rebars.

The proposed semi-analytical model provides a quick means for estimating the overall behaviour of beam–column sub-assemblies under concentrated loading condition. It simplifies the non-linear response into a piecewise multi-linear curve with some major points to be determined through equations. The assumptions of relatively good rotational restraint and bonding condition yield reasonable predictions as the model has been validated by test results. Strain energy as well as the maximum dynamic load can then be easily obtained from the simplified static response. Although the model is developed based on symmetric assumptions of structural and boundary conditions, it can be conservatively applied for non-symmetric sub-assemblages.

Acknowledgement
The authors gratefully acknowledge the funding entitled ‘Development of a design guideline and analytical tool to mitigate progressive collapse of buildings against explosive effects’, which is provided by the Ministry of Home Affairs, Singapore.
Appendix 1: derivations of displacements at points D and G

Point D
At point D, it is assumed that the beam sections at the middle and the end joints remain the maximum plastic moment capacities, $M_{\text{mid}}$ and $M_{\text{end}}$, besides the development of catenary action (Figure 6(a)). To calculate the plastic hinge lengths near the middle joint ($L_{\text{sp}}^{\text{mid}}$) and near the end joint ($L_{\text{sp}}^{\text{end}}$), the Paulay and Priestley (1992) method is applied. However, the equation of plastic hinge length from this method is proposed for a cantilever column subject to horizontal displacement at the free end. To employ this equation, therefore, equivalent cantilever spans, $L_{n}^{\text{mid}}$ and $L_{n}^{\text{end}}$ are derived, which represent the beam spans of segments $XY$ and $YZ$ (Figure 6(a)). As a result, $L_{n}^{\text{mid}}$ and $L_{n}^{\text{end}}$ can be computed from Equation 25. Thereafter, plastic hinge lengths $L_{\text{sp}}^{\text{mid}}$ and $L_{\text{sp}}^{\text{end}}$ are defined using Equations 26 and 27, respectively. Subsequently, yield and plastic displacements of segment $XY$ can be obtained by Equations 28 and 29, respectively.

25. \[ \frac{L_{n}^{\text{mid}}}{L_{n}^{\text{end}}} = \frac{M_{\text{mid}}}{M_{\text{end}}} \]

26. \[ L_{\text{sp}}^{\text{mid}} = 0.08L_{n}^{\text{mid}} + 0.022d_{\text{mid}}f_{y} \]
27. \[ L_{\text{sp}}^{\text{end}} = 0.08L_{n}^{\text{mid}} + 0.022d_{\text{end}}f_{y} \]

where \( d_{\text{mid}} \) and \( d_{\text{end}} \) are the diameters of longitudinal tensile reinforcement at the middle and the end joints.

28. \[ \Delta_{y}^{\text{mid}} = \frac{\phi'_{y}L_{n}^{\text{mid}2}}{3} \]

29. \[ \Delta_{y}^{\text{mid}} = \left( \phi'_{y} - \phi'_{u} \right)L_{n}^{\text{mid}} \left( L_{n}^{\text{mid}} - 0.5L_{\text{sp}} \right) \]

where \( \phi'_{y} \) and \( \phi'_{u} \) are the yield and ultimate curvatures of the concrete beam section at the middle joint under sagging moment. These values depend on the cross-sectional geometry and material properties of the concrete and reinforcement stress-strain curve. Park and Paulay (1975) have developed a rational method to calculate these values to construct the moment-curvature response of a flexural section. Nowadays, there are several convenient tools to obtain this response curve. In this study, a cross-sectional design tool integrated in the structural analysis software SAP 2000 (Habibullah and Wilson, 2005) is employed to determine the yield and ultimate values of curvature at different beam sections based on the material and geometry properties. The input data used in SAP 2000 are based on concrete cylinder strength, yield strength and ultimate tensile strength of reinforcement with corresponding strains. The simple parametric stress-strain curves of the reinforcement and of the unconfined concrete model are automatically generated by SAP 2000. After obtaining \( \Delta_{y}^{XY}, \Delta_{y}^{YZ} \) is defined using Equation 30.

30. \[ \Delta_{y}^{XY} = \frac{L_{n}^{\text{mid}}}{L_{n}^{\text{end}}} \]

### Appendix 2: Comparisons Between the Simplified Model and the FEM Results

Parameters derived from the proposed model such as displacements and loads at controlling points are compared to FEM predictions of 120 cases (Table 1). Basically, the values from the proposed model are either close to, or more conservative than, the numerical predictions.

#### For Point A

A comparison of \( d_{A} \) between the proposed model and the corresponding FEM displacement for the same applied load \( P_{A} \) is shown in Figure 14(a).

#### For Point C

Results of \( P_{C} \) and \( d_{C} \) are compared between FEM analyses and the proposed equations. The comparisons show relatively good agreement. Values from the simplified model are more conservative than FEM predictions in most cases (Figures 14(b) and 14(c)).

#### For Line FG

In Equation 14, \( \beta_{F}F_{\text{ap}}/L_{n} \) represents the gradient of FG while \( \beta_{d_{\text{beam}}} \) is the root of the equation \( P_{FG}(d) = 0 \), which represents the location of FG. Comparisons of the gradient
and the root from the function of line FG between numerical and simplified methods are carried out and illustrated in Figures 14(d) and 14(e). The predicted values for the gradient are relatively close to the FEM results. However, the proposed model gives more conservative values of the root compared to FEM when $k_{hor}$ is relatively small. Thus, the use of this model will provide safe results for the case of weak axial restraints.

For point E

Figure 14(f) describes a comparison of $P_E/P_D$ between the numerical predictions and the results from Equation 18. The errors are within an acceptable range.

Displacement at D and G

Comparisons are made between FEM and analytical results for the displacements $d_D$ and $d_G$. The results are shown in...
Figures 14(g) and 14(h). Basically, the simplified method has good agreement with numerical predictions for $d_d$ and gives more conservative results for $d_C$.

**Maximum tensile reaction, $H_{\text{max}}$**

The analytical values of $H_{\text{max}}$ show relatively good agreement with those computed by FEM (Figure 14(i)).

**REFERENCES**


---

**How can you contribute?**

To discuss this paper, please submit up to 500 words to the editor at journals@ice.org.uk. Your contribution will be forwarded to the author(s) for a reply and, if considered appropriate by the editorial board, it will be published as a discussion in a future issue of the journal.