<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Asymptotically locally optimal weight vector design for a tighter correlation lower bound of quasi-complementary sequence sets</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Liu, Zilong; Guan, Yong Liang; Mow, Wai Ho</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>Liu, Z., Guan, Y. L., &amp; Mow, W. H. (2017). Asymptotically locally optimal weight vector design for a tighter correlation lower bound of quasi-complementary sequence sets. IEEE Transactions on Signal Processing, 65(12), 3107-3119. doi:10.1109/TSP.2017.2684740</td>
</tr>
<tr>
<td><strong>Date</strong></td>
<td>2017</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/10220/49529">http://hdl.handle.net/10220/49529</a></td>
</tr>
<tr>
<td><strong>Rights</strong></td>
<td>© 2017 IEEE. Personal use of this material is permitted. Permission from IEEE must be obtained for all other uses, in any current or future media, including reprinting/republishing this material for advertising or promotional purposes, creating new collective works, for resale or redistribution to servers or lists, or reuse of any copyrighted component of this work in other works. The published version is available at: <a href="https://doi.org/10.1109/TSP.2017.2684740">https://doi.org/10.1109/TSP.2017.2684740</a></td>
</tr>
</tbody>
</table>
Asymptotically Locally Optimal Weight Vector Design for a Tighter Correlation Lower Bound of Quasi-Complementary Sequence Sets

Zilong Liu, Yong Liang Guan, Member, IEEE, and Wai Ho Mow, Senior Member, IEEE

Abstract—A quasi-complementary sequence set (QCSS) refers to a set of two-dimensional matrices with low nontrivial aperiodic auto- and cross-correlation sums. For multicarrier code-division multiple-access applications, the availability of large QCSSs with low correlation sums is desirable. The generalized Levenshtein bound (GLB) is a lower bound on the maximum aperiodic correlation sum of QCSSs. The bounding expression of GLB is a fractional quadratic function of a weight vector $w$ and is expressed in terms of three additional parameters associated with QCSS: the set size $K$, the number of channels $M$, and the sequence length $N$. It is known that a tighter GLB (compared to the Welch bound) is possible only if the condition $M \geq 2$ and $K \geq K + 1$, where $K$ is a certain function of $M$ and $N$, is satisfied. A challenging research problem is to determine if there exists a weight vector that gives rise to a tighter GLB for all (not just some) $K \geq K + 1$ and $M \geq 2$, especially for large $N$, i.e., the condition is asymptotically both necessary and sufficient. To achieve this, we analytically optimize the GLB which is (in general) nonconvex as the numerator term of this problem is an indefinite quadratic function of the weight vector. Our key idea is to apply the frequency domain decomposition of the circulant matrix (in the numerator term) to convert the nonconvex problem into a convex one. Following this optimization approach, we derive a new weight vector meeting the aforementioned objective and prove that it is a local minimizer of the GLB under certain conditions.

Index Terms—Fractional quadratic programming, convex optimization, Welch bound, Levenshtein bound, perfect complementary sequence set (PCSS), quasi-complementary sequence set (QCSS), Golay complementary pair.

I. INTRODUCTION

In recent years, multicarrier code-division multiple-access (MC-CDMA) based on the quasi-/perfect-complementary sequence set (in abbreviation, QCSS/PCSS) has attracted much attention due to its potential to achieve low-/zero-interference multiuser performance [1], [2]. Here, a QCSS (or PCSS) refers to a set of two-dimensional matrices with low (or zero) non-trivial auto- and cross-correlation sums [3]–[5]. In this paper, a complementary sequence is also called a complementary matrix, and vice versa.

To deploy a QCSS (or PCSS) in an MC-CDMA system, every data symbol of a specific user is spread by a complementary matrix by simultaneously sending out all of its row sequences over a number of non-interfering subcarrier channels. Because of this, the number of row sequences of a complementary matrix, denoted by $M$, is also called the number of channels. At a matched-filter based receiver, de-spreading operations are performed separately in each subcarrier channel, followed by summing the correlator outputs of all the subcarrier channels to attain a correlation sum which will be used for detection.

A PCSS may also be called a mutually orthogonal complementary sequence set (MOCSS) [6]–[9], a concept extended from mutually orthogonal Golay complementary pairs (GCPs) [10]–[13]. However, a drawback of PCSS is its small set size [14]. Specifically, the set size (denoted by $K$) of PCSS is upper bounded by the number of channels, i.e., $K \leq M$. ¹ This means that a PCSS based MC-CDMA system with $M$ subcarriers can support at most $M$ users only. Against such a backdrop, there have been two approaches aiming to provide a larger set size, i.e., $K > M$. The first approach is to design zero- or low- correlation zone (ZCZ/LCZ) based complementary sequence sets, called ZCZ-CSS [15], [16] or LCZ-CSS [17]. A ZCZ-CSS (LCZ-CSS) based MC-CDMA system is capable of achieving zero-(low-) interference performance but requires a closed-control loop to dynamically adjust the timings of all users such that the received signals can be quasi-synchronously aligned within the ZCZ (LCZ). A second approach is to design QCSS which has uniformly low correlation sums over all non-trivial time-shifts.

¹It is noted that the upper bound of set size $K$ is independent of sequence length $N$. 

Manuscript received October 11, 2016; revised January 16, 2017; accepted February 21, 2017. Date of publication March 20, 2017; date of current version April 18, 2017. The associate editor coordinating the review of this manuscript and approving it for publication was Dr. Pengfei Xia. The work of Z. Liu and Y. L. Guan was supported by the Advanced Communications Research Program under grant contract DSQCL14095, and the NRF-NSFC joint grant NRF2016NRFGNSFC001-089. The work of W. H. Mow was supported by the General Research Fund from the Research Grants Council of the Hong Kong Special Administrative Region, China, under Project 16233816. (Corresponding author: Zilong Liu.)

Z. Liu and Y. L. Guan are with the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore (e-mail: zilongliu@ntu.edu.sg; eylguan@ntu.edu.sg).

W. H. Mow is with the Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Hong Kong (e-mail: w.mow@ieee.org).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TSP.2017.2684740
where every quasi-complementary sequence is a matrix of order $M \times N$ (thus, every row sequence has length of $N$) with assumed energy of $MN$. The aforementioned set size upper bound of PCSS, namely, $K \leq M$, can also be obtained from (1) by setting $\delta_{\text{max}} = 0$. On the other hand, if $0 < \delta_{\text{max}} \leq MN$, one can show that $K > M$, meaning that a larger set size can be supported by QCSS.

Recently, a generalized Levenshtein bound (GLB) for QCSS has been derived by Liu, Guan and Mow in [5, Theorem 1]. The key idea behind the GLB (including the Levenshtein bound [19]) is that the weighted mean square aperiodic correlation of any sequence subset over the complex roots of unity should be equal to or greater than that of the whole set which includes all possible complex roots-of-unity sequences. The Levenshtein bound was extended from binary sequences to complex roots-of-unity sequences by Boztas [20]. A lower bound for aperiodic LCZ sequence sets was derived in [21] by an approach similar to Levenshtein’s.

In its bounding equation, GLB is a function of the “simplex” weight vector $w$, the set size $K$, the number of channels $M$, and the row sequence length $N$. A necessary condition (shown in [5, Theorem 2]) for the GLB to be tighter than the Welch bound is that $K \geq K + 1$, where $K$ is a function of $K, M, N$ and will be formally defined in Section II. The main objective of this paper is to optimize and then tighten the GLB for all $K \geq K + 1$ (instead of some). For this, we are to find a (locally) optimal weight vector which is used in the bounding equation. A similar research problem was raised in [19] for traditional binary sequences (i.e., non-QCSS with $M = 1$). See [22] for more details. The optimization of GLB on QCSS (with $M \geq 2$), however, is more challenging because an analytical solution to a non-convex GLB (in terms of weight vector $w$) for all possible cases of $(K, M)$ is in general intractable.

We first introduce a novel frequency-domain optimization approach in Subsection III-B to enable fractional quadratic programming of GLB in an analytical manner. This is achieved by properly exploiting the specific structure of the circulant quadratic matrix in the numerator of the fractional quadratic term of GLB. Following this optimization approach, we find a new weight vector which leads to a tighter GLB for all $(K, M)$ cases satisfying $K \geq K + 1$ and $M \geq 2$, asymptotically (in $N$). Our finding shows that the condition of $K \geq K + 1$, shown in [5, Theorem 2], is not only necessary but also sufficient, as $N$ tends to infinity. Moreover, in Subsection III-C, it is proved that the newly found weight vector is a local minimizer to the fractional quadratic function of GLB, asymptotically.

We then examine in Sections IV two weight vectors which were presented in [22] for the tightening of the Levenshtein bound on conventional single-channel (i.e., $M = 1$) sequence sets. We extend their tightening capability to GLB on multi-channel (i.e., $M \geq 2$) QCSS, although the proof is not straightforward. It is shown that each of these two weight vectors gives rise to a tighter GLB (over the Welch bound) for several small values of $M$ provided that $K \geq K + 1$. It is also noted that the GLB from the newly found weight vector is (in general) tighter than the GLBs from these two (earlier found) weight vectors, as shown by some numerical results.

II. PRELIMINARIES

In this section, we first present some necessary notations and define QCSS. Then, we give a brief review of GLB.

A. Introduction to QCSS

For two complex-valued sequences $a = [a_0, a_1, \cdots, a_{N-1}]$ and $b = [b_0, b_1, \cdots, b_{N-1}]$, their aperiodic correlation function at time-shift $\tau$ is defined as

$$\rho_{a,b}(\tau) = \begin{cases} \sum_{t=0}^{N-1-\tau} a_t b_{t+\tau}, & 0 \leq \tau \leq (N-1); \\ \sum_{t=0}^{N-1-\tau} a_t b_{t+\tau}, & (N-1) \leq \tau \leq -1; \\ 0, & |\tau| \geq N. \end{cases}$$

When $a \neq b$, $\rho_{a,b}(\tau)$ is called the aperiodic cross-correlation function (ACCf); otherwise, it is called the aperiodic auto-correlation function (AACf). For simplicity, the AACf of $a$ is denoted by $\rho_a(\tau)$.

Let $C = \{C^0, C^1, \cdots, C^{K-1}\}$ be a set of $K$ matrices, each of order $M \times N$ (where $M \geq 2$), i.e.,

$$C^\nu = \begin{bmatrix} c^\nu_0 & c^\nu_1 & \cdots & c^\nu_{N-1} \\ c^\nu_1 & c^\nu_2 & \cdots & c^\nu_{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ c^\nu_{M-1} & c^\nu_{M-1} & \cdots & c^\nu_{M-1} \end{bmatrix}_{M \times N}$$

(3)

where $0 \leq \nu \leq K - 1$. Define the “aperiodic correlation sum” of matrices $C^\nu$ and $C^\mu$ as follows,

$$\rho_{C^\nu, C^\mu}(\tau) = \sum_{m=0}^{M-1} \rho_{c^\mu, c^\nu}(\tau), 0 \leq \mu, \nu \leq K - 1.$$  

(4)

Also, define the aperiodic auto-correlation tolerance $\delta_a$ and the aperiodic cross-correlation tolerance $\delta_c$ of $C$ as

$$\delta_a \triangleq \max \left\{ \left| \rho_{C^\nu, C^\mu}(\tau) \right| : 0 < \tau \leq N - 1, \quad 0 \leq \mu \leq K - 1, \right\},$$

$$\delta_c \triangleq \max \left\{ \left| \rho_{C^\nu, C^\mu}(\tau) \right| : 0 < \tau \leq N - 1, \quad \mu \neq \nu, 0 \leq \mu, \nu \leq K - 1, \right\}$$

respectively. Moreover, define the aperiodic tolerance (also called the “maximum aperiodic correlation magnitude”) of $C$ as $\delta_{\text{max}} \triangleq \max \{\delta_a, \delta_c\}$. When $\delta_{\text{max}} = 0$, $C$ is called a perfect complementary sequence set (PCSS); otherwise, it is called a quasi-complementary sequence set (QCSS). In particular, when $M = 2$ and $K = 1$, a PCSS reduces to a matrix consisting of two row sequences which have zero out-of-phase aperiodic autocorrelation sums. Such matrices are called Golay complementary matrices (GCMs) or Golay complementary pairs (GCPs) in this paper, and either sequence in a GCP is called a Golay sequence.

Note that the transmission of a PCSS or a QCSS requires a multi-channel system. Specifically, every matrix in a PCSS (or a QCSS) needs $M \geq 2$ non-interfering channels for the separate

2QCSS can also be defined with respect to the "periodic correlation sums". The interested reader may refer to [4].
transmission of $M$ row sequences. This is different from the traditional single-channel sequences with $M = 1$ only.

B. Review of GLB

Let $w = [w_0, w_1, \ldots, w_{2N-2}]^T$ be a “simplex” weight vector which is constrained by

$$w_i \geq 0, \ i = 0, 1, \ldots, 2N - 2, \ \text{and} \ \sum_{i=0}^{2N-2} w_i = 1. \quad (5)$$

Define a quadratic function

$$Q(w, a) \triangleq w^T Q_a w$$

$$= a \sum_{i=0}^{2N-2} w_i^2 + \sum_{s, t=0}^{2N-2} \tau_{s,t,N} w_s w_t, \quad (6)$$

where $Q_a$ is a $(2N - 1) \times (2N - 1)$ circulant matrix with all of its diagonal entries equal to $a$, and its off-diagonal entries $Q_a(s, t) = \tau_{s,t,N}$, where $s \neq t$ and

$$0 \leq \tau_{s,t,N} \leq \min\{|t-s|, 2N-1-|t-s|\} \leq N-1. \quad (7)$$

We give an example below to show the circulant matrix structure of $Q_a$.

Example 1: Let $n = 5$. The $(s, t)$ element of $Q_a$ for $s \neq t$ is $Q_a(s, t) = \min\{|t-s|, 9-|t-s|\}$. The matrix $Q_a$ is

$$Q_a = \begin{bmatrix}
 a & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\
 1 & a & 1 & 2 & 3 & 4 & 3 & 2 \\
 2 & 1 & a & 1 & 2 & 3 & 4 & 3 \\
 3 & 2 & 1 & a & 1 & 2 & 3 & 4 \\
 4 & 3 & 2 & 1 & a & 1 & 2 & 3 \\
 4 & 3 & 4 & 3 & 2 & 1 & a & 1 \\
 2 & 3 & 4 & 3 & 2 & 1 & a & 1 \\
 1 & 2 & 3 & 4 & 3 & 2 & 1 & a
\end{bmatrix}. \quad (8)$$

Lemma 1: The GLB for QCSS over complex roots of unity [5], a fractional quadratic function of “simplex” weight vector $w$ [see (5)], is shown below.

$$\delta_\text{max}^2 \geq M \left[ N - \frac{Q(w, a)}{1 - \frac{a}{K}} \right], \quad (8)$$

in which $a = N(MN-1)/K$ is set in the quadratic function $Q(w, a)$ [see (6)]. A weaker simplified version of GLB with $a = MN^2/K$ is given below.

$$\delta_\text{max}^2 \geq M \left[ N - \frac{Q(w, MN^2)}{K} \right]. \quad (9)$$

Remark 1: Setting $w = \frac{1}{\sqrt{N}} [1, 1, \ldots, 1]^T$, the GLB reduces to the Welch bound for QCSS in (1).

Remark 2: [5, Theorem 2] For the GLB to be tighter than the corresponding Welch bound, it is necessary that $K \geq \overline{K} + 1$.

Remark 3: [5, Corollary 1] Applying the weight vector $w$ with

$$w_i = \begin{cases}
 \frac{1}{m}, & i \in \{0, 1, \ldots, m-1\}; \\
 0, & i \in \{m, m+1, \ldots, 2N-2\};
\end{cases} \quad (10)$$

where $1 \leq m \leq N$, to (8), we have

$$\delta_\text{max}^2 \geq \max_{1 \leq m \leq N} \frac{3MNKm - 3M^2N^2 - MK(m^2 - 1)}{3(mK-1)}. \quad (11)$$

The lower bound in (12) is tighter than the Welch bound for QCSS in (1) if one of the two following conditions is fulfilled:

1) $3M + 1 \leq K \leq 4M - 1$, $M \geq 2$ and

$$N \geq \frac{K - 1 + \sqrt{-3K^2 + (12M - 6)K + 12M + 1}}{2(K - 3M)} + 1; \quad (13)$$

2) $K \geq 4M$, $M \geq 2$ and $N \geq 2$.

III. PROPOSED WEIGHT VECTOR FOR TIGHTER GLB

A. Motivation

The necessary condition in Remark 2 implies that for a given $M,N$, the Welch bound for QCSS cannot be improved if $K \leq \overline{K}$, where $\overline{K}$ is defined in (10). On the other hand, the weight vector in (11) can only lead to a tighter GLB for $K \geq 3M + 1$. Because of this, the tightness of GLB is unknown in the following ambiguous zone.

$$\frac{\overline{K}}{M} < \frac{K}{M} < 3 + \frac{1}{M}. \quad (14)$$

For sufficiently large $N$, we have

$$\lim_{N \to \infty} \overline{K} = \left\lfloor \frac{\pi^2 M}{4} \right\rfloor. \quad (15)$$

and thus the above $K/M$ zone further reduces to

$$\frac{\pi^2 M}{4} < \frac{K}{M} < 3 + \frac{1}{M}, \quad (16)$$

by recalling (15). One may visualize this zone in the shaded area of Fig. 1 for $2 \leq M \leq 256$.

We are therefore interested in finding a weight vector which is capable of optimizing and tightening the GLB for all (rather than some) $K \geq \overline{K} + 1$. Relating this objective to Fig. 1, such a weight vector can give us a tighter GLB for the largest $K/M$ region right above the red diamond symbols.

However, the optimization of GLB in (8) is challenging because its fractional quadratic term (in terms of $w$) is indefinite. More specifically, the quadratic term $Q(w, a = \frac{N(MN-1)}{K})$ in the numerator is indefinite as some eigenvalues of the corresponding circulant matrix are negative when $K \geq \overline{K} + 1$ [5, Appendix B]. It is noted that indefinite quadratic programming
(QP) is NP-hard [23], even it has one negative eigenvalue only [24]. Moreover, checking local optimality of a feasible point in constrained QP is also NP-hard [25]. Although some optimality conditions for constrained QP have been derived by Bomze from the copositivity perspective [26]–[28], the situation becomes more complicated when indefinite fractional quadratic programming (FQP) problems are dealt with. According to [29], GLB may be classified as a standard FQP (StFQP) as the feasibility conditions for constrained QP have been derived by Bomze.

By [5, Appendix B], we have

\[ \lambda_l = a - \frac{1 - (-1)^l \cos \frac{\pi l}{N-1}}{2 \sin^2 \frac{\pi l}{2(N-1)}} \]

for \( l = 1, \ldots, 2N-2 \). Note that \( \lambda_l = \lambda_{2N-1-l} \) for \( l \in \{1, 2, \ldots, N-1\} \). Moreover, we remark that

\[ \lambda_l > \lambda_1, \text{ for } 2 \leq l \leq N-1. \]

This is because

1) if \( l \) is odd:

\[ \lambda_l - \lambda_1 = \frac{\sin^2 \frac{\pi l}{2(2N-1)} - \sin^2 \frac{\pi l}{2(2N-1)}}{4 \sin^2 \frac{\pi l}{2(2N-1)}} > 0. \]

2) if \( l \) is even:

\[ \lambda_l - \lambda_1 = \frac{\cos \frac{\pi l}{2N-1} + \cos \frac{\pi l}{2N-1}}{8 \sin^2 \frac{\pi l}{2(2N-1)} \cos^2 \frac{\pi l}{2(2N-1)}} > 0. \]

To maximize the GLB in (8), it is equivalent to consider the following optimization problem.
**Problem 1:**

\[
\min_v \lambda_0 + \sum_{l=1}^{2N-2} \lambda_1 |v_l|^2 \\
2N - 1 - \frac{\lambda_0}{K} - \frac{\lambda_1}{K} \sum_{l=1}^{2N-2} |v_l|^2.
\]

subject to \( w = \frac{1}{2N - 1} \Phi_{2N-1}^T v \geq 0. \) (29)

Since \( w \) is real-valued, \( v \) is conjugate symmetric, i.e., \( v_l = v_{2N-1-l}^* \) for \( l = 1, 2, \ldots, 2N - 2 \). Having this in mind, we define

\[
r^2 = \sum_{l=1}^{N-1} |v_l|^2 = \frac{2N-2}{K} \sum_{l=1}^{2N-2} |v_l|^2.
\]

Taking advantage of the fact that \( \lambda_1 = \lambda_{2N-2} \) are strictly smaller than other \( \lambda_l \)'s with nonzero \( l \) as shown in (26), we have

\[
\sum_{l=1}^{2N-2} \lambda_l |v_l|^2 = 2\lambda_1 r^2 + \sum_{l=2}^{2N-3} (\lambda_l - \lambda_1) |v_l|^2 \geq 2\lambda_1 r^2,
\]

where the equality is achieved if and only if \( v_l = 0 \) for \( l = 2, 3, \ldots, 2N - 3 \). Inspired by this observation, we relax the non-negativity constraint on \( w \), i.e., some negative \( |v_l|^2 \)'s may be allowed (but the sum of all elements of \( w \) must still be equal to 1). With this, the optimization problem in (29) can be translated to

\[
\min_r \min_{\sum_{l=1}^{2N-2} |v_l|^2 = r^2} \lambda_0 + \sum_{l=1}^{2N-2} \lambda_l |v_l|^2 \\
2N - 1 - \frac{1 + 2r^2}{K}.
\]

Taking advantage of the fact that \( \lambda_1 = \lambda_{2N-2} \) are strictly smaller than other \( \lambda_l \)'s with nonzero \( l \) as shown in (26), we have

\[
\sum_{l=1}^{2N-2} \lambda_l |v_l|^2 = 2\lambda_1 r^2 + \sum_{l=2}^{2N-3} (\lambda_l - \lambda_1) |v_l|^2 \geq 2\lambda_1 r^2,
\]

where the equality is achieved if and only if \( v_l = 0 \) for \( l = 2, 3, \ldots, 2N - 3 \). Inspired by this observation, we relax the non-negativity constraint on \( w \), i.e., some negative \( |v_l|^2 \)'s may be allowed (but the sum of all elements of \( w \) must still be equal to 1). With this, the optimization problem in (29) can be translated to

\[
\min_r \min_{\sum_{l=1}^{2N-2} |v_l|^2 = r^2} \lambda_0 + \sum_{l=1}^{2N-2} \lambda_l |v_l|^2 \\
2N - 1 - \frac{1 + 2r^2}{K}.
\]

where

\[
\lambda_0 = \frac{N(MN - 1)}{K} + N(N - 1),
\]

\[
\lambda_1 = \frac{N(MN - 1)}{K} - \frac{1}{4 \sin^2 \frac{\pi}{2(N - 1)}},
\]

From now on, we adopt the setting of

\[
v_1 = v_{2N-2} = r \exp \left( \sqrt{-1} \theta \right),
\]

\[
v_l = 0, \quad \text{for } l = 2, 3, \ldots, 2N - 3,
\]

where \( r, \theta \) denote the magnitude and phase of \( v_1 \), respectively. Since \( w = \frac{1}{2N - 1} \Phi_{2N-1}^T v \), we have

\[
w = \frac{1}{2N - 1} \begin{bmatrix} 1 + 2r \cos \theta, 1 + 2r \cos \left( \theta + \frac{2\pi}{2N - 1} \right), \\ \cdots, 1 + 2r \cos \left( \theta + \frac{2\pi(2N - 2)}{2N - 1} \right) \end{bmatrix}^T.
\]

To optimize the fractional function in (32), we have the following lemma.

**Lemma 2:** The fractional function \( \frac{\lambda_0 + 2\lambda_1 r^2}{2N - 1 - \frac{1 + 2r^2}{K}} \) in terms of \( r^2 \) in (32) is

1) monotonically decreasing in \( r^2 \) if \( K \geq \overline{K} + 1 \) and \( \lambda_0 / \lambda_1 \) \( < (2N - 1)K - 1 \);

2) monotonically increasing in \( r^2 \) if \( K \leq \overline{K} \), or \( K \geq \overline{K} + 1 \) and \( \lambda_0 / \lambda_1 \) \( \geq (2N - 1)K - 1 \).

**Proof:** To prove 1), we first show that \( \lambda_1 < 0 \) if and only if

\[
K \geq \overline{K} + 1 = \left( 4(MN - 1)N \sin^2 \frac{\pi}{2(N - 1)} \right) + 1,
\]

where \( \overline{K} \) is defined in (10). For ease of analysis, we write

\[
4(MN - 1)N \sin^2 \frac{\pi}{2(N - 1)} = n + \epsilon,
\]

where \( n \) is a positive integer and \( 0 \leq \epsilon < 1 \). Thus, \( \overline{K} + 1 = n + 1 \). Consequently, we have

\[
\lambda_1 = a - \frac{1}{4 \sin^2 \frac{\pi}{2(N - 1)}} \left( 1 - \frac{n + 1}{n + \epsilon} \right).
\]

< 0,

with which the proof of 1) follows. The proof of 2) can be easily obtained by following a similar argument.

For 2) of Lemma 2, it can be readily shown that the minimum of the fractional function \( \frac{\lambda_0 + 2\lambda_1 r^2}{2N - 1 - \frac{1 + 2r^2}{K}} \) in (32) is achieved at \( r = 0 \). Thus, the weight vector in (34) reduces to

\[
w = \frac{1}{2N - 1} [1, 1, \ldots, 1]^T,
\]

where the corresponding GLB reduces to the Welch bound in (1).

Next, let us focus on the application of 1) for GLB tightening. In this case, we wish to know the upper bound of \( r^2 \) in order to minimize the fractional function of \( r^2 \) in (32). Coming back to the constraint of \( w \) given in (5), \( r \) and \( \theta \) should satisfy

\[
1 + 2r \min_{i=0,1,\ldots,2N-2} \cos \left( \theta + \frac{2\pi i}{2N - 1} \right) \geq 0.
\]

Thus,

\[
0 < r \leq \max_{i=0,1,\ldots,2N-2} \min -1 \cos \left( \theta + \frac{2\pi i}{2N - 1} \right) = \frac{1}{2 \cos \left( \frac{\pi}{2N - 1} \right)},
\]

where the upper bound is achieved with equality when \( \theta = \frac{2\pi j}{2N - 1} \) for any integer \( j \). By substituting \( r = \frac{1}{2 \cos \left( \frac{\pi}{2N - 1} \right)} \) into (34), we obtain the following weight vector.

\[
w_i = \frac{1}{2N - 1} \begin{bmatrix} 1 + \cos \frac{2\pi(i + j)}{2N - 1} \cos \frac{\pi}{2N - 1} \end{bmatrix}^T,
\]

where \( i = 0, 1, \ldots, 2N - 2 \) and \( j \) is any integer. The resultant GLB from this weight vector is shown in the following lemma.

**Lemma 3:** For \( K \geq \overline{K} + 1 \) and \( \frac{\lambda_0}{\lambda_1} \) \( < (2N - 1)K - 1 \), we have

\[
\delta_{\text{max}}^2 \geq M \left[ N - \frac{K \left( \lambda_0 - \frac{|\lambda_1|}{4 \sin^2 \frac{\pi}{2(N - 1)}} \right)}{(2N - 1)K - 1 - \frac{1 + 2(2N - 2)/K}{K}} \right],
\]

where \( \lambda_0, \lambda_1 \) are given in (33).
To analyze the asymptotic tightness of the lower bound in (41), we note that when \( N \) is sufficiently large, the second condition in Lemma 3, i.e.,

\[
\frac{\lambda_0}{\lambda_1} < (2N - 1)K - 1,
\]

is true for \( K \geq \overline{K} + 1 \). To show this, we substitute \( \lambda_0, \lambda_1 \) into (42). After some manipulations, one can see that the inequality in (42) holds if and only if

\[
K > 4(MN - 1)N \sin^2 \frac{\pi}{2(2N - 1)} + \frac{4N(N - 1) \sin^2 \frac{\pi}{2N - 1} + 1}{2N - 1}.
\]

Carrying on the expression in (35), we require

\[
K - n \geq 1 > \epsilon \Rightarrow \frac{4N(N - 1) \sin^2 \frac{\pi}{2N - 1} + 1}{2N - 1},
\]

which is guaranteed to hold for sufficiently large \( N \) because \( \epsilon \) is strictly smaller than 1 by assumption. Furthermore, we note that

\[
\lim_{N \to \infty} \frac{K \lambda_0}{N} = (K + M)N,
\]

\[
\lim_{N \to \infty} \frac{K |\lambda_1|}{2N \cos^2 \frac{\pi}{2N - 1}} = \left( 2K \frac{\pi}{2} - \frac{M}{2} \right) N.
\]

Therefore,

\[
\lim_{N \to \infty} \frac{K \left( \lambda_0 - \frac{2 |\lambda_1|}{2N \cos^2 \frac{\pi}{2N - 1}} \right)}{N \cdot \left( (2N - 1)K - 1 - \frac{1}{2 \cos^2 \frac{\pi}{2N + 1}} \right)} = \frac{3M}{4K} + 1 - \frac{1}{\pi^2}.
\]

On the other hand, let us rewrite the Welch bound expression (1) as

\[
M^2N^2 \frac{K M - 1}{K(2N - 1) - 1} = M(N - R_1)
\]

with

\[
R_1 \triangleq \frac{N(MN - 1) + N(N - 1)K}{(2N - 1)K - 1}.
\]

Then,

\[
\lim_{N \to \infty} \frac{R_1}{N} = \frac{1}{2} + \frac{M}{2K}.
\]

With (48) and (49), one can show that the lower bound in Lemma 3 is asymptotically tighter than the Welch bound in (1) if and only if the following equation is satisfied.

\[
\frac{1}{2} + \frac{M}{2K} > \frac{3M}{4K} + 1 - \frac{1}{\pi^2}.
\]

Equivalently, we need to prove that for \( K = \lfloor \frac{\pi^2 M}{4} \rfloor + 1 \) (as \( N \to \infty \)), the following inequality holds.

\[
d_1(M) \triangleq \frac{\lfloor \frac{\pi^2 M}{4} \rfloor + 1}{M} - \frac{\pi^2}{4} > 0.
\]

One can readily show that the condition \( d_1(M) > 0 \) given in (51) is true for all \( M \geq 2 \). Therefore, we have the following theorem.

**Theorem 1:** The GLB in (41) which arises from the weight vector in (40) reduces to

\[
d^2_{\max} \geq MN \left[ \frac{1}{2} + \frac{1}{\pi^2} - 3M \frac{4}{4K} \right],
\]

for sufficiently large \( N \). Such an asymptotic lower bound is tighter than the Welch bound for all \( K \geq \overline{K} + 1 \) and for all \( M \geq 2 \).

Next, we prove the proposed weight vector in (40) is a local minimizer of the GLB in (8) under certain condition. We consider the weight vector \( w \) by setting \( j = 0 \) in (40) because other values of \( j \) will lead to identical value of GLB [cf. (22) and (23)].

\[
w_i = \frac{1}{2N - 1} \left( 1 + \frac{1}{2 \cos \frac{\pi}{2N - 1}} \right), \quad i \in \{0, 1, \cdots, 2N - 2\}.
\]

Note that the frequency domain vector \( v = F_{2N - 1}w \) has \( v_0 = 1, v_1 = v_{2N - 2} = \frac{1}{2 \cos \frac{\pi}{2N - 1}} \) and \( v_l = 0 \) for all \( l \in \{2, 3, \cdots, 2N - 3\} \). Our problem here can be formally cast as follows.

**Problem 2:** Define the fractional quadratic function \( f(x) \) as follows.

\[
f(x) \triangleq \frac{x^T Q_a x}{1 - \frac{1}{\pi} \cdot x^T x},
\]

where \( x_i \geq 0, i \in \{0, 1, \cdots, 2N - 2\}, \sum_{i=0}^{2N-2} x_i = 1, Q_a \) is the circulant matrix defined in (6) which has order \((2N - 1)\) and with \( a = (MN - 1)/K \). When \( K = \overline{K} + 1 \) and \( M, N \) becomes sufficiently large, prove that the weight vector \( w \) in (53) is a local minimizer of \( f(x) \), i.e.,

\[
f(w + e) \geq f(w),
\]

holds for any feasible perturbation \( e \) which has sufficiently small norm.

**Proof:** See Appendix A. \( \Box \)

**Remark 4:** Following a proof similar to the above, one can easily show that the weight vector \( w \) in (53) is also a local minimizer of the constrained QP of \( \min_w Q(w, \frac{N(MN - 1)}{K}) \) when \( K = \overline{K} + 1 \) and \( M, N \) are sufficiently large.

**IV. DISCUSSIONS AND COMPARISONS**

In this section, we first consider another two weight vectors and study the tightness of their resultant GLBs. Then, we compare them with the proposed weight vector in (40) by some numerical results.

\(^3\text{Note that } f(x) \text{ is essentially the fractional quadratic term in (8) by replacing } w \text{ with } x.\)
A. GLB From Weight Vector 2

In [22], Liu et al showed that the following “positive-cycle-of-sine” weight vector $w$

$$w_i = \begin{cases} 
\frac{\tan \frac{\pi}{2m} \sin \frac{\pi i}{m}}{m}, & i \in \{0, 1, \cdots, m-1\}; \\
0, & i \in \{m, m+1, \cdots, 2N-2\}, 
\end{cases}$$

where $2 \leq m \leq 2N-1$, asymptotically leads to a tighter Levenshtein bound (i.e., $M = 1$) for all $K \geq 3$ [19].

By [5, Proposition 1], one can show that the resultant GLB from the weight vector in (56) can be written as follows.

**Corollary 1:**

$$\delta_{\max}^2 \geq M \left[ N - \frac{N(MN-1)m \tan^2 \frac{\pi}{2m} + 2KQ(w,0)}{2K - m \tan^2 \frac{\pi}{2m}} \right],$$

where

$$Q(w,0) = \begin{cases} 
\frac{m}{4} \left(1 - \tan^2 \frac{\pi}{2m}\right), & \text{for } 2 \leq m \leq N, \\
\frac{-3m - 4N + 2}{4} \tan^2 \frac{\pi}{2m} + \frac{m - N - 1}{2} \cos \frac{N\pi}{m} + \frac{3}{4 \tan^2 \frac{\pi}{2m}} \sin \frac{N\pi}{m}, & \text{for } N < m \leq 2N-1. 
\end{cases}$$

(57)

It is noted that the numerator term of GLB in (57) is obtained based on the identity below.

$$Q\left(w, \frac{N(MN-1)}{K}\right) = \frac{N(MN-1)}{K} \cdot \frac{m}{2} \tan^2 \frac{\pi}{2m} + Q(w,0).$$

In what follows, we analyze the asymptotic tightness of the lower bound in (57).

Define $r \triangleq \lim_{N \to \infty} m/N$. Obviously, $r$ is a real-valued constant with $0 < r < 2$ when $m$ is on the same order of $rN$ (i.e., $m \sim rN$); and $r \to 0$ when $m$ is dominated by $N$ asymptotically (i.e., $m \sim o(N)$). Furthermore, define the fractional term in (57) as

$$\mathcal{R}_2 \triangleq \frac{N(MN-1)m \tan^2 \frac{\pi}{2m} + 2KQ(w,0)}{2K - m \tan^2 \frac{\pi}{2m}}.$$  

(59)

It is easy to see that the lower bound in (57) is tighter than the Welch bound in (1) if and only if

$$\mathcal{R}_1 > \min_{2 \leq m \leq 2N-1} \mathcal{R}_2,$$

where $\mathcal{R}_1$ is defined in (48). As $N$ tends to infinity, the inequality in (60) is equivalent to

$$\lim_{N \to \infty} \frac{\mathcal{R}_1}{N} > \lim_{N \to \infty} \min_{2 \leq m \leq 2N-1} \frac{\mathcal{R}_2}{N}.$$  

(61)

When $m \sim o(N)$, we have $r \to 0$ and $rN \in [2, \infty)$ as $N \to \infty$. In this case, one can show that

$$\lim_{N \to \infty} \frac{\mathcal{R}_1}{N} = \lim_{N \to \infty} \frac{N(MN-1)rN \tan^2 \frac{\pi}{2m} + 2K \cdot \frac{N}{r} \left(1 - \tan^2 \frac{\pi}{2m}\right)}{N(2K - rN \tan^2 \frac{\pi}{2m})} = \lim_{N \to \infty} \frac{MN(rN \tan^2 \frac{\pi}{2m}) + \frac{K}{r} \left(1 - \tan^2 \frac{\pi}{2m}\right)}{2K - rN \tan^2 \frac{\pi}{2m}} = \begin{cases} 
\infty, & \text{for } 2 \leq rN < \infty; \\
\frac{M\pi^2}{8Kr} + \frac{r}{4} \to \infty, & \text{for } rN \to \infty,
\end{cases}$$

(62)

which can be ignored without missing the minimum point of interest in the right-hand side of (61). Hence, we shall assume $r$ to be a non-vanishing real-valued constant with $0 < r < 2$, and rewrite (61) as

$$\lim_{N \to \infty} \frac{\mathcal{R}_1}{N} > \min_{0 < r < 2} \lim_{N \to \infty} \frac{\mathcal{R}_2}{N}.$$  

(63)

Here, the order of the limit and minimization operations can be exchanged because $\lim_{N \to \infty} \frac{\mathcal{R}_2}{N}$, as a function of $r$ exists, as shown below. Next, noting that $\lim_{N \to \infty} m \tan^2 \frac{\pi}{2m} = \lim_{N \to \infty} rN \tan^2 \frac{\pi}{2m} = 0$, we can express (59) as

$$\lim_{N \to \infty} \frac{\mathcal{R}_2}{N} = \lim_{N \to \infty} \frac{MN - 1}{2K} \cdot \frac{m}{2} \tan^2 \frac{\pi}{2m} + \lim_{N \to \infty} \frac{Q(w,0)}{N},$$

(64)

where

$$\lim_{N \to \infty} \frac{MN - 1}{2K} \cdot \frac{m}{2} \tan^2 \frac{\pi}{2m} = \frac{M\pi^2}{8Kr},$$

(65)

and after some manipulations,

$$f(r) \triangleq \lim_{N \to \infty} \frac{Q(w,0)}{N} = \begin{cases} 
\frac{r}{4}, & \text{for } 0 < r \leq 1; \\
\frac{4 - 3r}{4} + \frac{r - 1}{2} \cos \frac{\pi}{r} + \frac{3r}{2\pi} \sin \frac{\pi}{r}, & \text{for } 1 < r < 2.
\end{cases}$$

(66)

By (49), (65) and (66), it follows that (63) reduces to

$$\frac{1}{2} + \frac{M}{2K} > \min_{0 < r < 2} \left(\frac{M\pi^2}{8Kr} + f(r)\right).$$

(67)

Equivalently, we assert that the asymptotic lower bound in (57) is tighter than the Welch bound if and only if

$$\frac{K}{M} > \min_{0 < r < 2} L(r) \approx 2.483257,$$

(68)
Theorem 2: The GLB in (57) which arises from the weight vector in (56) reduces to

\[
\delta_{\text{max}}^2 \geq MN \left[ 1 - \min_{0 < r < 2} \left( \frac{M \pi^2}{8K} + f(r) \right) \right],
\]

for sufficiently large \( N \), where \( f(r) \) is given in (66). Such an asymptotic lower bound is tighter than the Welch bound for all \( K \geq K + 1 \) if and only if

\[
M \in \left\{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 20, 22, 24, 26, 28, 30, 31, 33, 35, 37, 39, 41, 43, 45, 60 \right\}.
\]

(73)

B. GLB From Weight Vector 3

The idea here is to optimize the weaker GLB in (9) with \( \alpha = MN^2 / K \). To this end, let us consider the weight vector obtained by minimizing the following function using the Lagrange multiplier.

\[
\mathcal{F}_{K,M,N,m}(w) = Q \left( \frac{MN^2}{K} \right) - 2\lambda \left( \sum_{i=0}^{m-1} w_i - 1 \right),
\]

(74)

where \( w_i = 0 \) for \( i \in \{ m, m+1, \cdots, 2N - 2 \} \) and \( 2 \leq m \leq 2N - 1 \). By relating the quadratic minimization solution of \( \mathcal{F}_{K,M,N,m}(w) \) to the Chebyshev polynomials of the second kind, one can obtain the weight vector\(^4\) below.

Let \( K \leq MN^2 \) and \( \cos \varphi = 1 - \frac{K}{MN^2} \). Also, let \( m \) be an even positive integer with \( m \varphi < \pi + \varphi \). For \( \varphi_0 = (\pi - m \varphi + \varphi) / 2 \), define the following weight vector

\[
w_i = \begin{cases} \sin \frac{\varphi_i}{M} \sin(\varphi_0 + i \varphi), & i \in \{ 0, 1, \cdots, m - 1 \}; \\ 0, & i \in \{ m, m+1, \cdots, 2N - 2 \}. \end{cases}
\]

(75)

Setting \( m = \left\lceil \frac{\pi}{2} \right\rceil + 1 \), one can minimize \( \mathcal{F}_{K,M,N,m}(w) \) in (74) over different \( m \) and get a generalized version of the Lev-enshtein bound in [19, Corollary 4] as follows.

**Corollary 2:**

\[
\delta_{\text{max}}^2 \geq M \left( N - \left\lceil \frac{\pi N}{\sqrt{8K} / M} \right\rceil \right), \quad \text{for } K \leq MN^2.
\]

(76)

As \( N \to \infty \), the lower bound in (76) is tighter than the Welch bound in (1) if and only if

\[
1 + \frac{M}{2K} > \lim_{N \to \infty} \frac{1}{N} \left\lceil \frac{\pi N}{\sqrt{8K} / M} \right\rceil = \frac{\pi}{\sqrt{8K} / M},
\]

(77)

or equivalently,

\[
\frac{K}{M} > \frac{\pi^2}{4} - 1 + \sqrt{\left( \frac{\pi^2}{8} - 1 \right) \frac{\pi^2}{2}} \approx 2.541303,
\]

(78)

where the right-hand side of (77) is obtained from (76).

**Remark 2:** For the GLB to be tighter than the corresponding Welch bound

\[
\frac{K}{M} \geq \lim_{N \to \infty} \frac{K + 1}{M} = \frac{\frac{\pi^2}{4} + 1}{M}.
\]

(79)

\(^4\)Although it looks similar to that in [22, Lemma 2], such a weight vector is more generic as it applies to QCSS with different \( M \geq 2 \).
Fig. 3. A plot of $d_2(M)$ in (71) and $d_3(M)$ in (81) versus $M$, where subplot (a) is a zoom-in of subplot (b), $d_2(M)$, $d_3(M)$ correspond to weight vectors 2 and 3, respectively. A positive $d_2(M)$ [or $d_3(M)$] indicates a tighter GLB over the Welch bound. It shows that weight vector 2 is superior than weight vector 3 as it leads to a tighter GLB for many more $M$ [c.f. (73) and (82)].

\[
\lim_{N \to \infty} K = \left\lfloor \frac{\pi^2 M^4}{4} \right\rfloor \leq \frac{\pi^2}{4} \approx 2.467401,
\]

which is smaller than the right-hand side of (78).

It can be asserted that the resultant GLB obtained from the weight vector in (75) with $m = \left\lceil \frac{\pi}{\varphi} \right\rceil + 1$ is tighter if and only if the value of $M$ satisfies the condition

\[
d_3(M) \triangleq \left\lfloor \frac{\pi^2 M^4}{4} \right\rfloor + 1 - \left[ \frac{\pi^2}{4} - 1 + \sqrt{\left( \frac{\pi^2}{8} - 1 \right) \frac{\pi^2}{2}} \right] > 0.
\]

This is because when condition (81) is satisfied, $K \geq \lim_{N \to \infty} K + 1$ is not only a necessary condition [c.f. (79)] but also a sufficient condition [c.f. (78)] for the GLB to be asymptotically tighter than the Welch bound.

In Fig. 3, $d_3(M)$ versus $M$ is plotted. By identifying $M$ satisfying $d_3(M) > 0$ [shown in (81)], we have the following theorem.

**Theorem 3:** The GLB in (76) which arises from the weight vector in (75) is asymptotically tighter than the Welch bound for all $K \geq K + 1$ if and only if

\[
M \in \{3, 5, 7, 9, 11\}.
\]

### C. Discussions

Denote by $B_1, B_2, B_3$ the optimized asymptotic lower bounds in (52), (72), (76), respectively. We remark that (1), Both $B_1$ and $B_2$ are greater than $B_3$ for any $M \geq 2$; (2), $B_1 > B_2$ except for $M \in \{3, 5, 7, 9\}$. The proof is omitted as it can be easily obtained from the tightness analysis in Subsection III-B and Subsection IV.

To further visualize their relative strengths of these three lower bounds, we calculate in Table I the ratio values of $B_1/B_W$, $B_2/B_W$, $B_3/B_W$ with $M \in \{2, 3, \cdots, 25\}$, where $N = 2048$, $K = K + 1$ and $B_W$ denotes the corresponding Welch bound. A ratio value which is larger than 1 corresponds to a tighter GLB (over the Welch bound). With Table I (in which the best bound is marked in red for each $M$), one may verify the three sets of $M$ for tighter GLB in Theorems 1-3 as well as the above-mentioned remark in this subsection. In particular, we can see that $B_1 > B_3$ for all $M \geq 2$, showing that weight vector 1 is superior than the other two as it is capable of tightening the GLB for all possible $M$, asymptotically.

### V. Conclusion

The generalized Levenshtein bound (GLB) in [5, Theorem 1] is an aperiodic correlation lower bound for quasi-complementary sequence sets (QCSSs) with number of channels not less than 2 (i.e., $M \geq 2$). Although GLB was shown to be
tighter than the corresponding Welch bound [i.e., (1)] for certain cases, there exists an ambiguous zone [shown in (14) and (16)] in which the tightness of GLB over Welch bound is unknown. Motivated by this, we aim at finding a properly selected weight vector in the bounding equation for a tighter GLB for all (other than some) $K \geq K + 1$, where $K$ denotes the set size, and $K$ is a value depending on $M$ and $N$ (the sequence length). As the GLB is in general a non-convex fractional quadratic function of the weight vector, the derivation of an analytical solution for a tighter GLB for all possible cases is a challenging task.

The most significant finding of this paper is weight vector 1 in (40) which is obtained from a frequency-domain optimization approach. We have shown that its resultant GLB in (41) is tighter than Welch bound for all $K \geq K + 1$ and for all $M \geq 2$, asymptotically. This finding is interesting as it explicitly shows that the GLB tighter condition given in [5, Theorem 2] is not only necessary but also sufficient, asymptotically, as shown in Theorem 1. Interestingly, we have proved in Subsection III-C that weight vector 1 in (40) is local minimizer of the GLB under certain asymptotic conditions.

We have shown that both weight vectors 2 and 3 [given in (56) and (75), respectively] lead to tighter GLBs for all $K \geq K + 1$ but only for certain small values of $M$ not less than 2. Note that although they were proposed in [22], the focus of [22] was on the tightening of Levenshtein bound for traditional single-channel (i.e., $M = 1$) sequence sets, whereas in this paper we have extended their tightening capability to GLB for multi-channel (i.e., $M \geq 2$) QCSS. Furthermore, we have shown in Theorem 2 and Theorem 3 that weight vector 2 is superior as its admissible set of $M$ [see (73)] is larger and subsumes that of weight vector 3.

**APPENDIX A**

**PROOF OF WEIGHT VECTOR 1 TO BE A LOCAL MINIMIZER**

To get started, we define

$$
\alpha(w, e) \triangleq w^{T}Q_{w}we^{T} - w^{T}w^{T}Q_{w}e,
$$

$$
\beta(w, e) \triangleq w^{T}Q_{w}we^{T} - w^{T}ww^{T}Q_{w}e,
$$

$$
\gamma(w, e) \triangleq \alpha(w, e) + 2\beta(w, e).
$$

(83)

It is easy to show that (55) is equivalent to the following inequality.

$$
2w^{T}Q_{w}e + e^{T}Q_{w}e + \gamma(w, e) \geq 0.
$$

(84)

Let $E = F_{2N-1}$. Since $e$ is a real vector, $E$ is conjugate symmetric in that $E_{i} = E^{*}_{2N-1-i}$ for $i = 1, 2, \cdots, 2N - 2$. By taking advantage of (20), we present the following properties which will be useful in the sequel.

$$
E_{0} = \sum_{i=0}^{2N-2} e_{i} = 0;
$$

$$
w + e \geq 0;
$$

$$
w^{T}Q_{w}e = \lambda_{1} \cdot \frac{E_{1} + E^{*}_{1}}{2(2N - 1) \cos \frac{\pi}{2N-1}};
$$

$$
e^{T}Q_{w}e = \frac{2}{2N - 1} \sum_{i=1}^{N-1} \lambda_{i}|E_{i}|^{2};
$$

$$
w^{T}Q_{w}w = \frac{1}{2N - 1} \left( \lambda_{0} + \frac{\lambda_{1}}{2 \cos^{2} \frac{\lambda}{2N-1}} \right);
$$

$$
e^{T}w = \frac{E_{1} + E^{*}_{1}}{2(2N - 1) \cos \frac{\pi}{2N-1}};
$$

$$
w^{T}w = \frac{1}{2N - 1} \left( 1 + \frac{1}{2 \cos^{2} \frac{\pi}{2N-1}} \right);
$$

$$
e^{T}e = \frac{2}{2N - 1} \sum_{i=1}^{N-1} |E_{i}|^{2}.
$$

(85a)

(85b)

(85c)

(85d)

(85e)

(85f)

(85g)

(85h)

By (85d), (85e), (85g) and (85h), we have

$$
\alpha(w, e) = \frac{2}{(2N - 1)^{2}} \left\{ \left( \lambda_{0} - \lambda_{1} \right) |E_{1}|^{2} + \sum_{i=2}^{N-1} \left( \lambda_{i} - \lambda_{1} - \lambda_{2} \right) |E_{i}|^{2} \right\}.
$$

(86)

By (85c), (85e), (85f) and (85g), we have

$$
\beta(w, e) = \frac{\lambda_{0} - \lambda_{1}}{2(2N - 1)^{2}} \cdot \frac{E_{1} + E^{*}_{1}}{\cos \frac{\pi}{2N-1}}.
$$

(87)

Therefore, $\gamma(w, e)$ can be expressed in the form shown in (88), shown at the bottom of the page. Since $e$ is a small perturbation, let us assume

$$
0 \leq 2 \sum_{i=1}^{N-1} |E_{i}|^{2} \ll 1.
$$

(89)

Next, we proceed with the following two cases.

1. **Case I**: If there exists $E_{i} \neq 0$ for $i \in \{2, 3, \cdots, N - 1\}$. Since we consider $K = K + 1$ with sufficiently large $M, N$, it is readily to show that $\lambda_{i} > 0$ holds for any $i \in \{2, 3, \cdots, N - 1\}$ [see (94) and (96)]. By (85d), let us write

$$
e^{T}Q_{w}e \cdot (2N - 1) = 2\lambda_{1}|E_{1}|^{2} + \xi,
$$

(90)
where \( \xi = 2 \sum_{i=2}^{N-1} \frac{\lambda_i |E_i|^2}{2} > 0 \). Furthermore, write
\[
\left[ 2w^T Q_0 e + e^T Q_a e + \gamma(w, e) \right] (2N-1) = \lambda_1 A + B.
\]
where
\[
A = 2 \left( 1 - \frac{1}{K(2N-1)} \right) |E_1|^2 + \frac{1}{K(2N-1)} \sum_{i=2}^{N-1} \frac{|E_i|^2}{\cos^2 \frac{\pi}{2N-1}}
\]
\[
B = \frac{\xi}{2} + 2\lambda_0 |E|^2 + \frac{1}{K(2N-1)} \sum_{i=2}^{N-1} \frac{\lambda_i - \lambda_0}{\cos^2 \frac{\pi}{2N-1}} |E_i|^2,
\]
By (94) and (96), we obtain
\[
\lim_{M \to \infty} \frac{B}{a} = \frac{\xi}{a} = \sum_{i=2}^{N-1} \frac{\lambda_i}{a} \cdot 2|E_i|^2 \geq \frac{2}{3} \cdot \left( 2 \sum_{i=2}^{N-1} |E_i|^2 \right).
\]
On the other hand,
\[
\lim_{M \to \infty} \frac{\lambda_1}{a} = \lim_{M \to \infty} \left[ 1 - \frac{|(4MN - 1)\sin^2 \frac{\pi}{2(N-1)} - 1|}{4N(MN - 1)\sin^2 \frac{\pi}{2(N-1)}} \right]
\]
\[
\to 0^-, \quad (98)
\]
where \( 0^- \) denotes a sufficiently small value (negative) that approaches zero from the left. Therefore, we have
\[
\lim_{M \to \infty} \frac{\lambda_1}{a} \cdot \lim_{M \to \infty} \frac{A + \lim_{M \to \infty} \frac{B}{a}}{\text{upper bounded}}
\]
By (97) and (98), we assert that when \( M \) is sufficiently large, the sign of the limit in (99) will be identical to that of \( \xi/a \) [cf. (97)] which is nonnegative. This shows that (55) [and (84)] holds for Case I, asymptotically.

2) Case II: If \( E_i = 0 \) for all \( i \in \{2, 3, \ldots, N-1\} \).

In this case, (92) reduces to
\[
A = 2 \left( 1 - \frac{1}{K(2N-1)} \right) |E|^2 + \frac{1}{K(2N-1)} \sum_{i=2}^{N-1} \frac{|E_i|^2}{\cos^2 \frac{\pi}{2N-1}}
\]
\[
+ \frac{1}{K(2N-1)} \sum_{i=2}^{N-1} \left( \lambda_i - \lambda_0 \cdot \frac{\lambda_i}{\cos^2 \frac{\pi}{2N-1}} \right) |E_i|^2 + \frac{1}{K(2N-1)} \lambda_0 |E|^2.
\]
Since \( E = F_{2N-1} e \), we have
\[
e_i = \frac{2}{2N-1} \Re \left\{ E_1 \exp \left( \frac{\sqrt{-1}2\pi i}{2N-1} \right) \right\}, \quad (101)
\]
where \( \Re \{x\} \) denotes the real part of complex data \( x \). Consider \( E_1 \) which takes the following form.
\[
E_1 = \frac{t}{2 \cos \frac{\pi}{2N-1}} \exp \left( \sqrt{-1}\psi \right), \quad (102)
\]
where \( 0 \leq t < 1 \) and \( \psi \) denotes the phase shift of \( E_1 \). As a result, \( e_i \) can be expressed as
\[
e_i = \frac{t}{(2N-1) \cos \frac{\pi}{2N-1} \cdot \cos \left( \frac{2\pi i}{2N-1} + \psi \right)}, \quad (103)
\]
Thus, 
\[
\lambda_1 A + B = \left(2|E_1|^2 + \frac{E_1 + E_i^*}{\cos \frac{\pi}{N-1}}\right) \left(\lambda_1 + \frac{\lambda_0 - \lambda_1}{K(2N-1)}\right).
\] (104)

Since \(\lambda_1 \sim O(N^2),\) \(\frac{\lambda_0 - \lambda_1}{K(2N-1)} \sim O\left(\frac{N^2}{K}\right) \sim O\left(\frac{N^2}{M}\right),\) we assert that for sufficiently large \(M, N,\)
\[
\lambda_1 + \frac{\lambda_0 - \lambda_1}{K(2N-1)} < 0,
\] (105)
holds because it will be dominated by the negative \(\lambda_1.\)

Our next task is to show that \(2|E_1|^2 + \frac{E_1 + E_i^*}{\cos \frac{\pi}{N-1}}\) does not change for all \(\psi,\)
\[
2|E_1|^2 + \frac{E_1 + E_i^*}{\cos \frac{\pi}{N-1}} \leq \frac{1}{2} \cos^2 \frac{\pi}{N-1} (t^2 + 2t \cos \psi).
\] (106)

It is required in (85b) that \(w_i + e_i \geq 0\) for all \(i,\)
\[
\cos \frac{\pi}{N-1} + \cos \frac{2\pi i}{2N-1} + t \cos \left(\frac{2\pi i}{2N-1} + \psi\right) \geq 0.
\] (107)

Setting \(i = N,\) we have
\[
\cos \frac{2\pi N}{2N-1} + \psi \geq 0 \rightarrow (\frac{1}{2} - \frac{1}{2N-1}) \pi \leq \psi \leq (\frac{3}{2} - \frac{1}{2N-1}).
\] (108)

Setting \(i = N,\) we have
\[
\cos \frac{2\pi (N-1)}{2N-1} + \psi \geq 0 \rightarrow (\frac{1}{2} + \frac{1}{2N-1}) \pi \leq \psi \leq (\frac{3}{2} + \frac{1}{2N-1}).
\] (109)

Therefore,
\[
(\frac{1}{2} + \frac{1}{2N-1}) \pi \leq \psi \leq (\frac{3}{2} + \frac{1}{2N-1}) \rightarrow -1 \leq \cos \psi < 0.
\] (110)

This shows \(t^2 + 2t \cos \psi \leq 0\) holds provided \(t \leq -2 \cos \psi.\) This can be easily satisfied by a sufficiently small \(t.\) Together with (104)-(106), we conclude that (55) \([\text{and (84)}]\) holds for Case II, asymptotically. This completes the proof of the local optimality of the proposed weight vector in (53).

REFERENCES


Zilong Liu received the Bachelor’s degree from the School of Electronics and Information Engineering from Huazhong University of Science and Technology, Wuhan, China, the Master’s degree from the Department of Electronic Engineering, Tsinghua University, Beijing, China, and the Ph.D degree from the School of Electrical and Electronic Engineering, Nanyang Technological University (NTU), Singapore, in 2004, 2007, and 2014, respectively. Since July 2008, he has been with the School of Electrical and Electronic Engineering, NTU, first as a Research Associate and since November 2014 a Research Fellow. He was a visitor in the University of Melbourne from May 2012 to February 2013 [hosted by Prof. U. Parampalli], and a visiting Ph.D student in the Hong Kong University of Science and Technology from June 2013 to July 2013 (hosted by Prof. W. H. Mow). He is generally interested in coding and signal processing for various communication systems, with emphasis on signal design and algebraic coding, error correction codes, robust/efficient multuser communications, and physical layer receiver design. His homepage is at http://www.ntu.edu.sg/home/zilongliu/.

Yong Liang Guan (M’94) received the Bachelor of Engineering degree with first class honors from the National University of Singapore, Singapore, and the Ph.D. degree from Imperial College of London, London, U.K. He is a tenured Associate Professor at the School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore. His research interests broadly include coding and signal processing for communication and storage systems. His homepage is at http://www3.ntu.edu.sg/home/eylguan/index.htm.

Wai Ho Mow (S’89–M’93–SM’99) received the M.Phil. and Ph.D. degrees in information engineering from the Chinese University of Hong Kong, Hong Kong, in 1991 and 1993, respectively. From 1997 to 1999, he was with Nanyang Technological University, Singapore. Since March 2000, he has been with Hong Kong University of Science and Technology. He received seven research/exchange fellowships from five countries, including the Humboldt Research Fellowship. His research interests include the areas of communications, coding, and information theory. He pioneered the lattice approach to signal detection problems (such as sphere decoding and complex lattice reduction-aided detection) and united all known constructions of perfect roots-of-unity (aka CAZAC) sequences (widely used as preambles and sounding sequences). He has published one book, and has coauthored more than 30 filed patent applications and more than 190 technical publications, among which he is the sole author of more than 40 publications. He coauthored two papers that received the ISITA2002 Paper Award for Young Researchers and the APCC2013 Best Paper Award. Since 2002, he has been the principal investigator of 16 funded research projects. In 2005, he chaired the Hong Kong Chapter of the IEEE Information Theory Society. He was the Technical Program Cochair of five conferences, and served the technical program committees of numerous conferences, such as ICC, Globecom, ITW, ISITA, VTC, and APCC. He was a Guest Associate Editor for numerous special issues of the IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences and is currently an Editor of the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. He was an industrial consultant for Huawei, ZTE, and Magnotech, Ltd. He was a member of the Radio Spectrum Advisory Committee, Office of the Telecommunications Authority, Hong Kong S.A.R. Government from 2003 to 2008. His homepage is at http://www.ee.ust.hk/~eewmow/.
