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Optimum Masking Levels and Coefficient Sparseness for Hilbert Transformers and Half-Band Filters Designed Using the Frequency-Response Masking Technique

Yong Ching Lim, *Fellow, IEEE*, Ya Jun Yu, *Member, IEEE*, and Tapio Saramäki, *Fellow, IEEE*

***Abstract*—Hilbert transformers and half-band filters are two very important special classes of finite-impulse response filters often used in signal processing applications. Furthermore, there exists a very close relationship between these two special classes of filters in such a way that a half-band filter can be derived from a Hilbert transformer in a straightforward manner and vice versa. It has been shown that these two classes of filters may be synthesized using the frequency-response masking (FRM) technique resulting in very efficient implementation when the filters are very sharp. While filters synthesized using the FRM technique has been characterized for the general low-pass case, Hilbert transformers and half-band filters synthesized using the FRM technique have not been characterized. The characterization of the two classes of filter is a focus of this paper. In this paper, we re-develop the FRM structure for the synthesis of Hilbert transformer from a new perspective. This new approach uses a frequency response correction term produced by masking the frequency response of a sparse coefficient filter, whose frequency response is periodic, to sharpen the bandedge of a low-order Hilbert transformer. Optimum masking levels and coefficient sparseness for the Hilbert transformers are derived; corresponding quantities for the half-band filters are obtained via the close relationship between these two classes of filters.**

***Index Terms*—Finite-impulse response (FIR) digital filter, frequency-response masking (FRM), Hilbert transformer, half-band filter, sparse coefficient filter.**

I. INTRODUCTION

THROUGHOUT this paper, the phrase "a filter $H(z)$" refers to a filter whose z-transform transfer function is $H(z)$. We shall also denote the frequency response of $H(z)$ by $H(e^{j\omega})$. Furthermore, by the phrase "a filter $H(e^{j\omega})$," we refer to the filter whose frequency response is $H(e^{j\omega})$.

The conventional frequency-response masking (FRM) structure shown in Fig. 1 was first introduced in [1]. It has been used very successfully for the synthesis of very sharp low-pass, high-pass, and bandpass filters with extremely low complexity. Further improvements on the structure, powerful specialized optimization algorithms, and interesting applications have been developed by many authors for the FRM [2]–[24] technique.

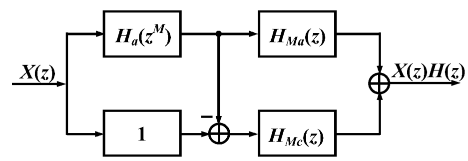

Fig. 1. Structure of a filter synthesized using the FRM technique. All the subfilters are assumed to be zero phase. Causality can be restored by introducing appropriate delays into the subfilters.

It was shown in [2] that the FRM technique can also be used to synthesize half-band filters. Since every other coefficient value of a half-band filter is zero, the various estimation formulae derived in [4] are no longer applicable.

A Hilbert transformer with odd length may be derived from a low-pass half-band filter by discarding the centre coefficient, multiplying its nth coefficient by $e^{jn\pi/2}$, and scale up the coefficient values by a factor of two. Making use of this close relationship between a Hilbert transformer and a half-band filter, [3] illustrated the synthesis of a Hilbert transformer using the FRM technique. Although [2] and [3] illustrated the application of the FRM technique for the synthesis of Hilbert transformer and half-band filter, important information on the coefficient sparseness (M) for the bandedge shaping filter and the optimum masking levels (K_{opt}) were not presented.

In this paper, we adopted a new approach toward the synthesis of filters using the FRM technique. In this new approach, the Hilbert transformer is synthesized as a parallel connection of two filters. One of these two filters is a low-order Hilbert transformer. The other one is a series connection of a masking filter and a sparse coefficient bandedge shaping filter producing a correction term that sharpens the bandedge of the low-order Hilbert transformer.

A Hilbert transformer with odd length can be derived from one with even length by replacing each z^{-1} in the z-transform transfer function of the even length Hilbert transformer by z^{-2}; this will cause the transition width to shrink by a factor of two. As a result of this straightforward relationship between the odd length and even length Hilbert transformers, the odd length Hilbert transformer will not be discussed in this paper.

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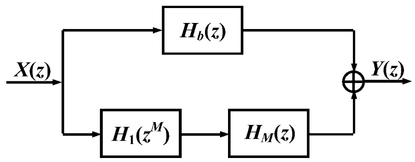

Fig. 2. Structure for the synthesis of a Hilbert transformer using the FRM technique.

Our new FRM approach is presented in Section II. In the FRM technique, each delay of a prototype bandedge shaping filter is replaced by M delays. The optimum estimate for M is presented in Section III together with the estimates for the resulting complexity and effective filter length. The application of our new FRM technique to the design of a Hilbert transformer for suppressing acoustic feedback is presented in Section IV; a factor of 9 reduction in the number of nontrivial coefficients is achieved when compared to the Chebyshev optimum design. Further reduction in the complexity of the filter may be achieved by considering a multi-level masking structure; this is presented in Section V. A two level-masking design is presented in Section VI to meet the requirement of the example of Section IV. The two-level masking approach produces a factor of 17 reduction in the number of nontrivial coefficients comparing to the Chebyshev optimum design.

II. Hilbert Transformer Synthesized Using FRM Technique

The z-transform transfer function $H(z)$ of a Hilbert transformer with length N, where N is an even integer, is given by

$$H(z) = e^{j\left(\frac{\pi}{2}-\frac{N-1}{2}\omega\right)} \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right) \sin\left(\left(n-\frac{1}{2}\right)\omega\right). \tag{1}$$

In (1), $h(n)$ is the nth impulse response. Since N is even, its group delay, given by $(N-1)/2$, is not an integer and the filter is said to have "half-sample delay".

Fig. 2 shows the structure for implementing the Hilbert transformer using the FRM technique. The z-transform transfer function for the overall filter is given by

$$H(z) = H_1(z^M)H_M(z) + H_b(z). \tag{2}$$

It is a system of subfilters consisting of a parallel connection of two branches. One of the parallel branches consists of a basic filter $H_b(z)$. The other branch consists of a cascade of a bandedge sharpening filter $H_1(z^M)$ and a masking filter $H_M(z)$. The basic filter $H_b(z)$ provides a low-order approximation (with wide transition width) to the desired specification. The cascade of $H_1(z^M)$ and $H_M(z)$ provides a transfer function correction term for sharpening the transition band.

Let the lengths of $H_b(z), H_1(z)$, and $H_M(z)$ be N_b, N_1, and N_M, respectively. The length of $H_1(z^M)H_M(z)$ will be $MN_1 + N_M - M$. The delay introduced by $H_b(z)$ and that introduced by $H_1(z^M)H_M(z)$ must be the same; otherwise, pure delays must be introduced into the shorter one to equalize them. In order to avoid inserting half-sample delay in the implementation, the parity of N_b and that of $MN_1 + N_M - M$ must be the same. Furthermore, $H_1(z^M)H_M(z)$ must have anti-symmetrical impulse response.

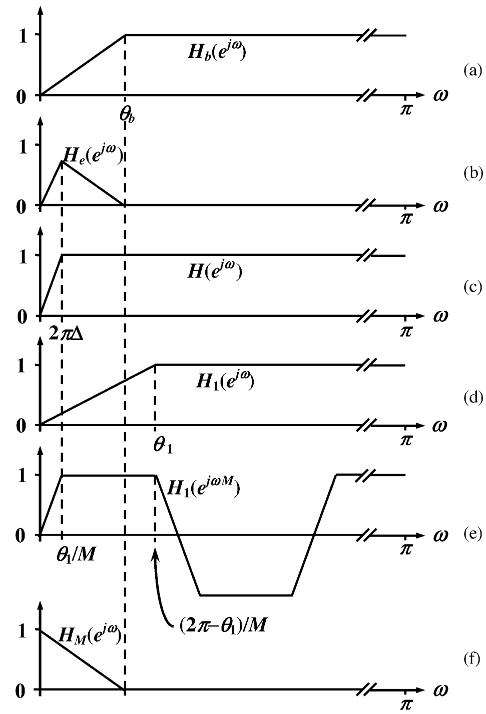

Fig. 3. Frequency responses of the subfilters for even length $H_b(z)$. Note that $\theta_1/M = 2\pi\Delta$. $2\pi\Delta$ is the transition width of the desired transfer function.

Consider a wide transition band Hilbert transformer $H_b(e^{j\omega})$ as shown in Fig. 3(a) where N_b is even. The computational complexity of $H_b(z)$ is low since its transition band is wide. Now consider a transition band correction filter $H_e(e^{j\omega})$ as shown in Fig. 3(b). When $H_e(e^{j\omega})$ is added to $H_b(e^{j\omega})$, a sharp transition band Hilbert transformer, $H(e^{j\omega})$, as shown in Fig. 3(c) is obtained. Our objective is to design a very low complexity filter $H_e(e^{j\omega})$ using the FRM technique. Consider a bandedge shaping filter $H_1(e^{j\omega})$ as shown in Fig. 3(d). The complexity of $H_1(z)$ is low because it has wide transition width. Replacing each delay of $H_1(z)$ by M delays, a filter $H_1(e^{jM\omega})$ as shown in Fig. 3(e) is obtained. A masking filter $H_M(e^{j\omega})$ as shown in Fig. 3(f) is used to mask the unwanted passbands of $H_1(e^{jM\omega})$ to produce the frequency response $H_e(e^{j\omega})$ as shown in Fig. 3(b). $H_M(e^{j\omega})$ has low complexity because its frequency response has a wide transition band. Since the length of $H_b(z)$ is even, the length of $H_1(z^M)H_M(z)$, i.e., $MN_1 + N_M - M$, must also be even. If M is even, N_M must also be even. If M is odd, N_M and N_1 must have different parities. By considering the gain of $H_M(e^{j\omega})$ in the vicinity of $\omega = 0$, it is clear that $H_M(z)$ has symmetrical impulse response. Thus, $H_1(z)$ must have anti-symmetrical impulse response to satisfy the condition that $H_1(z^M)H_M(z)$ must have anti-symmetrical impulse response.

The bandedges of $H_b(z)$ and $H_M(z)$ are the same. Let the bandedge of $H_b(z)$ be θ_b as shown in Fig. 3. Let the bandedge of $H_1(z)$ be θ_1. As can be seen from Fig. 3, $\theta_b \leq (2\pi - \theta_1)/M$.

III. Optimum M for Even Length Hilbert Transformers

Joint simultaneous optimization of $H_1(z^M), H_M(z)$, and $H_b(z)$ is a nonlinear optimization problem and can be solved by general purpose or specialized optimization packages. Nevertheless, regardless of the optimization packages used, it is necessary to determine the value of M before initiating the optimization process.

Let the transition width of the desired Hilbert transformer be Δf_s where f_s is the sampling frequency and let its passband ripple magnitude be δ. It is shown in Appendix I that, for $\Delta < 0.2$, the filter length, N_{opt}, of the Chebyshev optimum design is given by

$$N_{\text{opt}} \approx \frac{\Phi_H(\delta)}{\Delta} + 1 \approx \frac{\Phi_H(\delta)}{\Delta}. \tag{3}$$

The length of $H_M(z)$ may be estimated from the Chebyshev polynomial

$$P_N(x) = \begin{cases} \cos\left(\frac{N-1}{2}\cos^{-1}(x)\right), & |x| \le 1 \quad (4a) \\ \cosh\left(\frac{N-1}{2}\cosh^{-1}(x)\right), & |x| > 1 \quad (4b) \end{cases}$$

where

$$x = \frac{X_0+1}{2}\cos(\omega) + \frac{X_0-1}{2}. \tag{4c}$$

Equation (4c) maps ω from 0 to π into x from X_0 to -1. Let $H_M(z)$ has unity gain at $z=1$ and let the peak stopband ripple magnitude of $H_M(z)$ be δ. It can be shown that

$$\frac{3-\cos(\theta_b)}{1+\cos(\theta_b)} = \cosh\left(\frac{2}{N-1}\cosh^{-1}\left(\frac{1}{\delta}\right)\right). \tag{5}$$

Let $\Delta_b = (\theta_b/2\pi)$. For $\Delta_b < 0.2$

$$\begin{aligned} N_M &\approx \frac{\frac{1}{\pi}\cosh^{-1}\left(\frac{1}{\delta}\right)}{\Delta_b} + 1 \approx \frac{\frac{1}{\pi}\cosh^{-1}\left(\frac{1}{\delta}\right)}{\Delta_b} \\ &\approx \frac{\ln(2)-\ln(\delta)}{\pi} \approx \Delta_b \approx \frac{\Phi_M(\delta)}{\Delta_b} \end{aligned} \tag{6a}$$

where

$$\Phi_M(\delta) = 0.220\,64 - 0.732\,94\log_{10}(\delta). \tag{6b}$$

Since the desired transition width is Δf_s, the transition width of $H_1(z)$ is $M\Delta f_s$. Thus, its length is

$$N_1 = \frac{\Phi_H(\delta)}{M\Delta}. \tag{7}$$

The transition width of $H_b(z)$ is $(1-M\Delta)f_s/M$ and N_b is given by

$$N_b = \frac{\Phi_H(\delta)}{\frac{1}{M}-\Delta}. \tag{8}$$

The transition width of $H_M(z)$ is the same as that of $H_b(z)$. Thus

$$N_M = \frac{\Phi_M(\delta)}{\frac{1}{M}-\Delta}. \tag{9}$$

The total number of nontrivial coefficients, $N_{\text{total}}(\delta)$, is given by

$$\begin{aligned} N_{\text{total}}(\delta) &= N_1 + N_b + N_M \\ &= \frac{\Phi_H(\delta)}{M\Delta} + \frac{\Phi_H(\delta)}{\frac{1}{M}-\Delta} + \frac{\Phi_M(\delta)}{\frac{1}{M}-\Delta}. \end{aligned} \tag{10}$$

TABLE I
Values of $\Phi_H(\delta), \Phi_M(\delta)$ and $\Phi_M(\delta)/\Phi_H(\delta)$

δ	0.1	0.01	0.001	0.0001	0.00001
$\Phi_H(\delta)$	0.534	1.166	1.830	2.510	3.189
$\Phi_M(\delta)$	0.939	1.687	2.419	3.152	3.885
$\frac{\Phi_M(\delta)}{\Phi_H(\delta)}$	1.76	1.45	1.32	1.26	1.22

It can be shown by differentiating (10) with respect to M and equating the derivative to zero that, for small Δ, the value of M, denoted by M_{opt}, corresponding to the minimum value of $N_{\text{total}}(\delta)$ is given by

$$M_{\text{opt}} \approx \sqrt{\frac{\Phi_H(\delta)}{(\Phi_H(\delta)+\Phi_M(\delta))\Delta}}. \tag{11a}$$

Several values of $\Phi_H(\delta), \Phi_M(\delta)$ and $\Phi_M(\delta)/\Phi_H(\delta)$ for δ ranging from 0.1 to 0.000 01 are tabulated in Table I. As can be seen from Table I, for $0.01 > \delta > 0.000\,01, \Phi_M(\delta)/\Phi_H(\delta) \approx 1.32$. This further simplifies (11a) to

$$M_{\text{opt}} \approx \sqrt{\frac{0.43}{\Delta}}. \tag{11b}$$

For M_{opt} given by (11a)

$$N_1 \approx \sqrt{\frac{\Phi_H(\delta)(\Phi_H(\delta)+\Phi_M(\delta))}{\Delta}}. \tag{12a}$$

For M_{opt} given by (11a) and $1/M_{\text{opt}} \gg \Delta$

$$N_b + N_M \approx \sqrt{\frac{\Phi_H(\delta)(\Phi_H(\delta)+\Phi_M(\delta))}{\Delta}}. \tag{12b}$$

For $\Phi_M(\delta)/\Phi_H(\delta) = 1.32$, it can be easily shown from (3) and (12) that $N_1+N_b+N_M < N_{\text{opt}}$ for $\Delta < 0.108$. Thus, our new FRM technique will produce a design with a smaller number of nontrivial coefficients than that of the Chebyshev optimum design if the transition width of the desired Hilbert transformer is less than $0.108 f_s$. For an audio signal sampled at 48 kHz (44.1 kHz), our new FRM technique will produce a saving in complexity comparing to the Chebyshev optimum design if frequency components below 5.2 kHz (4.8 kHz) must be faithfully transformed.

The effective length N_{eff} of the filter designed using our new FRM technique is given by

$$N_{\text{eff}} \approx M_{\text{opt}} N_1 + N_M. \tag{13}$$

From (3), (7), (9), (11), and (13)

$$\begin{aligned} N_{\text{eff}} &\approx N_{\text{opt}}\left(1 + \frac{\Phi_M(\delta)}{\Phi_H(\delta)}\sqrt{\frac{\Phi_H(\delta)\Delta}{\Phi_H(\delta)+\Phi_M(\delta)}}\right) \\ &\approx N_{\text{opt}}(1+0.86\sqrt{\Delta}). \end{aligned} \tag{14}$$

In (14), the term 0.86 $\sqrt{\Delta}$ represents the fractional increase in filter length when compared to the Chebyshev optimum design. The fractional increase is less than 10% for $\Delta < 0.013$.

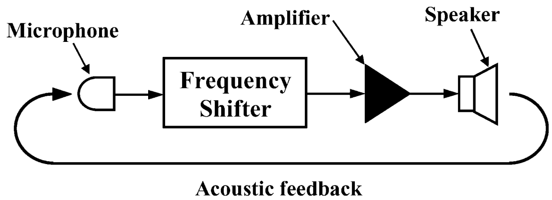

Fig. 4.Frequency shifter is inserted to suppress oscillation due to acoustic feedback.

TABLE II
COEFFICIENT VALUES OF $H_b(z)$

$h_b(-47) = 0.00014858 = -h_b(47)$	$h_b(-23) = 0.01266332 = -h_b(23)$
$h_b(-45) = 0.00016629 = -h_b(45)$	$h_b(-21) = 0.01582046 = -h_b(21)$
$h_b(-43) = 0.00028316 = -h_b(43)$	$h_b(-19) = 0.01970981 = -h_b(19)$
$h_b(-41) = 0.00042792 = -h_b(41)$	$h_b(-17) = 0.02461925 = -h_b(17)$
$h_b(-39) = 0.00161558 = -h_b(39)$	$h_b(-15) = 0.03073442 = -h_b(15)$
$h_b(-37) = 0.00175315 = -h_b(37)$	$h_b(-13) = 0.03856408 = -h_b(13)$
$h_b(-35) = 0.00295337 = -h_b(35)$	$h_b(-11) = 0.04904002 = -h_b(11)$
$h_b(-33) = 0.00374368 = -h_b(33)$	$h_b(-9) = 0.06315979 = -h_b(9)$
$h_b(-31) = 0.00488766 = -h_b(31)$	$h_b(-7) = 0.08529546 = -h_b(7)$
$h_b(-29) = 0.00615065 = -h_b(29)$	$h_b(-5) = 0.12344694 = -h_b(5)$
$h_b(-27) = 0.00805283 = -h_b(27)$	$h_b(-3) = 0.21037431 = -h_b(3)$
$h_b(-25) = 0.01006458 = -h_b(25)$	$h_b(-1) = 0.63529816 = -h_b(1)$

IV. EXAMPLE

A problem often encountered in an acoustic feedback system is oscillation due to feedback. Such oscillation can be suppressed by inserting a frequency shifter into the system as shown in Fig. 4. The frequency shifter shifts the frequency by an adjustable amount ranging from 0 to 5 Hz. A shift of 2 Hz produces no noticeable distortion to most people. The implementation of the shifter requires a sharp Hilbert transformer if low frequency components must also be faithfully shifted.

In a particular system, the application required a Hilbert transformer with the following specifications. Sampling rate: 32 kHz. Lower-bandedge: 20 Hz. Peak ripple magnitude: 0.0001.

The above requirements correspond to $\Delta = 0.000\,625$. In order to reduce the arithmetic complexity of the Hilbert transformer further, we chose an odd length filter with passband spanning from 20 to 15 980 Hz. The design was done by first designing an even length Hilbert transformer with $\Delta = 0.000\,625 \times 2 = 0.001\,25$. The even length Hilbert transformer was then converted to an odd length Hilbert transformer by replacing the variable z of its z-transform transfer function by z^2. The transition width shrank from 0.001 25 to 0.000 625 when z was replaced by z^2. The estimated length of a direct form even length Chebyshev optimum design meeting the specification was 2000. Using our FRM technique, the value of M suggested by (11a) was 18.8. The values of N_1, N_b, and N_M estimated from (7), (8), and (9) for $M = 19$ were 106.6, 48.8, and 61.4, respectively. The optimization algorithm reported in [19] was used to optimize the subfilters and a design with $N_1 = 106, N_b = 48$, and $N_M = 59$ was obtained. The total number of nontrivial coefficient was $106 + 48 + 59 = 213$. It was only about 10.7% of the 2000 nontrivial coefficients of the

TABLE III
COEFFICIENT VALUES OF $H_M(z)$

$h_M(-58) = 0.00076333 = h_M(58)$	$h_M(-28) = 0.01630893 = h_M(28)$
$h_M(-56) = 0.00069048 = h_M(56)$	$h_M(-26) = 0.01880688 = h_M(26)$
$h_M(-54) = 0.00131371 = h_M(54)$	$h_M(-24) = 0.02113358 = h_M(24)$
$h_M(-52) = 0.00160856 = h_M(52)$	$h_M(-22) = 0.02372854 = h_M(22)$
$h_M(-50) = 0.00208650 = h_M(50)$	$h_M(-20) = 0.02391805 = h_M(20)$
$h_M(-48) = 0.00255934 = h_M(48)$	$h_M(-18) = 0.02720104 = h_M(18)$
$h_M(-46) = 0.00335705 = h_M(46)$	$h_M(-16) = 0.02762120 = h_M(16)$
$h_M(-44) = 0.00409132 = h_M(44)$	$h_M(-14) = 0.02990513 = h_M(14)$
$h_M(-42) = 0.00503851 = h_M(42)$	$h_M(-12) = 0.03173558 = h_M(12)$
$h_M(-40) = 0.00612590 = h_M(40)$	$h_M(-10) = 0.03375364 = h_M(10)$
$h_M(-38) = 0.00738429 = h_M(38)$	$h_M(-8) = 0.03471639 = h_M(8)$
$h_M(-36) = 0.00892903 = h_M(36)$	$h_M(-6) = 0.03611482 = h_M(6)$
$h_M(-34) = 0.01057801 = h_M(34)$	$h_M(-4) = 0.03701503 = h_M(4)$
$h_M(-32) = 0.01242610 = h_M(32)$	$h_M(-2) = 0.03760300 = h_M(2)$
$h_M(-30) = 0.01455075 = h_M(30)$	$h_M(0) = 0.03787069$

TABLE IV
COEFFICIENT VALUES OF $H_1(z^{19})$

$h_1(-1995) = 0.00012105 = -h_1(1995)$	$h_1(-969) = 0.00532561 = -h_1(969)$
$h_1(-1957) = 0.00001606 = -h_1(1957)$	$h_1(-931) = 0.00593894 = -h_1(931)$
$h_1(-1919) = 0.00011230 = -h_1(1919)$	$h_1(-893) = 0.00660967 = -h_1(893)$
$h_1(-1881) = 0.00009540 = -h_1(1881)$	$h_1(-855) = 0.00734753 = -h_1(855)$
$h_1(-1843) = 0.00015646 = -h_1(1843)$	$h_1(-817) = 0.00815578 = -h_1(817)$
$h_1(-1805) = 0.00017722 = -h_1(1805)$	$h_1(-779) = 0.00904647 = -h_1(779)$
$h_1(-1767) = 0.00023753 = -h_1(1767)$	$h_1(-741) = 0.01002558 = -h_1(741)$
$h_1(-1729) = 0.00028299 = -h_1(1729)$	$h_1(-703) = 0.01110894 = -h_1(703)$
$h_1(-1691) = 0.00035519 = -h_1(1691)$	$h_1(-665) = 0.01230689 = -h_1(665)$
$h_1(-1653) = 0.00042427 = -h_1(1653)$	$h_1(-627) = 0.01364168 = -h_1(627)$
$h_1(-1615) = 0.00051575 = -h_1(1615)$	$h_1(-589) = 0.01513135 = -h_1(589)$
$h_1(-1577) = 0.00061115 = -h_1(1577)$	$h_1(-551) = 0.01680929 = -h_1(551)$
$h_1(-1539) = 0.00072803 = -h_1(1539)$	$h_1(-513) = 0.01870760 = -h_1(513)$
$h_1(-1501) = 0.00085411 = -h_1(1501)$	$h_1(-475) = 0.02088073 = -h_1(475)$
$h_1(-1463) = 0.00100256 = -h_1(1463)$	$h_1(-437) = 0.02338879 = -h_1(437)$
$h_1(-1425) = 0.00116478 = -h_1(1425)$	$h_1(-399) = 0.02632959 = -h_1(399)$
$h_1(-1387) = 0.00135144 = -h_1(1387)$	$h_1(-361) = 0.02982433 = -h_1(361)$
$h_1(-1349) = 0.00155626 = -h_1(1349)$	$h_1(-323) = 0.03407189 = -h_1(323)$
$h_1(-1311) = 0.00178839 = -h_1(1311)$	$h_1(-285) = 0.03934913 = -h_1(285)$
$h_1(-1273) = 0.00204323 = -h_1(1273)$	$h_1(-247) = 0.04613329 = -h_1(247)$
$h_1(-1235) = 0.00232900 = -h_1(1235)$	$h_1(-209) = 0.05519160 = -h_1(209)$
$h_1(-1197) = 0.00264243 = -h_1(1197)$	$h_1(-171) = 0.06800972 = -h_1(171)$
$h_1(-1159) = 0.00299119 = -h_1(1159)$	$h_1(-133) = 0.08754305 = -h_1(133)$
$h_1(-1121) = 0.00337321 = -h_1(1121)$	$h_1(-95) = 0.12115935 = -h_1(95)$
$h_1(-1083) = 0.00379594 = -h_1(1083)$	$h_1(-57) = 0.19117149 = -h_1(57)$
$h_1(-1045) = 0.00425837 = -h_1(1045)$	$h_1(-19) = 0.40604989 = -h_1(19)$
$h_1(-1007) = 0.00476822 = -h_1(1007)$	

Chebyshev optimum design. The effective length of the Chebyshev optimum design (after replacing z by z^2) was 3999. The effective length of our FRM design (after replacing z by z^2) was 4107; it was only about 2.7% longer than the Chebyshev optimum design. The coefficient values of the subfilters (after replacing z by z^2 or z^{2M} which ever is appropriate) of the odd length Hilbert transformer are tabulated in Tables II–IV. The passband ripple of the overall filter is shown in Fig. 5 and the frequency responses of the various subfilters are shown in Fig. 6.

V. MULTI-LEVEL FRM

If the transition width of the required filter is very narrow, the lengths of $H_1(z^M), H_M(z)$ and $H_b(z)$ may still be very long. In

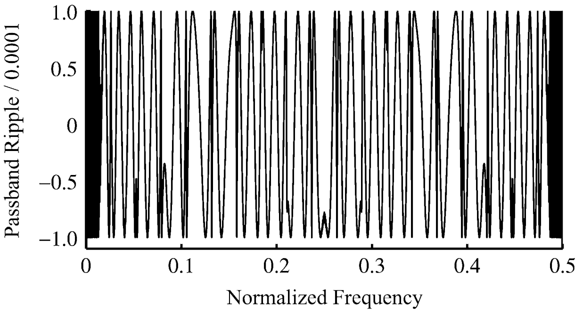

Fig. 5. Passband ripples of the Hilbert transformer.

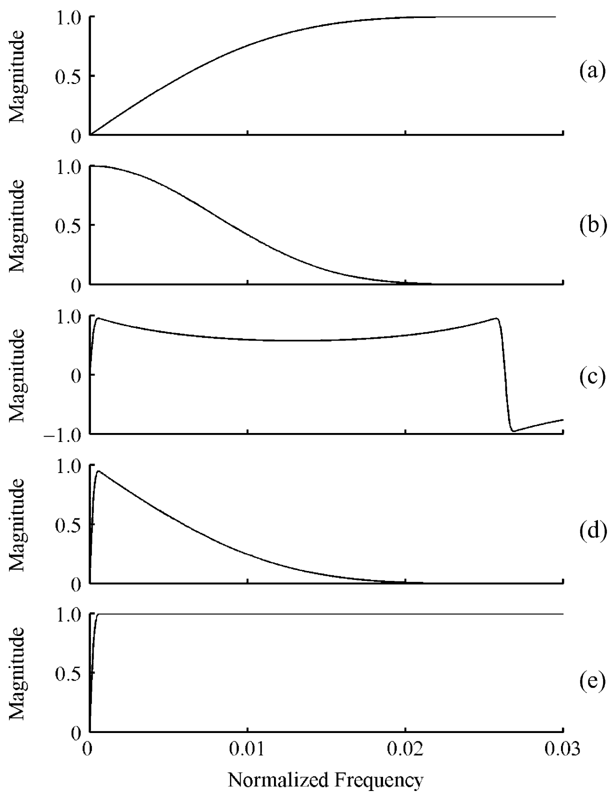

Fig. 6. Frequency responses of the individual subfilters. (a) $H_b(e^{j\omega})$, (b) $H_M(e^{j\omega})$, (c) $H_1(e^{jM\omega})$, (d) $H_M(e^{j\omega})H_1(e^{jM\omega})$, (e) $H_b(e^{j\omega}) + H_M(e^{j\omega})H_1(e^{jM\omega})$.

this case, a multi-level FRM technique may be employed to further reduce the complexity. In the multi-level FRM technique, we define

$$H^{(K)}\left(z^{\prod_{i=1}^{K} M_i}\right) = H^{(K+1)}\left(z^{\prod_{i=1}^{K+1} M_i}\right) H_M^{(K+1)} \times \left(z^{\prod_{i=1}^{K} M_i}\right) + H_b^{(K+1)}\left(z^{\prod_{i=1}^{K} M_i}\right) \quad (15)$$

where $\prod_{i=1}^{0} M_i$ is defined as unity.

Consider a K-level masking structure. The case for $K = 3$ is shown in Fig. 7. Let the lengths of $H^{(K)}(z), H_M^{(i)}(z)$, and $H_b^{(i)}(z)$ be $N^{(K)}, N_M^{(i)}$, and $N_b^{(i)}$, respectively. Estimates for $N^{(K)}, N_M^{(i)}$, and $N_b^{(i)}$ for even values of $N^{(K)}$ and $N_b^{(i)}$ are given in

$$N^{(K)} = \frac{\Phi_H(\delta)}{\Delta \prod_{i=1}^{K} M_i} \quad (16a)$$

$$N_b^{(k)} = \frac{\Phi_H(\delta)}{\frac{1}{M_i} - \Delta \prod_{i=1}^{k-1} M_i}, \qquad k = 1, \ldots, K \quad (16b)$$

$$N_M^{(k)} = \frac{\Phi_M(\delta)}{\frac{1}{M_i} - \Delta \prod_{i=1}^{k-1} M_i}, \qquad k = 1, \ldots, K. \quad (16c)$$

The total number of nontrivial coefficients, $N_{\mathrm{total}}^{(K)}(\delta)$ of a K-level masking design is given by

$$N_{\mathrm{total}}^{(K)}(\delta) = N^{(K)} + \sum_{i=1}^{K} N_b^{(i)} + \sum_{i=1}^{K} N_M^{(i)} = \frac{\Phi_H(\delta)}{\Delta \prod_{i=1}^{K} M_i} + \sum_{k=1}^{K} \frac{\Phi_H(\delta) + \Phi_M(\delta)}{\frac{1}{M_k} - \Delta \prod_{i=1}^{k-1} M_i}. \quad (17)$$

The optimum values of $M_i, i = 1, 2, \ldots, K$ are equal to $M_{\mathrm{opt}}(K)$ where

$$M_{\mathrm{opt}}(K) \approx \left(\frac{\Phi_H(\delta)}{(\Phi_H(\delta) + \Phi_M(\delta))\Delta}\right)^{\frac{1}{K+1}} \approx \left(\frac{0.43}{\Delta}\right)^{\frac{1}{K+1}} \quad \text{for } \frac{\Phi_M(\delta)}{\Phi_H(\delta)} = 1.32. \quad (18)$$

For $M_i = M_{\mathrm{opt}}(K)$ given in (18), we have

$$N^{(K)} = N_M^{(K)} + N_b^{(K)} \approx \left(\frac{\Phi_H(\delta)(\Phi_H(\delta) + \Phi_M(\delta))^K}{\Delta}\right)^{\frac{1}{K+1}} \approx \frac{2.32^{\frac{K}{K+1}} \Phi_H(\delta)}{\Delta^{\frac{1}{K+1}}}. \quad (19)$$

The total number of nontrivial coefficients, $N_{\mathrm{total}}^{(K)}(\delta)$, of a K-level masking design is given by

$$N_{\mathrm{total}}^{(K)}(\delta) = N^{(K)} + \sum_{i=1}^{K}\left(N_b^{(i)} + N_M^{(i)}\right) = (K+1)\left(\frac{\Phi_H(\delta)(\Phi_H(\delta) + \Phi_M(\delta))^K}{\Delta}\right)^{\frac{1}{K+1}} \quad (20a)$$

$$= \Phi_H(\delta)(K+1)2.32^{K/(K+1)}\Delta^{-1/(K+1)} \quad \text{for } \Phi_M(\delta)/\Phi_H(\delta) = 1.32. \quad (20b)$$

The values of $(K + 1)2.32^{K/(K+1)}\Delta^{-1/(K+1)}$ for several values of Δ and K are tabulated in Table V. It can be seen from Table V that, for a given value of $\Delta, (K + 1)2.32^{K/(K+1)}\Delta^{-1/(K+1)}$ decreases initially with increasing K until it reaches a minimum and then increases with increasing K. The value of K that will yield the minimum

TABLE V
VALUES OF $(K+1)2.32^{K/(K+1)}\delta^{-1/(K+1)}$ FOR SEVERAL VALUES OF δ AND K. RESULTS FOR $K = 0$ CORRESPOND TO THOSE WITHOUT USING OUR PROPOSED FRM TECHNIQUE

	Δ = 0.01	**Δ = 0.001**	**Δ = 0.0001**
***K* = 0**	**100**	**1000**	**10000**
***K* = 1**	**30.5**	**96.3**	**304.6**
***K* = 2**	**24.4**	**52.6**	**113.3**
***K* = 3**	**23.8**	**42.3**	**75.2**
***K* = 4**	**24.6**	**39.0**	**61.9**
***K* = 5**	**26.1**	**38.3**	**56.2**
***K* = 6**	**27.8**	**38.6**	**53.7**
***K* = 7**	**29.7**	**39.6**	**52.8**
***K* = 8**	**31.7**	**41.0**	**52.9**
***K* = 9**	**33.8**	**42.6**	**53.6**

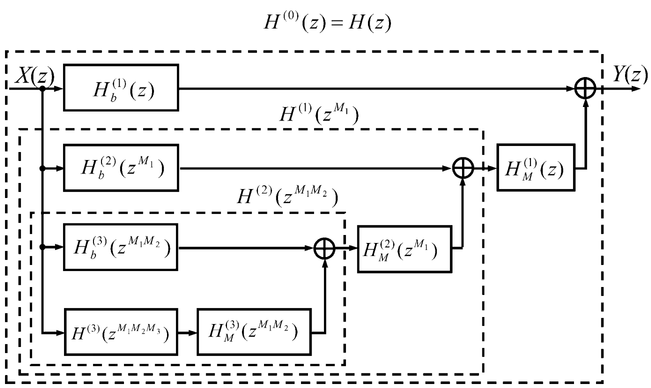

Fig. 7. Three-level masking structure.

value of $(K+1)2.32^{K/(K+1)}\Delta^{-1/(K+1)}$ depends on the value of Δ.

It can be shown by manipulating (20a) that $N_{\text{total}}^{(K+1)}(\delta) > N_{\text{total}}^{(K)}(\delta)$ if

$$\Delta > \beta(\delta, K) \tag{21a}$$

where

$$\begin{aligned}\beta(\delta, K) &= \frac{\Phi_H(\delta)}{\Phi_H(\delta)+\Phi_M(\delta)}\left(\frac{K+1}{K+2}\right)^{(K+1)(K+2)} \\ &\approx 0.43\left(\frac{K+1}{K+2}\right)^{(K+1)(K+2)}.\end{aligned} \tag{21b}$$

The value of K that minimizes $N_{\text{total}}^{(K)}(\delta)$, denoted by K_{opt}, satisfies the constraint

$$\beta(\delta, K_{\text{opt}}-1) > \Delta > \beta(\delta, K_{\text{opt}}). \tag{22}$$

The optimum values of K for various transition widths are tabulated in Table VI. If K is selected to minimize $N_{\text{total}}^{(K)}(\delta)$ based on (20), it can be shown by considering (20a) and (18) that $M_{\text{opt}}(K)$ is bounded by

$$\left(\frac{K_{\text{opt}}+1}{K_{\text{opt}}}\right)^{K_{\text{opt}}} < M_{\text{opt}}(K_{\text{opt}}) < \left(\frac{K_{\text{opt}}+2}{K_{\text{opt}}+1}\right)^{K_{\text{opt}}+2}. \tag{23}$$

TABLE VI
RANGES OF Δ CORRESPONDING TO VARIOUS VALUES OF K_{opt} FOR MINIMUM $N_{\text{total}}^{(K_{\text{opt}})}(\delta)$

***K**opt*	**Δ = Transition width / Sampling frequency**
1	**0.108 ~ 0.038**
2	**0.038 ~ 0.0136**
3	**0.0136 ~ 0.0050**
4	**0.0050 ~ 0.00181**
5	**0.00181 ~ 0.00066**
6	**0.00066 ~ 0.000243**
7	**0.000243 ~ 0.000089**
8	**0.000089 ~ 0.000033**
9	**0.000033 ~ 0.0000120**
10	**0.0000120 ~ 0.0000044**

Note that both

$$\left(\frac{K_{\text{opt}}+1}{K_{\text{opt}}}\right)^{K_{\text{opt}}} \quad \text{and} \quad \left(\frac{K_{\text{opt}}+2}{K_{\text{opt}}+1}\right)^{K_{\text{opt}}+2}$$

approach e (the base of the natural logarithm) as K_{opt} approaches infinity. Thus, $M_{\text{opt}}(K_{\text{opt}})$ approaches e as Δ approaches zero. It is also interesting to note that $\beta(\delta, K_{\text{opt}}-1)/\beta(\delta, K_{\text{opt}})$ also approaches e as K_{opt} approaches infinity.

The effective length of the filter $N_{\text{eff}}(K)$ is given by

$$\begin{aligned}N_{\text{eff}}(K) &= N^{(K)}\prod_{i=1}^{K} M_i + \sum_{k=1}^{K} N_M^{(k)}\prod_{i=1}^{k-1} M_i \\ &= N_{\text{opt}}\left(1+\Delta\frac{M_{\text{opt}}^{K+1}(K)-M_{\text{opt}}(K)}{M_{\text{opt}}(K)-1}\right).\end{aligned} \tag{24}$$

In (24), the term $\Delta(M_{\text{opt}}^{K+1}(K)-M_{\text{opt}}(K))/(M_{\text{opt}}(K)-1)$ represents the fractional increase in filter length when compared to the Chebyshev optimum design. For $\Delta \rightarrow 0$, we have $K_{\text{opt}} \rightarrow \infty$, $M_{\text{opt}}^{K_{\text{opt}}+1}(K_{\text{opt}}) - M_{\text{opt}}(K_{\text{opt}}) \rightarrow M_{\text{opt}}^{K_{\text{opt}}+1}(K_{\text{opt}})$, $M_{\text{opt}}(K_{\text{opt}}) \rightarrow 2.7$, and applying the result of (18)

$$N_{\text{eff}}(K_{\text{opt}}) \rightarrow N_{\text{opt}}\left(1+\frac{\Phi_H(\delta)}{1.7(\Phi_H(\delta)+\Phi_M(\delta))}\right). \tag{25}$$

For $\Phi_M(\delta)/\Phi_H(\delta) = 1.32$, $(\Phi_H(\delta))/(1.7(\Phi_H(\delta)+\Phi_M(\delta))) \approx 0.25$. Thus, as $K_{\text{opt}} \rightarrow \infty$ for the case where $\Delta \rightarrow 0$, the FRM technique will produce a Hilbert transformer whose length is about 25% longer than that of the Chebyshev optimum design.

VI. TWO-LEVEL MASKING EXAMPLE

For the acoustic feedback suppression example in Section IV, the optimum value of K as listed in Table VI was 5 (corresponding to $\Delta = 0.001\,25$). The value of $N_{\text{total}}^{(5)}(0.0001)$ predicted from (20a) was 90; it is smaller than the Chebyshev optimum design of 2000 by a factor of 22! However, the actual selection of the value of K depends on several factors. One of the most important factors influencing the selection of K is the availability of user friendly and reliable optimization packages. The difficulty faced in the design process increases

TABLE VII
COEFFICIENT VALUES OF $H_b^{(1)}(z)$

$h_b^{(1)}(-17) = 0.00035664 = -h_b^{(1)}(17)$	$h_b^{(1)}(-7) = 0.05116141 = -h_b^{(1)}(7)$
$h_b^{(1)}(-15) = 0.00112735 = -h_b^{(1)}(15)$	$h_b^{(1)}(-5) = 0.09558504 = -h_b^{(1)}(5)$
$h_b^{(1)}(-13) = 0.00571987 = -h_b^{(1)}(13)$	$h_b^{(1)}(-3) = 0.19152228 = -h_b^{(1)}(3)$
$h_b^{(1)}(-11) = 0.01278547 = -h_b^{(1)}(11)$	$h_b^{(1)}(-1) = 0.63050716 = -h_b^{(1)}(1)$
$h_b^{(1)}(-9) = 0.02680092 = -h_b^{(1)}(9)$	

TABLE VIII
COEFFICIENT VALUES OF $H_M^{(1)}(z)$

$h_M^{(1)}(-20) = 0.00204909 = h_M^{(1)}(20)$	$h_M^{(1)}(-8) = 0.06345628 = h_M^{(1)}(8)$
$h_M^{(1)}(-18) = 0.00468652 = h_M^{(1)}(18)$	$h_M^{(1)}(-6) = 0.07736226 = h_M^{(1)}(6)$
$h_M^{(1)}(-16) = 0.01000542 = h_M^{(1)}(16)$	$h_M^{(1)}(-4) = 0.09310812 = h_M^{(1)}(4)$
$h_M^{(1)}(-14) = 0.01825014 = h_M^{(1)}(14)$	$h_M^{(1)}(-2) = 0.10265599 = h_M^{(1)}(2)$
$h_M^{(1)}(-12) = 0.03019052 = h_M^{(1)}(12)$	$h_M^{(1)}(0) = 0.10639832$
$h_M^{(1)}(-10) = 0.04503649 = h_M^{(1)}(10)$	

TABLE IX
COEFFICIENT VALUES OF $H_b^{(2)}(z^7)$

$h_b^{(2)}(-105) = 0.00077414 = -h_b^{(2)}(105)$	$h_b^{(2)}(-49) = 0.04783235 = -h_b^{(2)}(49)$
$h_b^{(2)}(-91) = 0.00533037 = -h_b^{(2)}(91)$	$h_b^{(2)}(-35) = 0.08969270 = -h_b^{(2)}(35)$
$h_b^{(2)}(-77) = 0.01192049 = -h_b^{(2)}(77)$	$h_b^{(2)}(-21) = 0.17271575 = -h_b^{(2)}(21)$
$h_b^{(2)}(-63) = 0.02492091 = -h_b^{(2)}(63)$	$h_b^{(2)}(-7) = 0.40522743 = -h_b^{(2)}(7)$

TABLE X
COEFFICIENT VALUES OF $H_M^{(2)}(z^7)$

$h_M^{(2)}(-140) = 0.00215243 = h_M^{(2)}(140)$	$h_M^{(2)}(-56) = 0.06351735 = h_M^{(2)}(56)$
$h_M^{(2)}(-126) = 0.00451527 = h_M^{(2)}(126)$	$h_M^{(2)}(-42) = 0.07706411 = h_M^{(2)}(42)$
$h_M^{(2)}(-112) = 0.00940580 = h_M^{(2)}(112)$	$h_M^{(2)}(-28) = 0.09314295 = h_M^{(2)}(28)$
$h_M^{(2)}(-98) = 0.01720234 = h_M^{(2)}(98)$	$h_M^{(2)}(-14) = 0.10409780 = h_M^{(2)}(14)$
$h_M^{(2)}(-84) = 0.02935234 = h_M^{(2)}(84)$	$h_M^{(2)}(0) = 0.10870994$
$h_M^{(2)}(-70) = 0.04519465 = h_M^{(2)}(70)$	

with increasing K; optimization algorithms become less user friendly as K increases. Instead of choosing $K = 5$, we choose $K = 2$ to illustrate the advantage that can be gained in multi-level masking. For two-level masking, the estimated values were $M_1 = M_2 = 7, N^{(2)} = 41, N_M^{(1)} = 23.5, N_M^{(2)} = 22.3, N_b^{(1)} = 18.7, N_b^{(2)} = 17.6$. A design with $M_1 = M_2 = 7, N^{(2)} = 42, N_M^{(1)} = N_M^{(2)} = 21$, and $N_b^{(1)} = 18$, and $N_b^{(2)} = 16$ was obtained. The total number of nontrivial coefficient was $42+21+21+18+16 = 118$. This is only 6% of the 2000 nontrivial coefficients of the Chebyshev optimum design; the saving was a factor of 17. The effective length of our FRM design (after replacing z by z^2) was 4339; it was only 8.5% longer than the Chebyshev optimum design. The coefficient values of the subfilters (after replacing z by z^2, z^{2M_1}, z^{2M_2} or $z^{2M_1M_2}$ which ever is appropriate) are tabulated in Tables VII– XI. The passband ripple of the overall filter is shown in Fig. 8 and the frequency responses of the various subfilters are shown in Fig. 9.

TABLE XI
COEFFICIENT VALUES OF $H^{(2)}(z^{49})$

$h^{(2)}(-2009) = 0.00017441 = -h^{(2)}(2009)$	$h^{(2)}(-931) = 0.01548193 = -h^{(2)}(931)$
$h^{(2)}(-1911) = 0.00024335 = -h^{(2)}(1911)$	$h^{(2)}(-833) = 0.02027347 = -h^{(2)}(833)$
$h^{(2)}(-1813) = 0.00052237 = -h^{(2)}(1813)$	$h^{(2)}(-735) = 0.02636264 = -h^{(2)}(735)$
$h^{(2)}(-1715) = 0.00086057 = -h^{(2)}(1715)$	$h^{(2)}(-637) = 0.03422554 = -h^{(2)}(637)$
$h^{(2)}(-1617) = 0.00140905 = -h^{(2)}(1617)$	$h^{(2)}(-539) = 0.04463223 = -h^{(2)}(539)$
$h^{(2)}(-1519) = 0.00215043 = -h^{(2)}(1519)$	$h^{(2)}(-441) = 0.05904263 = -h^{(2)}(441)$
$h^{(2)}(-1421) = 0.00319015 = -h^{(2)}(1421)$	$h^{(2)}(-343) = 0.08044706 = -h^{(2)}(343)$
$h^{(2)}(-1323) = 0.00456441 = -h^{(2)}(1323)$	$h^{(2)}(-245) = 0.11620845 = -h^{(2)}(245)$
$h^{(2)}(-1225) = 0.00638009 = -h^{(2)}(1225)$	$h^{(2)}(-147) = 0.18938801 = -h^{(2)}(147)$
$h^{(2)}(-1127) = 0.00871393 = -h^{(2)}(1127)$	$h^{(2)}(-49) = 0.41325457 = -h^{(2)}(49)$
$h^{(2)}(-1029) = 0.01170195 = -h^{(2)}(1029)$	

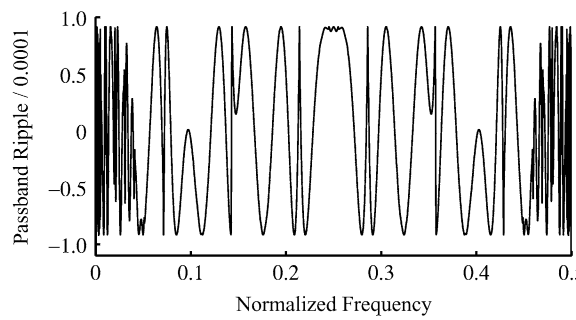

Fig. 8. Passband ripples of the Hilbert transformer. The frequency axis is normalized against the sampling frequency.

VII. HALF-BAND FILTER

An odd length low-pass half-band filter may be derived from an even length Hilbert transformer by scaling all the coefficient values of the Hilbert transformer by a factor of half $(1/2)$, inserting a zero value coefficient between two coefficients, multiplying its nth coefficient by $je^{-jn\pi/2}$, and replacing the centre coefficient by $1/2$. The factor of two reduction in transition width due to inserting a zero between every two coefficients is nullified by the factor of two increase in transition width when the centre coefficient $(\text{value} = 1/2)$ is inserted to raise the frequency response of the Hilbert transformer to become that of the half-band filter. Since the coefficient values have been scaled by half, the ripple magnitude of the resultant half-band filter is half that of the original Hilbert transformer. Taking this into consideration, the set of equations for the half-band filter may be obtained from that for the Hilbert transformer by replacing δ by 2δ. Specifically, for a half-band filter, (18) becomes

$$M_{\text{opt}}(K) \approx \left(\frac{\Phi_H(2\delta)}{(\Phi_H(2\delta)+\Phi_M(2\delta))\Delta}\right)^{\frac{1}{K+1}} \approx \left(\frac{0.43}{\Delta}\right)^{\frac{1}{K+1}} \quad \text{for } \frac{\Phi_M(2\delta)}{\Phi_H(2\delta)} = 1.32 \tag{26}$$

and (22) becomes

$$\beta(2\delta, K_{\text{opt}}-1) > \Delta > \beta(2\delta, K_{\text{opt}}). \tag{27}$$

VIII. $H_b(z)$ WITH SPECIAL PROPERTIES

In the examples presented in Sections IV and VI, $H_b(z)$ was optimized to minimize the overall complexity of the Hilbert transformer. However, it should be noted that $H_b(z)$ can be any low-order Hilbert transformer implemented using any structure

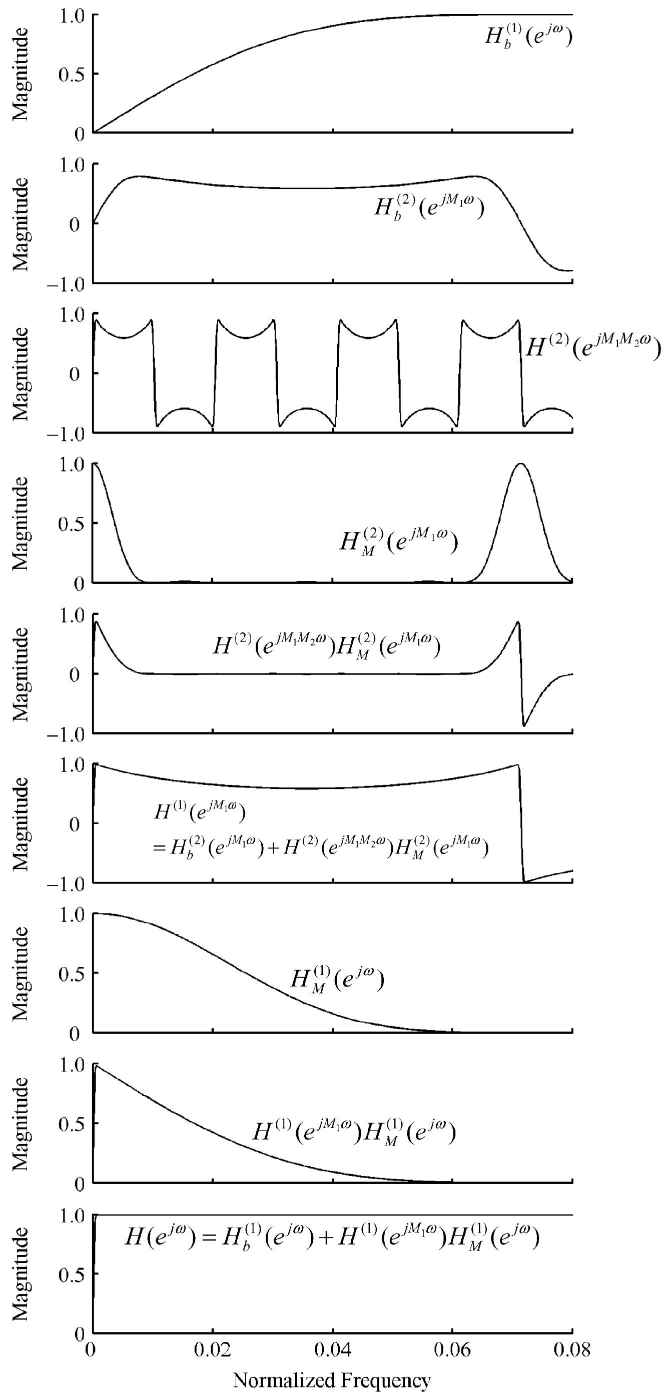

Fig. 9. Frequency responses of the individual subfilters and the overall filter.

such as those in [25]–[27] and can be designed independent on $H_1(z^M)$ and $H_M(z)$. The frequency responses of $H_1(z^M)$ and $H_M(z)$ are than optimized as a correction term to sharpen the frequency response of the overall filter. In this case, the frequency response of the overall filter at the vicinity of $\omega = 0$ is determined mainly by $H_1(z^M)$ and $H_M(z)$. If the attenuation of $H_M(z)$ is very high for frequencies far away from $\omega = 0$, the characteristics of the frequency response of the overall filter will be similar to that of $H_b(z)$ for frequencies far away from $\omega = 0$. Nevertheless, the overall complexity of the filter will be higher than that where $H_b(z), H_1(z^M)$, and $H_M(z)$ are jointly optimized.

IX. Conclusion

A set of equations for characterizing the Hilbert transformer based on the implementation structure shown in Fig. 7 have been derived. It is also demonstrated that, in an acoustic feedback oscillation suppression example, the FRM technique produced a Hilbert transformer with reduction in the number of nontrivial coefficients by a factor of 17 when compared to the Chebyshev optimum design.

Appendix I

The expression [28] relating the length (N_0), of a low-pass filter to its passband and stopband ripples, δ_p and δ_s, respectively, and its normalized transition width (β) is

$$\begin{aligned} N_0 &\approx \frac{\Phi_1(\delta_p,\delta_s)}{\beta} - \Phi_2(\delta_p,\delta_s)\beta + 1 \\ \Phi_1(\delta_p,\delta_s) &= [a_1(\log_{10}\delta_p)^2 + a_2\log_{10}\delta_p + a_3] \times \log_{10}\delta_s \\ &\quad + [a_4(\log_{10}\delta_p)^2 + a_5\log_{10}\delta_p + a_6] \\ a_1 &= 5.309 \times 10^{-3} \\ a_2 &= 7.114 \times 10^{-2} \\ a_3 &= -4.761 \times 10^{-1} \\ a_4 &= -2.66 \times 10^{-3} \\ a_5 &= -5.941 \times 10^{-1} \\ a_6 &= -4.278 \times 10^{-1} \\ \Phi_2(\delta_p,\delta_s) &= 11.012\,17 + 0.512\,44 \times (\log_{10}\delta_p - \log_{10}\delta_s). \end{aligned}$$

A Hilbert transformer may be derived from a low-pass half-band filter by discarding the centre coefficient, multiplying its nth coefficient by $-je^{jn\pi/2}$, discarding trivial coefficients, and scale up the coefficient values by a factor of two. Replacing $\delta_p = \delta_s = \delta/2$ we have

$$N_{\text{opt}} \approx \frac{\Phi_H(\delta)}{\Delta} + 1$$

where $\Phi_H(\delta) = 0.002\,655(\log_{10}(\delta))^3 + 0.031\,843(\log_{10}(\delta))^2 - 0.554\,993\log_{10}(\delta) - 0.049\,788$.

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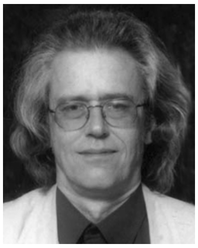

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