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<td>Date</td>
<td>2019</td>
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<td>URL</td>
<td><a href="http://hdl.handle.net/10220/48714">http://hdl.handle.net/10220/48714</a></td>
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Some fractional integral inequalities of type Hermite–Hadamard through convexity

Shahid Qaisar¹, Jamshed Nasir², Saad Ihsan Butt³, Asma Asma¹, Farooq Ahmad³,⁴, Muhammad Iqbal⁵ and Sajjad Hussain³,⁴

Abstract


MSC: 26A15; 26A51; 26D10

Keywords: Hermite–Hadamard's inequality; Convex functions; Power-mean inequality; Riemann–Liouville fractional integration

1 Introduction

A function $g : I \subset \mathbb{R} \to \mathbb{R}$ is called convex in the classical sense, if the inequality

$$g(\omega x + (1 - \omega)y) \leq \omega g(x) + (1 - \omega)g(y)$$

holds for all $x, y \in I$ and $\omega \in [0, 1]$. In fact a large number of articles have been written on inequalities using classical convexity but one of the most important and well known is Hermite–Hadamard’s inequality. This double inequality is stated as follows [4]. Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $I$ of real numbers and $x, y \in I$ with $x < y$. Then

$$g\left(\frac{x + y}{2}\right) \leq \frac{1}{y - x} \int_x^y g(t) dt \leq \frac{g(x) + g(y)}{2}.$$  

Both inequalities hold in the reversed direction for $g$ to be concave. Several improvements and extensions of Hermite–Hadamard’s type inequality to different kinds of convexity were established by different researchers.

First we recall some important definitions and results which we have used in this paper.

Definition 1 For $g \in L_1[a,b]$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_a^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t) dt, \quad a < x,$$
and

\[ J_0^b g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t) \, dt, \quad x < b, \]

respectively, where \( \Gamma(\cdot) \) is Gamma function and its definition is \( \Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} \, du \). It is to be noted that \( J_0^b g(x) = J_0^b g(x) = g(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

Properties relating to this operator can be found in [5] and for useful details on Hermite-Hadamard type inequalities connected with fractional integral inequalities, we refer the reader to [5–17] and the references therein.

In [18] Dragomir and Agarwal, obtained inequalities for differentiable convex mappings which are connected with the right-hand side of Hermite-Hadamard’s (trapezoid) inequality and applied them to obtain some elementary inequalities for real numbers and in numerical integration as follows.

**Theorem 1** Let \( g : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable mapping on \( I^o \) where \( x, y \in I \) with \( x < y \). If \( |g'|^q \) is convex on \([x, y]\), for some \( q \geq 1 \) then the following inequality holds:

\[
\left| \frac{g(x) + g(y)}{2} - \frac{1}{y-x} \int_x^y f(u) \, du \right| \leq \frac{y-x}{8} \left( |g'(x)| + |g'(y)| \right).
\]  

(1)

In [2] Dragomir, obtained inequalities for a Lipschitzian mapping which are in connection with the right-hand side of Hermite-Hadamard’s (trapezoid) inequality.

**Theorem 2** Let \( g : I \subset \mathbb{R} \rightarrow \mathbb{R} \) be a M-Lipschitzian mapping on \( I \) where \( x, y \in I \) with \( x < y \), then we have the following inequality:

\[
\left| \frac{g(x) + g(y)}{2} - \frac{1}{y-x} \int_x^y f(u) \, du \right| \leq \frac{M}{3} (y-x).
\]

(2)

In [3] Yang, obtained Hermite–Hadamard’s (trapezoid) inequalities for differentiable mapping for concave function.

**Theorem 3** Let \( I \subset \mathbb{R} \) be an open interval, \( l, m, n, P, Q \in I \) with \( l \leq P \leq n \leq Q \leq m \) (\( n \neq l, m \)) \( l, m, n \in \mathbb{R} \) and \( g : [x, y] \rightarrow \mathbb{R} \) be a differentiable function. If \( |g'|^q \) is concave on \([x, y]\) and \( 1 \leq \theta \leq q \), then

\[
\left| (P - l)g(l) + (m - Q)g(m) + (Q - P)g(n) - \int_x^y g(u) \, du \right| \\
\leq K(P, Q, n, \theta) \cdot J(P, Q, n, \theta),
\]

(3)

where

\[
K(P, Q, n, \theta) = \left( \frac{1}{2} \left[ (P - l)^2 + (n - P)^2 + (Q - n)^2 + (m - Q)^2 \right] \right)^{\frac{\theta-1}{\theta}}
\]
and

\[
J(P, Q, n, \theta) = \left( \frac{1}{2} \right) \left[ (P - l)^2 + (n - P)^2 \right] \left[ g' \left( \frac{(P - l)^2 + (n - P)^2}{3[(P - l)^2 + (n - P)^2]} + l \right) \right]^{(m-1)} + \left( \frac{1}{2} \right) \left[ (Q - n)^2 + (m - Q)^2 \right] \left[ g' \left( m - \frac{(Q - n)^2(3m - 2n - Q) + (m - Q)^3}{3[(Q - n)^2 + (m - Q)^2]} \right) \right]^{(m-1)}.
\]

**Corollary 1** Under the assumptions of Theorem 3 with \( P = Q = n = (l + m)/2 \) and \( \theta = 1 \), we get the following inequality:

\[
\left| \frac{g(x) + g(y)}{2} - \frac{1}{y-x} \int_x^y g(u) \, du \right| \leq \frac{y-x}{8} \left[ \left| g' \left( \frac{5x + y}{6} \right) \right| + \left| g' \left( \frac{x + 5y}{6} \right) \right| \right].
\]  

(4)

The goal of this article is to establish Hermite–Hadamard type inequalities for the Riemann–Liouville fractional integral using convexity as well as concavity, for functions whose absolute values of the first derivative are convex. Here we will derive a general integral inequality for the Riemann–Liouville fractional integral.

### 2 Main results

Before going on our main result first we prove the following integral inequality.

**Lemma 1** Let \( I \subset \mathbb{R} \) be an open interval, \( a, b \in I \) with \( a < b \) and \( f : [a, b] \to \mathbb{R} \) be a differentiable function such that \( f' \) is integrable and \( 0 < \alpha \leq 1 \) on \( (a, b) \) with \( a < b \). If \( |f'| \) is convex on \([a, b]\), then we have the following inequality:

\[
\left[ \frac{(b - a)\alpha - (x - a)\alpha}{(b - a)\alpha} + \frac{(b - x)\alpha}{(b - a)\alpha} \right] f'(b) + \frac{(b - a)\alpha - (b - x)\alpha}{(b - a)\alpha} + \frac{(x - a)\alpha}{(b - a)\alpha} f(a) - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ \int_b^a f'(b) + \int_a^b f(a) \right] = \frac{1}{2} \sum_{k=1}^4 I_{1k},
\]

where

\[
I_{11} = \frac{(x - a)^{\alpha+1}}{(b - a)^\alpha} \int_0^1 (t^{\alpha} - 1) f' (tx + (1 - t)a) \, dt,
\]

\[
I_{12} = \frac{(b - x)^{\alpha+1}}{(b - a)^\alpha} \int_0^1 (1-t^\alpha) f' (tx + (1-t)b) \, dt,
\]

\[
I_{13} = \frac{(b - x)^{\alpha+1}}{(b - a)^\alpha} \int_0^1 \left[ \frac{a - b}{x - b} - \frac{a - x}{x - b} \right] f' (tx + (1-t)b) \, dt,
\]

\[
I_{14} = \frac{(x - a)^{\alpha+1}}{(b - a)^\alpha} \int_0^1 \left[ \frac{b - x}{x - a} - \frac{b - a}{x - a} \right] f' (tx + (1-t)a) \, dt.
\]

**Proof** Integrating by parts

\[
I_{11} = \frac{(x - a)^{\alpha+1}}{(b - a)^\alpha} \int_0^1 (t^{\alpha} - 1) f' (tx + (1-t)a) \, dt
\]
holds \[ \text{Analogously differentiable function such that } f \text{ is } \left\{ \begin{array}{l}
\frac{(t^\alpha - 1)f(tx + (1 - t)a)}{x - a} \mid_0^1 \\
+ \frac{\alpha}{x - a} \int_0^1 (t^\alpha - 1)f(tx + (1 - t)a) \, dt \\
= \frac{(x - a)^{\alpha + 1}}{(b - a)^\alpha} \left\{ \frac{f(a)}{x - a} - \frac{\alpha}{x - a} \int_a^x (u - a)^{\alpha - 1} f(u) \, du \right\} \\
= \frac{f(a)(x - a)^\alpha}{(b - a)^\alpha} - \frac{\alpha}{(b - a)^\alpha} \int_a^x (u - a)^{\alpha - 1} f(u) \, du, \\
I_{13} = \frac{(b - x)^{\alpha + 1}}{(b - a)^\alpha} \int_0^1 \left\{ \left( \frac{a - b}{b - x} - t \right)^\alpha - \left( \frac{a - x}{b - x} \right)^\alpha \right\} f'(tx + (1 - t)b) \, dt \\
= \frac{(b - x)^{\alpha + 1}}{(b - a)^\alpha} \left[ \frac{(a - x)^{\alpha - 1} f(b)}{b - x} - \frac{\alpha}{(b - x)^{\alpha + 1}} \int_b^x (u - a)^{\alpha - 1} f(u) \, du \right]. \\
\right. \\
\text{Analogously} \\
I_{12} = \frac{f(b)(b - x)^\alpha}{(b - a)^\alpha} - \frac{\alpha}{(b - a)^\alpha} \int_b^x (b - u)^{\alpha - 1} f(u) \, du, \\
I_{14} = \frac{(x - a)^{\alpha + 1}}{(b - a)^\alpha} \left\{ \frac{-(b - x)^\alpha + (b - a)^\alpha}{(x - a)^{\alpha + 1}} f(a) - \frac{\alpha}{(x - a)^{\alpha + 1}} \int_a^x (u - a)^{\alpha - 1} f(u) \, du \right\}. \\
\right. \\
\text{Adding the above equalities, we get} \\
I_{11} + I_{13} = \frac{(x - a)^\alpha}{(b - a)^\alpha} f(a) + \left( 1 - \frac{(x - a)^\alpha}{(b - a)^\alpha} \right) f(b) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} \int_a^x f(a), \\
I_{12} + I_{14} = \frac{(b - x)^\alpha}{(b - a)^\alpha} f(b) + \left( 1 - \frac{(b - x)^\alpha}{(b - a)^\alpha} \right) f(a) - \frac{\Gamma(\alpha + 1)}{(b - a)^\alpha} \int_a^x f(b). \\
\right. \\
\text{The proof is completed.} \quad \square \\
\textbf{Theorem 4} \quad \text{Let } I \subset \mathbb{R} \text{ be an open interval, } a, b \in I \text{ with } a < b \text{ and } f : [a, b] \to \mathbb{R} \text{ be a differen} \\
tiable function such that } f' \text{ is integrable and } 0 < \alpha \leq 1 \text{ on } (a, b) \text{ with } a < b. \text{ If } |f'\text{}} \\
is convex on } [a, b], \text{ then the following inequality for Riemann–Liouville fractional integrals holds:} \\
\left[ \left( \frac{(b - a)^\alpha - (x - a)^\alpha}{(b - a)^\alpha} + \frac{(b - x)^\alpha}{(b - a)^\alpha} \right) f(b) + \left( \frac{(b - a)^\alpha - (b - x)^\alpha}{(b - a)^\alpha} + \frac{(x - a)^\alpha}{(b - a)^\alpha} \right) f(a) - \frac{\Gamma(\alpha + 1)}{2(b - a)^\alpha} \left[ \int_a^x f(b) + \int_a^x f(a) \right] \right]
\[
\frac{(x - a)^{\alpha+1}}{(b - a)^{\alpha+1}} \left[ A |f'(x)| + B |f'(a)| \right] + \frac{(b - x)^{\alpha+1}}{(b - a)^{\alpha+1}} \left[ A |f'(x)| + B |f'(b)| \right] + \frac{(b - x)^{\alpha+1}}{(b - a)^{\alpha+1}} \left[ C |f'(b)| + D |f'(b)| \right] + \frac{(x - a)^{\alpha+1}}{(b - a)^{\alpha+1}} \left[ E |f'(x)| + F |f'(a)| \right] \leq \frac{\alpha}{2(\alpha + 2)},
\]

where

\[
A = \int_{0}^{1} |t^{\alpha}| \, dt = \frac{\alpha}{2(\alpha + 2)},
\]

\[
B = \int_{0}^{1} |1 - t^{\alpha}|(1 - t) \, dt = \frac{\alpha}{\alpha + 1} - \frac{\alpha}{2(\alpha + 2)} = \frac{\alpha(\alpha + 3)}{2(\alpha + 1)(\alpha + 2)},
\]

\[
C = \int_{0}^{1} t \left( \frac{a - b}{x - b} - t \right)^{\alpha} - \left( \frac{a - x}{x - b} \right)^{\alpha} \, dt
\]

\[
= -\frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{a - x}{x - b} \right)^{\alpha+2} + \frac{1}{2} \left( \frac{a - x}{x - b} \right)^{\alpha+1} + \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{a - x}{x - b} \right)^{\alpha+1} - \frac{1}{\alpha + 1} \left( \frac{a - x}{x - b} \right)^{\alpha+1} - \frac{1}{2} \left( \frac{a - x}{x - b} \right)^{\alpha}
\]

\[
D = \int_{0}^{1} \left( \frac{a - b}{x - b} - t \right)^{\alpha} - \left( \frac{a - x}{x - b} \right)^{\alpha} \, (1 - t) \, dt
\]

\[
= \frac{1}{\alpha + 1} \left( \frac{a - b}{x - b} \right)^{\alpha+1} + \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{a - x}{x - b} \right)^{\alpha+2} - \frac{1}{\alpha + 1} \left( \frac{a - x}{x - b} \right)^{\alpha+1} - \frac{1}{2} \left( \frac{a - x}{x - b} \right)^{\alpha+1} - \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{a - b}{x - b} \right)^{\alpha+2}.
\]

**Proof** Here, utilizing the properties of the modulus in Lemma 1 and convexity of $|f'|$, we have

\[
|K_1| = \frac{(x - a)^{\alpha+1}}{(b - a)^{\alpha}} \int_{0}^{1} (t^{\alpha} - 1) f'(tx + (1 - t)a) \, dt,
\]

\[
|K_1| \leq \frac{(x - a)^{\alpha+1}}{(b - a)^{\alpha}} \int_{0}^{1} (1 - t^{\alpha}) |f'(tx + (1 - t)a)| \, dt
\]

\[
\leq \frac{(x - a)^{\alpha+1}}{(b - a)^{\alpha}} \int_{0}^{1} (1 - t^{\alpha}) \left[ t |f'(x)| + (1 - t) |f'(a)| \right] \, dt,
\]

\[
|K_1| = \frac{(x - a)^{\alpha+1}}{(b - a)^{\alpha}} \left[ A |f'(x)| + B |f'(a)| \right],
\]

and analogously

\[
|K_2| = \frac{(b - x)^{\alpha+1}}{(b - a)^{\alpha}} \int_{0}^{1} (t^{\alpha} - 1) f'(tx + (1 - t)b) \, dt,
\]

\[
|K_2| \leq \frac{(b - x)^{\alpha+1}}{(b - a)^{\alpha}} \int_{0}^{1} (1 - t^{\alpha}) |f'(tx + (1 - t)b)| \, dt
\]

\[
\leq \frac{(b - x)^{\alpha+1}}{(b - a)^{\alpha}} \int_{0}^{1} (1 - t^{\alpha}) \left[ t |f'(x)| + (1 - t) |f'(b)| \right] \, dt,
\]

\[
|K_2| = \frac{(b - x)^{\alpha+1}}{(b - a)^{\alpha}} \left[ A |f'(x)| + B |f'(b)| \right],
\]
using the convexity on $|f'|$ and the fact that, for $\alpha \in (0,1]$ and $\forall t \in [0,1]$, 

$$|K_3| \leq \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 \left( \frac{a-b}{x-b} - t \right)^{\alpha} \left( \frac{a-x}{x-b} \right)^{\alpha} |f'(tx + (1-t)b)| \, dt$$

$$\leq \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 \left( \frac{a-b}{x-b} - t \right)^{\alpha} \left( \frac{a-x}{x-b} \right)^{\alpha} \left| t |f'(x)| + (1-t) |f'(b)| \right| \, dt,$$

$$|K_3| = \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \left| C |f'(x)| + D |f'(b)| \right|,$$

and analogously

$$|K_4| \leq \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 \left( \frac{b-x}{x-a} - t \right)^{\alpha} \left( \frac{b-a}{x-a} \right)^{\alpha} |f'(tx + (1-t)a)| \, dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 \left( \frac{b-x}{x-a} - t \right)^{\alpha} \left( \frac{b-a}{x-a} \right)^{\alpha} \left| t |f'(x)| + (1-t) |f'(a)| \right| \, dt$$

$$\leq \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} \left| E |f'(x)| + F |f'(a)| \right|.$$

The proof is completed. \[\square\]

**Remark 1** On letting $\alpha = 1$, $x = \frac{a+b}{2}$ in Theorem 4, inequality (5) reduces to inequality (1).

**Theorem 5** Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f'$ is integrable and $0 < \alpha \leq 1$ on $(a, b)$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, $q \geq 1$ then the following inequality holds:

$$\left[ \left( \frac{(b-a)^{\alpha} - (x-a)^{\alpha}}{(b-a)^{\alpha}} + \frac{(b-x)^{\alpha}}{(b-a)^{\alpha}} \right) f(b) + \left( \frac{(b-a)^{\alpha} - (b-x)^{\alpha}}{(b-a)^{\alpha}} + \frac{(x-a)^{\alpha}}{(b-a)^{\alpha}} \right) f(a) \right]$$

$$\leq \left[ \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} (g_1)^{1-q} \left( A |f'(x)|^q + B |f'(a)|^q \right) \right]^{1/q}$$

$$+ \left( \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} (g_2)^{1-q} \left( A |f'(x)|^q + B |f'(b)|^q \right) \right]^{1/q}$$

$$+ \left( \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} (g_3)^{1-q} \left( C |f'(x)|^q + D |f'(b)|^q \right) \right]^{1/q}$$

$$+ \left( \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} (g_4)^{1-q} \left( E |f'(x)|^q + F |f'(a)|^q \right) \right]^{1/q},$$

where

$$E = \int_0^1 t \left( \frac{b-a}{x-a} - t \right)^{\alpha} \left( \frac{b-x}{x-a} \right)^{\alpha} \, dt$$

$$= \left( \frac{b-x}{2(x-a)} \right) - \frac{1}{\alpha + 1} \left( \frac{b-x}{x-a} \right)^{\alpha+1} - \frac{b-x}{(x-a)}$$

$$- \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{b-x}{x-a} \right)^{\alpha+2} + \frac{1}{(\alpha + 1)(\alpha + 2)} \left( \frac{b-a}{x-a} \right)^{\alpha+2},$$
By using the properties of the modulus in Lemma 1, we have

\[
\begin{align*}
F &= \int_0^1 \left| \left( \frac{b-a}{x-a} - t \right)^a - \left( \frac{b-x}{x-a} \right)^a \right| (1-t) \, dt \\
&= \frac{1}{(\alpha + 1)(x-a)^{\alpha+2}} \left( \frac{b-x}{2} \right) + \frac{1}{(\alpha + 1)(x-a)^{\alpha+2}} \left( b-x \right) - E,
\end{align*}
\]

\[
\gamma_1 = \int_0^1 |t^a - 1| \, dt = \frac{\alpha}{(\alpha + 1)},
\]

\[
\gamma_2 = \int_0^1 |1 - t^a| \, dt = \frac{\alpha}{(\alpha + 1)},
\]

\[
\gamma_3 = \int_0^1 \left| \left( \frac{a-b}{x-b} - t \right)^a - \left( \frac{a-x}{x-b} \right)^a \right| \, dt
\]

\[
= -\frac{1}{(\alpha + 1)} \left( \frac{a-x}{x-b} \right)^{\alpha+1} - \left( \frac{a-x}{x-b} \right)^\alpha + \frac{1}{(\alpha + 1)} \left( \frac{a-b}{x-b} \right)^\alpha,
\]

\[
\gamma_4 = \int_0^1 \left| \left( \frac{b-x}{x-a} - t \right)^a - \left( \frac{b-x}{x-a} \right)^a \right| \, dt
\]

\[
= \left( \frac{b-x}{x-a} \right)^a + \frac{1}{(\alpha + 1)} \left( \frac{b-x}{x-a} \right)^{\alpha+1} - \frac{1}{(\alpha + 1)} \left( \frac{b-a}{x-a} \right)^{\alpha+1}.
\]

**Proof** By using the properties of the modulus in Lemma 1, we have

\[
\left| \left( \frac{(b-a)^a - (x-a)^a}{(b-a)^a} + \frac{(b-x)^a}{(b-a)^a} \right) f(b) + \left( \frac{(b-a)^a - (b-x)^a}{(b-a)^a} + \frac{(x-a)^a}{(b-a)^a} \right) f(a) \right| \leq \sum_{k=1}^4 |J_k|
\]

and using convexity of \( |f'| \), we have

\[
|J_1| \leq \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 (1-t^a) \left| f'(tx + (1-t)a) \right| \, dt
\]

\[
\leq \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} \left( \int_0^1 (1-t^a) \, dt \right)^{\frac{1}{2}} \left( \int_0^1 (1-t') \left| f'(tx + (1-t)a) \right|^2 \, dt \right)^{\frac{1}{2}}
\]

\[
= \frac{(x-a)^{\alpha+1}}{(b-a)^{\alpha}} \left( \gamma_1 \right)^{\frac{1}{2}} \left( A |f'(x)|^a + B |f'(a)|^a \right)^{\frac{1}{2}}
\]

and analogously

\[
|J_3| \leq \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 \left| \left( \frac{a-b}{x-b} - t \right)^a - \left( \frac{a-x}{x-b} \right)^a \right| \, dt
\]

\[
\leq \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \left( \int_0^1 \left| \left( \frac{a-b}{x-b} - t \right)^a - \left( \frac{a-x}{x-b} \right)^a \right| \, dt \right)^{\frac{1}{2}}
\]

\[
\times \left( \int_0^1 \left| \left( \frac{a-b}{x-b} - t \right)^a - \left( \frac{a-x}{x-b} \right)^a \right| \, dt \right)^{\frac{1}{2}}
\]

\[
= \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \left( \gamma_3 \right)^{\frac{1}{2}} \left( C |f'(x)|^a + D |f'(b)|^a \right)^{\frac{1}{2}},
\]

\[
|J_4| \leq \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \int_0^1 \left| \left( \frac{b-x}{x-a} - t \right)^a - \left( \frac{b-x}{x-a} \right)^a \right| \, dt
\]

\[
= \frac{(b-x)^{\alpha+1}}{(b-a)^{\alpha}} \left( \gamma_4 \right)^{\frac{1}{2}} \left( E |f'(x)|^a + F |f'(b)|^a \right)^{\frac{1}{2}}.
\]
using the convexity and the fact that, for \( \alpha \in (0, 1] \) and \( \forall t \in [0, 1] \),
\[
|J_2| \leq \frac{(b-x)^{\alpha +1}}{(b-a)^{\alpha}} \gamma_2 \left[ A|f'(x)|^q + B|f'(b)|^q \right]^{\frac{1}{q}}
\]
and similarly
\[
|J_4| \leq \frac{(b-x)^{\alpha +1}}{(b-a)^{\alpha}} \int_0^1 \left\{ \frac{(b-a)}{x-a} - t \right\}^\alpha \left[ \frac{(b-x)}{x-a} \right]^\alpha \left| f'(tx + (1-t)a) \right| dt
\]
\[
\leq \frac{(b-x)^{\alpha +1}}{(b-a)^{\alpha}} \left( \int_0^1 \left\{ \frac{(b-a)}{x-a} - t \right\}^\alpha \left[ \frac{(b-x)}{x-a} \right]^\alpha \left| f'(tx + (1-t)a) \right|^q dt \right)^{\frac{1}{q}}
\]
\[
\times \left( \int_0^1 \left\{ \frac{(b-a)}{x-a} - t \right\}^\alpha \left[ \frac{(b-x)}{x-a} \right]^\alpha \left| f'(tx + (1-t)a) \right| dt \right)^{\frac{1}{q}}
\]
\[
= \frac{(b-x)^{\alpha +1}}{(b-a)^{\alpha}} \gamma_4 \left[ E|f'(x)|^q + F|f'(a)|^q \right]^{\frac{1}{q}}.
\]

The proof is completed. \( \square \)

**Corollary 2** On letting \( \alpha = 1 \), \( x = \frac{a+b}{2} \) and \( |f'(a)| = |f'(b)| \leq M \) in Theorem 5, inequality (6) reduces to the inequality
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{M}{4} (b-a).
\]
(7)

**Remark 2** The obtained inequality (7) is an improvement of the inequality as in (2).

In the following, we obtain an estimate of the Hermite–Hadamard inequality for concave functions.

**Theorem 6** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function on \((a, b)\) such that \( f' \in L_1[a, b] \). If \(|f'|^q\) is concave on \([a, b]\), for some fixed \( p > 1 \) with \( q = \frac{p}{p+1} \), the following inequality for fractional integrals holds:
\[
\left| \left\{ \frac{(b-a)^{\alpha}}{(b-a)^{\alpha}} - (x-a)^{\alpha} \right\} + \frac{(b-x)^{\alpha}}{(b-a)^{\alpha}} \right| \left( \left| f'(x) \right|^q + \left| f'(b) \right|^q \right) \frac{1}{2}
\]
\[
\leq \left[ \gamma_1 \int f'(\alpha + 1) \left( \frac{Ax + Bb}{\alpha} \right) \, dx + \gamma_2 \int f'(\alpha + 1) \left( \frac{Ax + Bb}{\alpha} \right) \, dx \right]
\]
\[
+ \gamma_3 \int f'(\alpha + 1) \left( \frac{Cx + Db}{\alpha} \right) \, dx + \gamma_4 \int f'(\alpha + 1) \left( \frac{Ex + Fa}{\alpha} \right) \, dx \right].
\]
(8)

**Proof** Using the concavity of \(|f'|^q\) and the power-mean inequality, we obtain
\[
\left| f'(tx + (1-t)y) \right|^q \geq q \left| f'(x) \right|^q + (1-q) \left| f'(y) \right|^q
\]
\[
\geq q \left| f'(x) \right|^q + (1-q) \left| f'(y) \right|^q.
\]
Hence

$$|f'(tx + (1-t)y)| \geq t|f'(x)| + (1-t)|f'(y)|,$$

so \( |f'| \) is also concave. By the Jensen integral inequality, we have

$$|I_1| \leq \frac{(x-a)^{a+1}}{(b-a)^{a+1}} \left( \int_0^1 |1-t^a| \cdot \frac{f'(tx + (1-t)a) \ dt}{\int_0^1 |1-t^a| \ dt} \right),$$

and similarly

$$|I_2| \leq \frac{(b-x)^{a+1}}{(b-a)^{a+1}} \gamma_2 |f'(Ax + Ba)\),$$

$$|I_3| \leq \frac{(b-x)^{a+1}}{(b-a)^{a+1}} \left( \int_0^1 \left( \frac{a-b}{x-b} - t \right)^a - \left( \frac{a-x}{x-b} \right)^a \right) \left| f' \left( \frac{b-x}{x-b} - t \right)^a - \left( \frac{a-x}{x-b} \right)^a \right| \ dt,$$

and

$$|I_4| \leq \frac{(x-a)^{a+1}}{(b-a)^{a+1}} \left( \int_0^1 \left( \frac{b-x}{x-a} - \left( \frac{b-a}{x-a} - t \right)^a \right) \right) \left| f' \left( \frac{b-x}{x-a} - \left( \frac{b-a}{x-a} - t \right)^a \right) \ dt,$$

$$|I_4| \leq \frac{(Ex + Fb}{\gamma_4} |f'(Ax + Ba)\).$$

The proof is completed. \( \square \)

Remark 3 On letting \( \alpha = 1, x = \frac{a+b}{2} \) in Theorem 6, inequality (8) reduces to inequality (4).

3 Conclusion

In this article, based on a more general inequality, the authors have determined a few inequalities of Hermite–Hadamard type for functions that possess a first derivative on the interior of an interval of real numbers, by utilizing the Hölder inequality and the assumptions that the mappings \(|f'|^q, q \geq 1 \) are convex and concave. The outcomes exhibited here surely give refinements of those outcomes demonstrated in [1, 2] and [3], and we can get many intriguing results for \( \alpha = 1 \) and \( x = \frac{a+b}{2} \).

Acknowledgements

The first author is grateful to Prof. Dr. S.M. Junaid Zaidi, Executive Director and Prof. Dr. Raheel Qamar Rector, COMSATS University Islamabad Sahiwal Campus, Pakistan, for providing excellent research facilities. Fifth and seventh authors are grateful to Prof. Dr. K.T. Ooi, HOD of MAE and Dr. K.C. Leong, Asso. Prof. MAE, Nanyang Technological University Singapore for providing excellent research facilities for this research work too.
Funding
This research [Grant No. 5325/Federal/NRPU/R&D/HEC/2016] was partially supported by the Higher Education Commission of Pakistan.

Competing interests
The authors declare to have no conflict of interest.

Authors’ contributions
Dr. SQ, JN and Prof. Dr. FA analyzed the problem and suggested mathematical modeling. Dr. SIB and Dr. AA generalized the results by proposing various lemmas. This manuscript has been written and revised by Prof. Dr. SH and Dr. MI, who also made some necessary corrections regarding mathematical formulations in the original. Dr. AA, Dr. MI and Prof. Dr. SH prepared a revised version of the manuscript. All authors read and approved the final manuscript.

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Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 11 December 2018 Accepted: 8 April 2019 Published online: 18 April 2019

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