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ON EXTREMAL k-GRAPHS WITHOUT REPEATED COPIES OF 2-INTERSECTING EDGES*

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Abstract. The problem of determining extremal hypergraphs containing at most r isomorphic copies of some element of a given hypergraph family was first studied by Boros et al. in 2001. There are not many hypergraph families for which exact results are known concerning the size of the corresponding extremal hypergraphs, except for those equivalent to the classical Turán numbers. In this paper, we determine the size of extremal k-uniform hypergraphs containing at most one pair of 2-intersecting edges for $k \in \{3, 4\}$. We give a complete solution when $k = 3$ and an almost complete solution (with eleven exceptions) when $k = 4$.

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AMS subject classifications. 05B05, 05B07, 05B40, 05D05

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1. Introduction.

A *set system* is a pair $G = (X, \mathcal{A})$, where X is a finite set and $\mathcal{A} \subseteq 2^X$. The members of X are called *vertices* or *points*, and the members of $\mathcal{A}$ are called *edges* or *blocks*. The *order* of G is the number of vertices $|X|$, and the *size* of G is the number of edges $|\mathcal{A}|$. The set K is called a *set of block sizes* for G if $|A| \in K$ for all $A \in \mathcal{A}$. G is called a k-*uniform hypergraph* (or k-*graph*) if $\{k\}$ is a set of block sizes for G. A 2-graph is also known simply as a *graph*.

A pair of edges is said to be t-*intersecting* if they intersect in at least t points. The k-graph of size two whose two edges intersect in exactly t points is denoted $\Lambda(k, t)$.

Let $\mathcal{F}$ be a family of k-graphs. Boros et al. [2] introduced the function $T(n, \mathcal{F}, r)$, which denotes the maximum number of edges in a k-graph of order n containing no r isomorphic copies of a member of $\mathcal{F}$. So $T(n, \mathcal{F}, 1)$ is just the classical Turán number $\text{ex}(n, \mathcal{F})$ [1]. A family of k-graphs $\mathcal{F}$ is said to *grow polynomially* if there exist $c > 0$ and a nonnegative integer s such that, for every m, there are at most cm^s members in $\mathcal{F}$ having exactly m edges. The following theorem is established in [2].

THEOREM 1.1 (Boros et al. [2]). *Let $\mathcal{F}$ be a family of k-graphs which grows polynomially with parameters c and s. Then, for n sufficiently large,*

$$T(n, \mathcal{F}, r) < \text{ex}(n, \mathcal{F}) + (c \cdot (r-1) \cdot s! + 1)\text{ex}(n, F)^{(s+1)/(s+2)} + 2(c \cdot (r-1) \cdot s! + 1)^2 \text{ex}(n, \mathcal{F})^{s/(s+2)}.$$

For $k \geq 3$, let $\mathcal{F}(k)$ be the family of k-graphs of two 2-intersecting edges; that is, $\mathcal{F}(k) = \{\Lambda(k, t) : 2 \leq t \leq k-1\}$. $T(n, \mathcal{F}(k), 1)$, which is the Turán number

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$\mathrm{ex}(n, \mathcal{F}(k))$, is equal to the following well studied parameters in design theory and coding theory:

- $D(n, k, 2)$, the maximum number of blocks in a 2-$(n, k, 1)$ packing [11], and
- $A(n, 2(k-1), k)$, the maximum number of codewords in a binary code of length n, minimum distance $2(k-1)$, and constant weight k [10].

Despite much effort, the exact value of $T(n, \mathcal{F}(k), 1)$ is known for all n only when $k = 3$ [14, 15] and $k = 4$ [3]. Even for $k = 5$, there are an infinite number of n for which $T(n, \mathcal{F}(5), 1)$ is not yet determined. In this paper, we determine $T(n, \mathcal{F}(k), 2)$ for all n when $k = 3$ and for all but 11 values of n when $k = 4$.

2. Design-theoretic preliminaries. Our determination of $T(n, \mathcal{F}(k), 2)$, $k \in \{3, 4\}$, makes extensive use of combinatorial designs. In this section, we review some design-theoretic constructs and review some prior results that are needed in our solution.

For positive integers $i \le j$, the set $\{i, i+1, \ldots, j\}$ is denoted $[i, j]$. The set $[1, j]$ is further abbreviated as $[j]$. A k-graph $(X, \mathcal{A})$ of order n is a packing of pairs by k-tuples, or more commonly known as a 2-$(n, k, 1)$ *packing* if every 2-subset of X is contained in at most one block of $\mathcal{A}$. The *leave* of $(X, \mathcal{A})$ is the graph $L = (X, \mathcal{E})$, where $\mathcal{E}$ consists of all 2-subsets of X that are not contained in any blocks of $\mathcal{A}$. We also say that $(X, \mathcal{A})$ is a 2-$(n, k, 1)$ packing *leaving* L. Given a graph G, the maximum size of a 2-$(n, k, 1)$ packing whose leave contains G is denoted $m(n, k, G)$. Note that the maximum size of a 2-$(n, k, 1)$ packing, $D(n, k, 2)$, is the quantity $m(n, k, G)$ when G is the empty graph.

THEOREM 2.1 (Schönheim [14], Spencer [15]). *For all $n \ge 0$, we have*

$$D(n,3,2) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1 & \text{if } n \equiv 5 \pmod 6, \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor & \text{otherwise.} \end{cases}$$

THEOREM 2.2 (Brouwer [3]). *For all $n \ge 0$, we have*

$$D(n,4,2) = \begin{cases} \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 1 & \text{if } n \equiv 7 \text{ or } 10 \pmod{12} \text{ and } n \notin \{10, 19\}, \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 1 & \text{if } n \in \{9, 17\}, \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 2 & \text{if } n \in \{8, 10, 11\}, \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor - 3 & \text{if } n = 19, \\ \left\lfloor \frac{n}{4} \left\lfloor \frac{n-1}{3} \right\rfloor \right\rfloor & \text{otherwise.} \end{cases}$$

A *pairwise balanced design* (PBD) is a set system $(X, \mathcal{A})$ such that every 2-subset of X is contained in exactly one block of $\mathcal{A}$. If a PBD is of order n and has a set of block sizes K, we denote it by PBD(n, K). If a member $k \in K$ is superscripted with a "$\star$" (written "$k^\star$"), it means that the PBD has exactly one block of size k. We require the following result on the existence of PBDs.

THEOREM 2.3 (Fort and Hedlund [5]). *There exists a* PBD$(n, \{3, 5^\star\})$ *if and only if* $n \equiv 5 \pmod 6$.

THEOREM 2.4 (Rees and Stinson [13]). *There exists a* PBD$(n, \{4, f^\star\})$ *if and only if* $n \ge 3f + 1$, *and*

(i) $n \equiv 1$ *or* $4 \pmod{12}$ *and* $f \equiv 1$ *or* $4 \pmod{12}$ *or*

(ii) $n \equiv 7$ *or* $10 \pmod{12}$ *and* $f \equiv 7$ *or* $10 \pmod{12}$.

Let $(X, \mathcal{A})$ be a set system, and let $\mathcal{G} = \{G_1, \ldots, G_s\}$ be a partition of X into subsets, called *groups*. The triple $(X, \mathcal{G}, \mathcal{A})$ is a *group divisible design* (GDD) when every 2-subset of X not contained in a group appears in exactly one block, and

$|A \cap G| \le 1$ for all $A \in \mathcal{A}$ and $G \in \mathcal{G}$. We denote a GDD $(X, \mathcal{G}, \mathcal{A})$ by K-GDD if K is a set of block sizes for $(X, \mathcal{A})$. The *type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $[|G| : G \in \mathcal{G}]$. When more convenient, we use the exponentiation notation to describe the type of a GDD: A GDD of type $g_1^{t_1} \dots g_s^{t_s}$ is a GDD where there are exactly t_i groups of size g_i, $i \in [s]$. The following results on the existence of $\{4\}$-GDDs are useful.

THEOREM 2.5 (Hanani [7]). *There exists a* $\{3\}$-GDD *of type* g^t *if and only if* $t \ge 3$, $g^2\binom{t}{2} \equiv 0 \pmod 3$, *and* $g(t-1) \equiv 0 \pmod 2$.

THEOREM 2.6 (Brouwer, Schrijver, and Hanani [4]). *There exists a* $\{4\}$-GDD *of type* g^t *if and only if* $t \ge 4$ *and*

(i) $g \equiv 1$ *or* $5 \pmod 6$ *and* $t \equiv 1$ *or* $4 \pmod{12}$ *or*
(ii) $g \equiv 2$ *or* $4 \pmod 6$ *and* $t \equiv 1 \pmod 3$ *or*
(iii) $g \equiv 3 \pmod 6$ *and* $t \equiv 0$ *or* $1 \pmod 4$ *or*
(iv) $g \equiv 0 \pmod 6$,

with the two exceptions of types 2^4 *and* 6^4, *for which* $\{4\}$-GDD*s do not exist.*

THEOREM 2.7 (Brouwer [3]). *A* $\{4\}$-GDD *of type* $2^u 5^1$ *exists if and only if* $u = 0$, *or* $u \equiv 0 \pmod 3$ *and* $u \ge 9$.

THEOREM 2.8 (see [9]). *There exists a* $\{4\}$-GDD *of type* $3^t u^1$ *if and only if* $t = 0$, *or* $t \ge (2u+3)/3$ *and*

(i) $t \equiv 0$ *or* $1 \pmod 4$ *and* $u \equiv 0$ *or* $6 \pmod{12}$ *or*
(ii) $t \equiv 0$ *or* $3 \pmod 4$ *and* $u \equiv 3$ *or* $9 \pmod{12}$.

THEOREM 2.9 (Ge and Ling [6]). *There exists a* $\{4\}$-GDD *of type* $2^t u^1$ *for* $t = 0$ *and for each* $t \ge 6$ *with* $t \equiv 0 \pmod 3$, $u \equiv 2 \pmod 3$, *and* $2 \le u \le t-1$, *except for* $(t, u) = (6, 5)$ *and except possibly for* $(t, u) \in \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$.

THEOREM 2.10 (Ge and Ling [6]). *There exists a* $\{4\}$-GDD *of type* $12^t u^1$ *for* $t = 0$ *and for each* $t \ge 4$ *and* $u \equiv 0 \pmod 4$ *such that* $0 \le u \le 6(t-1)$.

An *incomplete transversal design of group size* n, *block size* k, and *hole size* h is a quadruple $(X, \mathcal{G}, H, \mathcal{A})$ such that

(i) $(X, \mathcal{A})$ is a k-graph of order nk;
(ii) $\mathcal{G}$ is a partition of X into k subsets (called *groups*), each of cardinality n;
(iii) $H \subseteq X$, with the property that, for each $G \in \mathcal{G}$, $|G \cap H| = h$; and
(iv) every 2-subset of X is
- contained in the *hole* H and not contained in any blocks or
- contained in a group and not contained in any blocks or
- contained in neither a hole nor a group and contained in exactly one block of $\mathcal{A}$.

Such an incomplete transversal design is denoted $\mathrm{TD}(k, n) - \mathrm{TD}(k, h)$.

THEOREM 2.11 (Heinrich and Zhu [8]). *For* $n > h > 0$, *a* $\mathrm{TD}(4, n) - \mathrm{TD}(4, h)$ *exists if and only if* $n \ge 3h$ *and* $(n, h) \ne (6, 1)$.

3. Packings with leaves containing specified graphs. In this section, we relate the problem of determining $T(n, \mathcal{F}(k), 2)$ to that of determining $m(n, k, G)$ for G isomorphic to $K_4 - e$, $K_5 - e$, and $2 \circ K_4$ (edge-gluing of two K_4's) when $k \in \{3, 4\}$. These graphs are shown in Figures 3.1–3.3, respectively.

LEMMA 3.1. *There exists a* 3*-graph of order* n *and size* m *containing exactly one copy of an element of* $\mathcal{F}(3)$ *if and only if there exists a* 2-$(n, 3, 1)$ *packing of size* $m-2$ *with a leave containing* $K_4 - e$ *as a subgraph.*

Proof. $\mathcal{F}(3)$ contains only a single 3-graph, $\Lambda(3, 2)$. Let $(X, \mathcal{A})$ be a 3-graph of order n and size m containing exactly one copy of $\Lambda(3, 2)$. Then there exist exactly two blocks $A, B \in \mathcal{A}$, with $|A \cap B| = 2$. Let $P = (X, \mathcal{A} \setminus \{A, B\})$. Then P is a

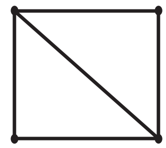

FIG. 3.1. $K_4 - e$.

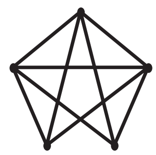

FIG. 3.2. $K_5 - e$.

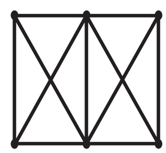

FIG. 3.3. $2 \circ K_4$.

2-$(n, 3, 1)$ packing of size $m-2$ with a leave containing the 2-subsets in X that occurs in A and B, which together form a $K_4 - e$. This construction is reversible. □

COROLLARY 3.2. *The following holds:*

$$T(n, \mathcal{F}(3), 2) = \max\{T(n, \mathcal{F}(3), 1), m(n, 3, K_4 - e) + 2\}.$$

Proof. If a 3-graph contains no two isomorphic copies of $\Lambda(3, 2)$, then either it contains no copies, in which case its maximum size is given by $T(n, \mathcal{F}(3), 1)$, or else it contains exactly one copy, in which case its maximum size is given by $m(n, 3, K_4 - e) + 2$. □

The proofs for the following two lemmas are similar to that for Lemma 3.1 and are thus omitted.

LEMMA 3.3. *There exists a* 4*-graph of order* n *and size* m *containing exactly one copy of* $\Lambda(4, 2)$ *if and only if there exists a* 2-$(n, 4, 1)$ *packing of size* $m - 2$ *with a leave containing* $2 \circ K_4$ *as a subgraph.*

LEMMA 3.4. *There exists a* 4*-graph of order* n *and size* m *containing exactly one copy of* $\Lambda(4, 3)$ *if and only if there exists a* 2-$(n, 4, 1)$ *packing of size* $m - 2$ *with a leave containing* $K_5 - e$ *as a subgraph.*

COROLLARY 3.5. *The following holds:*

$$T(n, \mathcal{F}(4), 2) = \max\{T(n, \mathcal{F}(4), 1), m(n, 4, 2 \circ K_4) + 2, m(n, 4, K_5 - e) + 2\}.$$

Proof. $\mathcal{F}(4)$ contains the graphs $\Lambda(4, 2)$ and $\Lambda(4, 3)$. So if a 4-graph contains no two isomorphic copies of an element of $\mathcal{F}(4)$, then either it contains none of them, in which case its maximum size is given by $T(n, \mathcal{F}(4), 1)$, or else it contains exactly one of $\Lambda(4, 2)$ or $\Lambda(4, 3)$. In the former case, its maximum size is $m(n, 4, 2 \circ K_4) + 2$ by Lemma 3.3, and, in the latter case, its maximum size is $m(n, 4, K_5 - e)$ by Lemma 3.4. □

4. Determining $T(n, \mathcal{F}(3), 2)$. When $n \equiv 1$ or 3 (mod 6), a 2-$(n, 3, 1)$ packing of size $T(n, \mathcal{F}(3), 1)$ has the property that every pair of distinct points is contained in exactly one block. Such a 2-$(n, 3, 1)$ packing is called a *Steiner triple system* of order n and is denoted STS(n).

Let $P = (X, \mathcal{A})$ be a 2-$(n, 3, 1)$ packing. When $n \equiv 1$ or 3 (mod 6), the leave $L = (X, \mathcal{E})$ of P must satisfy:

(i) $|\mathcal{E}| \equiv 0$ (mod 3), and

(ii) the degree of every vertex in L is even.

Any L containing $K_4 - e$ as a subgraph and satisfying conditions (i) and (ii) above has at least nine edges. Hence, the maximum size of a 2-$(n, 3, 1)$ packing with a leave containing $K_4 - e$ is at most $\frac{1}{3}(\binom{n}{2} - 9)$. We show below that there indeed exists such a 2-$(n, 3, 1)$ packing of size $\frac{1}{3}(\binom{n}{2} - 9)$.

LEMMA 4.1. *There exists a* 2-$(n, 3, 1)$ *packing of size* $\frac{1}{3}(\binom{n}{2} - 9)$*, with a leave containing* $K_4 - e$*, for every* $n \equiv 1$ *or* 3 (mod 6).

Proof. Let $(X, \mathcal{A})$ be an STS(n). Suppose there exist three blocks in $\mathcal{A}$ of the form $\{1, 2, 3\}$, $\{1, 4, 5\}$, and $\{3, 4, a\}$. Then deleting these three blocks gives a 2-$(n, 3, 1)$ packing of size $\frac{1}{3}(\binom{n}{2} - 9)$ with a leave containing $K_4 - e$. Hence, it suffices to show that we can always find such a 3-block configuration in any STS(n). To see that this is true, pick any two intersecting blocks in an STS(n), say, $\{1, 2, 3\}$ and $\{1, 4, 5\}$. As the third block, take the unique block containing the 2-subset $\{3, 4\}$. ∎

Next, we consider $n \equiv 5$ (mod 6). In this case, $\binom{n}{2} \equiv 1$ (mod 3). So if the leave of a 2-$(n, 3, 1)$ packing contains $K_4 - e$, then it must contain at least seven edges. Therefore, such a packing can have at most $\frac{1}{3}(\binom{n}{2} - 7)$ blocks. We show below that this upper bound can be met using pairwise balanced designs.

LEMMA 4.2. *There exists a* 2-$(n, 3, 1)$ *packing of size* $\frac{1}{3}(\binom{n}{2} - 7)$*, with a leave containing* $K_4 - e$*, for every* $n \equiv 5$ (mod 6).

Proof. Let $(X, \mathcal{A})$ be a PBD$(n, \{3, 5^\star\})$ with $[5]$ as the block of size five. The existence of such a PBD is provided by Theorem 2.3. Deleting the block of size five from this PBD and adding the block $\{1, 2, 3\}$ yield the desired 2-$(n, 3, 1)$ packing. ∎

For $n \equiv 0$, 2, or 4 (mod 6), every vertex in the leave L of a 2-$(n, 3, 1)$ packing is of odd degree. If L contains $K_4 - e$, then L must have at least four vertices of degree at least three. The minimum possible number of edges in L, if L contains $K_4 - e$, is therefore $n/2 + 4$. It follows that the number of blocks in a 2-$(n, 3, 1)$ packing with a leave containing $K_4 - e$ is at most $\left\lfloor \frac{1}{3}(\binom{n}{2} - \frac{n}{2} - 4) \right\rfloor$.

LEMMA 4.3. *There exists a* 2-$(n, 3, 1)$ *packing of size* $\frac{1}{3}(\binom{n}{2} - \frac{n}{2} - 4)$*, with a leave containing* $K_4 - e$*, for every* $n \equiv 4$ (mod 6).

Proof. Let $(X, \mathcal{A})$ be a PBD$(n + 1, \{3, 5^\star\})$ which exists by Theorem 2.3. Let x be a point contained in the block of size five. Then $(X \setminus \{x\}, \mathcal{B})$, where

$$\mathcal{B} = \{A \in \mathcal{A} : x \notin A \text{ and } |A| = 3\}$$

is the desired 2-$(n, 3, 1)$ packing. ∎

LEMMA 4.4. *There exists a* 2-$(n, 3, 1)$ *packing of size* $\frac{1}{3}(\binom{n}{2} - \frac{n}{2} - 6)$*, with a leave containing* $K_4 - e$*, for every* $n \equiv 0$ *or* 2 (mod 6).

Proof. Consider a $\{3\}$-GDD of type $2^{n/2}$, which exists whenever $n \equiv 0$ or 2 (mod 6) by Theorem 2.5. Without loss of generality, we may assume $\{1, 2\}$ is a group and $\{1, 3, 4\}$ is a block in this GDD. There is a unique block of the form $\{2, 3, a\}$. Deleting the blocks $\{1, 3, 4\}$ and $\{2, 3, a\}$ from this GDD gives a 2-$(n, 3, 1)$ packing of size $\frac{1}{3}(\binom{n}{2} - \frac{n}{2} - 6)$, with a leave containing $K_4 - e$. ∎

This completes our determination of $m(n, 3, K_4 - e)$. We summarize our results above as follows.

THEOREM 4.5. *For all $n \geq 0$, we have $m(n, 3, K_4 - e) = \frac{1}{3}(\binom{n}{2} - f(n))$, where*

$$f(n) = \begin{cases} n/2 + 6 & \text{if } n \equiv 0 \text{ or } 2 \pmod 6, \\ 9 & \text{if } n \equiv 1 \text{ or } 3 \pmod 6, \\ n/2 + 4 & \text{if } n \equiv 4 \pmod 6, \\ 7 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

5. Determining $T(n, \mathcal{F}(4), 2)$.

We now determine $T(n, \mathcal{F}(4), 2)$.

5.1. The case $n \equiv 1$ or $4 \pmod{12}$.

The leave $L = (X, \mathcal{E})$ of a 2-$(n, 4, 1)$ packing must satisfy:

(i) $|\mathcal{E}| \equiv 0 \pmod 6$, and

(ii) every vertex in L has degree $\equiv 0 \pmod 3$.

Any leave of P containing $K_5 - e$ or $2 \circ K_4$ as a subgraph and satisfying conditions (i) and (ii) above has at least 18 edges. So $m(n, 4, G) \leq \frac{1}{6}(\binom{n}{2} - 18)$ for $G \in \{K_5 - e, 2 \circ K_4\}$. We show below that this bound can be met with a finite number of possible exceptions.

The *cocktail party graph* CP(n) is the unique $(2n-2)$-regular graph on $2n$ vertices. We begin with an observation on CP(4) (shown in Figure 5.1).

LEMMA 5.1. CP(4) *contains an edge-disjoint union of a $K_5 - e$ and a K_4.*

Proof. Without loss of generality, we may take the vertex set and edge set of the CP(4) as $[8]$ and $\{A \subset [8] : |A| = 2\} \setminus \{\{i, i+4\} : i \in [4]\}$, respectively. Consider the subsets of edges $\mathcal{E}_1 = \{A \subset \{1, 2, 3, 5, 8\} : |A| = 2\} \setminus \{\{1, 5\}\}$ and $\mathcal{E}_2 = \{A \subset \{2, 4, 6, 7\} : |A| = 2\}$. $\mathcal{E}_1$ is the edge set of a $K_5 - e$, $\mathcal{E}_2$ is the edge set of a K_4, and they are disjoint. ∎

LEMMA 5.2. CP(4) *contains an edge-disjoint union of a $2 \circ K_4$ and a K_4.*

Proof. Without loss of generality, we may take the vertex set and edge set of the CP(4) as $[8]$ and $\{A \subset [8] : |A| = 2\} \setminus \{\{i, i+4\} : i \in [4]\}$, respectively. Consider the subsets of edges $\mathcal{E}_1 = \{A \subset [4] : |A| = 2\} \cup (\{A \subset [3, 6] : |A| = 2\} \setminus \{\{3, 4\}\})$ and $\mathcal{E}_2 = \{A \subset \{1, 6, 7, 8\} : |A| = 2\}$. $\mathcal{E}_1$ is the edge set of a $2 \circ K_4$, $\mathcal{E}_2$ is the edge set of a K_4, and they are disjoint. ∎

LEMMA 5.3. *Let $G \in \{K_5 - e, 2 \circ K_4\}$ and $n \equiv 1$ or $4 \pmod{12}$. If there exists a 2-$(n, 4, 1)$ packing leaving* CP(4)*, then there exists a 2-$(n, 4, 1)$ packing of size $\frac{1}{6}(\binom{n}{2} - 18)$ with a leave containing G.*

Proof. A 2-$(n, 4, 1)$ packing whose leave is CP(4) has size $\frac{1}{6}(\binom{n}{2} - 24)$. We have seen from Lemmas 5.1 and 5.2 that we can add one more block of size four to this packing to give a 2-$(n, 4, 1)$ packing with a leave containing G. ∎

In view of the above lemma, we now focus on constructing 2-$(n, 4, 1)$ packings leaving CP(4).

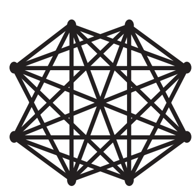

FIG. 5.1. CP(4).

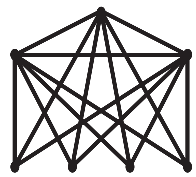

FIG. 5.2. $K_{3,4} + 3e$.

LEMMA 5.4. *Let* $n \geq 6$. *If there exists a* PBD$(n + f, \{4, f^\star\})$*, then there exists a* 2-$(4n + f, 4, 1)$ *packing leaving* CP(4).

Proof. Take a TD$(4, n)-$TD$(4, 2)$ $(X, \mathcal{G}, H, \mathcal{A})$, which exists by Theorem 2.11, and for each $G \in \mathcal{G}$, let $(G \cup F, \mathcal{A}_G)$ be a PBD$(n+f, \{4, f^\star\})$, where F is the block of size f in the PBD. Consider the set system $(Y, \mathcal{B})$, where $Y = X \cup F$, and $\mathcal{B} = \mathcal{A} \cup (\cup_{G \in \mathcal{G}} \mathcal{A}_G)$ (note that the block of size F is included only once). $(Y, \mathcal{B})$ is a 4-graph of order $4n+f$ having the property that every 2-subset of $X \cup F$ is contained in exactly one block of $\mathcal{B}$, except for those 2-subsets $\{a, b\}$, with $a \in G \cap H$ and $b \in G' \cap H$ for distinct $G, G' \in \mathcal{G}$, which are not contained in any blocks of $\mathcal{B}$. $(Y, \mathcal{B})$ therefore gives the required 2-$(4n + f, 4, 1)$ packing leaving CP(4). ◻

LEMMA 5.5. *Let* $n \equiv 1$ *or* 4 (mod 12) *such that* $n \geq 40$ *and* $n \notin \{73, 76, 85\}$. *Then there exists a* 2-$(n, 4, 1)$ *packing leaving* CP(4).

Proof. Taking a PBD$(n + f, \{4, f^\star\})$, with $(n, f) \in \{$(9,4), (12,1), (13,0), (15,1), (16,0), (21,4), (24,1), (25,0), (27,1), (28,0)$\}$, whose existence is provided by Theorem 2.4, and applying Lemma 5.4 give 2-$(n, 4, 1)$ packings leaving CP(4) for $n \in \{$40, 49, 52, 61, 64, 88, 97, 100, 109, 112$\}$. By Theorem 2.4, there exists a PBD$(n, \{4, 40^\star\})$ for all $n \equiv 1$ or 4 (mod 12) and $n \geq 121$. Break up the block of size 40 in this PBD with the blocks of a 2-$(40, 4, 1)$ packing leaving CP(4) to obtain a 2-$(n, 4, 1)$ packing leaving CP(4). ◻

COROLLARY 5.6. *Let* $n \equiv 1$ *or* 4 (mod 12) *such that* $n \geq 40$ *and* $n \notin \{73, 76, 85\}$. *Then* $m(n, 4, G) = \frac{1}{6}(\binom{n}{2} - 18)$ *for* $G \in \{K_5 - e, 2 \circ K_4\}$.

5.2. The case $n \equiv 7$ or 10 (mod 12). The leave $L = (X, \mathcal{E})$ must satisfy:

(i) $|\mathcal{E}| \equiv 3$ (mod 6), and

(ii) every vertex in L has degree $\equiv 0$ (mod 3).

We first consider the case when L contains $K_5 - e$. Any such L satisfying the conditions (i) and (ii) above must have at least 15 edges. So $m(n, 4, K_5 - e) \leq \frac{1}{6}(\binom{n}{2} - 15)$.

When L contains $2 \circ K_4$, L must also have at least 15 edges. Suppose L contains $2 \circ K_4$ and has 15 edges. Then L must have at least two vertices, each of degree at least six. Let a be the number of degree three vertices, and let b be the number of vertices with degree greater than three in L. Then we have $3a + 6b \leq 30$ (counting the edges), $b \geq 2$ (considering the two vertices of degree five in $2 \circ K_4$), and $a + b \geq 7$ (considering the presence of vertices with degree at least six). These inequalities imply that $2 \leq b \leq 3$ and $a + b \leq 8$. So the possible degree sequences for L are $\mathcal{D}_1 = (6, 6, 6, 3, 3, 3, 3)$ and $\mathcal{D}_2 = (6, 6, 3, 3, 3, 3, 3, 3)$. Note that we suppress including vertices of degree zero in the degree sequence of L. There is a unique graph with degree sequence $\mathcal{D}_1$, namely, the graph in Figure 5.2, obtained by adding to $K_{3,4}$ three edges connecting the vertices in the part of the bipartition with three vertices. This graph does not contain $2 \circ K_4$. Hence, L cannot have degree sequence $\mathcal{D}_1$. If L contains $2 \circ K_4$ and has degree sequence $\mathcal{D}_2$, then since $2 \circ K_4$ has degree sequence

$(5,5,3,3,3,3)$, the two vertices of nonzero degree not in $2 \circ K_4$ cannot both be adjacent to the two vertices of degree five in $2 \circ K_4$. But this prevents these two vertices having degree three, a contradiction. Hence L cannot have degree sequence $\mathcal{D}_2$. It follows that the leave of any 2-$(n,4,1)$ packing containing $2 \circ K_4$ must have at least 21 edges, and we have $m(n,4,2 \circ K_4) \le \frac{1}{6}(\binom{n}{2} - 21)$.

The following shows that these bounds can be met.

LEMMA 5.7. *K_7 contains an edge-disjoint union of a $K_5 - e$ and a K_4.*

Proof. Take the vertex set of the K_7 as $[7]$. Consider the subsets of edges $\mathcal{E}_1 = \{A \subset [5] : |A| = 2\} \setminus \{\{4,5\}\}$ and $\mathcal{E}_2 = \{A \subset [4,7] : |A| = 2\}$. Then $\mathcal{E}_1$ is the edge set of a $K_5 - e$, $\mathcal{E}_2$ is the edge set of a K_4, and they are disjoint. □

LEMMA 5.8. *Let $n \equiv 7$ or 10 (mod 12) such that $n \ge 7$ and $n \notin \{10, 19\}$. Then $m(n,4,K_5 - e) = \frac{1}{6}(\binom{n}{2} - 15)$.*

Proof. Let $(X, \mathcal{A})$ be a PBD$(n, \{4, 7^\star\})$ with F as the block of size seven, whose existence is provided by Theorem 2.4, and let B be any 4-subset of F. Then $(X, (\mathcal{A} \cup \{B\}) \setminus \{F\})$ is a 2-$(n,4,1)$ packing of size $\frac{1}{6}(\binom{n}{2} - 15)$ leaving $K_7 - K_4$, which contains $K_5 - e$ by Lemma 5.7. □

LEMMA 5.9. *Let $n \equiv 7$ or 10 (mod 12) such that $n \ge 7$ and $n \notin \{10, 19\}$. Then $m(n,4,2 \circ K_4) = \frac{1}{6}(\binom{n}{2} - 21)$.*

Proof. Observe that any 2-$(n,4,1)$ packing leaving K_7 has size $\frac{1}{6}(\binom{n}{2} - 21)$. The theorem now follows for $n = 7$ trivially and for $n \ge 22$ from the existence of a PBD$(n, \{4, 7^\star\})$ provided by Theorem 2.4. □

5.3. The case $n \equiv 2$, 5, 8, or 11 (mod 12). The leave $L = (X, \mathcal{E})$ must have vertices all of degree 1 (mod 3). Furthermore, $|\mathcal{E}| \equiv 1$ (mod 6) when $n \equiv 2$ or 11 (mod 12), and $|\mathcal{E}| \equiv 4$ (mod 6) when $n \equiv 5$ or 8 (mod 12).

If L contains $K_5 - e$, then L must have at least five vertices, each of degree at least four and the remaining vertices each of degree at least one. Hence, L must have at least $\frac{1}{2}(n+15)$ edges when $n \equiv 5$ or 11 (mod 12) and at least $\frac{1}{2}(n+24)$ edges when $n \equiv 2$ or 8 (mod 12). Consequently,

$$m(n,4,K_5 - e) \le \begin{cases} \frac{1}{6}(\binom{n}{2} - \frac{n+15}{2}) & \text{if } n \equiv 5 \text{ or } 11 \pmod{12}, \\ \frac{1}{6}(\binom{n}{2} - \frac{n+24}{2}) & \text{if } n \equiv 2 \text{ or } 8 \pmod{12}. \end{cases}$$

If L contains $2 \circ K_4$, then L must have at least two vertices, each of degree at least seven, at least four vertices each of degree at least four, and the rest of the vertices each of degree one. Hence, L must have at least $\frac{1}{2}(n+24)$ edges when $n \equiv 2$ or 8 (mod 12) and at least $\frac{1}{2}(n+27)$ edges when $n \equiv 5$ or 11 (mod 12). Consequently,

$$m(n,4,2 \circ K_4) \le \begin{cases} \frac{1}{6}(\binom{n}{2} - \frac{n+24}{2}) & \text{if } n \equiv 2 \text{ or } 8 \pmod{12}, \\ \frac{1}{6}(\binom{n}{2} - \frac{n+27}{2}) & \text{if } n \equiv 5 \text{ or } 11 \pmod{12}. \end{cases}$$

These bounds can be met with the following constructions.

5.3.1. The value of $m(n, 4, K_5 - e)$.

LEMMA 5.10. *Let $n \equiv 5$ or 11 (mod 12) such that $n = 5$ or $n \ge 23$. Then we have $m(n,4,K_5 - e) = \frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n+15))$.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $2^{(n-5)/2}5^1$, which exists by Theorem 2.7. Then $(X, \mathcal{A})$ is a 2-$(n,4,1)$ packing of size $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n+15))$ with a leave containing K_5, and hence $K_5 - e$. □

LEMMA 5.11. *There exists a 2-$(14,4,1)$ packing of size 12 having a leave containing $K_5 - e$.*

Proof. Let $(X, \mathcal{A})$ be a maximum 2-$(13, 4, 1)$ packing, which has size 13 by Theorem 2.2. Let $\infty \notin X$ and $A \in \mathcal{A}$. Then $(X \cup \{\infty\}, \mathcal{A} \setminus \{A\})$ is a 2-$(14, 4, 1)$ packing of size 12 with a leave containing K_5 (whose edges are the 2-subsets of $A \cup \{\infty\}$). ☐

LEMMA 5.12. *Let $n \equiv 2$ or $8 \pmod{12}$ such that $n = 14$ or $n \geq 44$. Then we have $m(n, 4, K_5 - e) = \frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 24))$.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $2^{(n-14)/2}14^1$, which exists by Theorem 2.9. Let $G \in \mathcal{G}$ be the group of cardinality 14, and let $(G, \mathcal{B})$ be a 2-$(14, 4, 1)$ packing of size 12 having a leave containing $K_5 - e$, whose existence is provided by Theorem 5.11. Then $(X, \mathcal{A} \cup \mathcal{B})$ is a 2-$(n, 4, 1)$ packing having a leave containing $K_5 - e$. The size of this packing is $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n - 14) - \binom{14}{2}) + 12 = \frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 24))$. ☐

5.3.2. The value of $m(n, 4, 2 \circ K_4)$.

LEMMA 5.13. *If there exists a $\{4\}$-GDD of type $[g_1, \ldots, g_s]$ with $s \geq 3$ and a $\{4\}$-GDD of type $2^{g_i/2+1}$ for each $i \in [s]$, then there exists a 2-$(n, 4, 1)$ packing of size $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 24))$ with a leave contaning $2 \circ K_4$, where $n = 2 + \sum_{i=1}^{s} g_i$.*

Proof. Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a $\{4\}$-GDD of type $[g_1, \ldots, g_s]$, where $\mathcal{G} = \{G_1, \ldots, G_s\}$ and $|G_i| = g_i$ for $i \in [s]$. Let $Y = \{\infty_1, \infty_2\}$, where $\infty_1, \infty_2 \notin X$, and let $(G_i \cup Y, \mathcal{H}_{G_i}, \mathcal{A}_{G_i})$ be a $\{4\}$-GDD of type $2^{g_i/2+1}$ such that

$$\begin{cases} Y \in \mathcal{H}_{G_i} & \text{if } i \in [s-2], \\ Y \text{ is contained in a block } A_{G_i} \in \mathcal{A}_{G_i} & \text{if } i \in \{s-1, s\}. \end{cases}$$

Construct a 4-graph $(X \cup Y, \mathcal{B})$ of order $2 + \sum_{i=1}^{s} g_i$, where

$$\mathcal{B} = \mathcal{A} \cup \left(\bigcup_{i=1}^{s} \mathcal{A}_{G_i} \right) \setminus \{A_{G_{s-1}}, A_{G_s}\}.$$

It is easy to see that $(X \cup Y, \mathcal{B})$ is a 2-$(2 + \sum_{i=1}^{s} g_i, 4, 1)$ packing. Also, the 2-subsets of $A_{G_{s-1}}$ and A_{G_s} are not contained in any blocks of $\mathcal{B}$. So the leave of $(X \cup Y, \mathcal{B})$ contains $2 \circ K_4$ as a subgraph. It remains to compute the size of $(X \cup Y, \mathcal{B})$. The 2-subsets of $X \cup Y$ that are not contained in any blocks of $\mathcal{B}$ are precisely the elements of $\mathcal{H}_{G_i}$ for $i \in [s]$ and the 2-subsets of $A_{G_{s-1}}$ and A_{G_s}. Since Y appears precisely s times among these 2-subsets, the total number of distinct 2-subsets of $X \cup Y$ that are not contained in any blocks of $\mathcal{B}$ is $\sum_{i=1}^{s}(g_i/2 + 1) + 12 - (s - 1) = n/2 + 12$, where $n = 2 + \sum_{i=1}^{s} g_i$. Hence $|\mathcal{B}| = \frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 24))$, as required. ☐

LEMMA 5.14. *If there exists a $\{4\}$-GDD of type $[g_1, \ldots, g_s]$ with $s \geq 3$, a $\{4\}$-GDD of type $2^{g_i/2+1}$ for each $i \in [s-1]$, and a $\{4\}$-GDD of type $2^{(g_s-3)/2}5^1$, then there exists a 2-$(n, 4, 1)$ packing of size $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 27))$ with a leave containing $2 \circ K_4$, where $n = 2 + \sum_{i=1}^{s} g_i$.*

Proof. Suppose that $(X, \mathcal{G}, \mathcal{A})$ is a $\{4\}$-GDD of type $[g_1, \ldots, g_s]$, where $\mathcal{G} = \{G_1, \ldots, G_s\}$ and $|G_i| = g_i$ for $i \in [s]$. Let $Y = \{\infty_1, \infty_2\}$, where $\infty_1, \infty_2 \notin X$, and let $(G_i \cup Y, \mathcal{H}_{G_i}, \mathcal{A}_{G_i})$ be a $\{4\}$-GDD of type $2^{g_i/2+1}$ such that

$$\begin{cases} Y \in \mathcal{H}_{G_i} & \text{if } i \in [s-3], \\ Y \text{ is contained in a block } A_{G_i} \in \mathcal{A}_{G_i} & \text{if } i \in \{s-2, s-1\}. \end{cases}$$

Further, let $(G_s \cup Y, \mathcal{H}_{G_s}, \mathcal{A}_{G_s})$ be a $\{4\}$-GDD of type $2^{(g_s-3)/2}5^1$ such that Y is contained in the group $H \in \mathcal{H}_{G_s}$ of cardinality five. Now form the 4-graph $(X \cup Y, \mathcal{B})$ of order $2 + \sum_{i=1}^{s} g_i$, where

$$\mathcal{B} = \mathcal{A} \cup \left(\bigcup_{i=1}^{s} \mathcal{A}_{G_i} \right) \cup \{H \setminus \{\infty_1\}\} \setminus \{A_{G_{s-2}}, A_{G_{s-1}}\}.$$

It is easy to see that $(X \cup Y, \mathcal{B})$ is a 2-$(2 + \sum_{i=1}^{s} g_i, 4, 1)$ packing. Also, the 2-subsets of $A_{G_{s-1}}$ and A_{G_s} are not contained in any blocks of $\mathcal{B}$. So the leave of $(X \cup Y, \mathcal{B})$ contains $2 \circ K_4$ as a subgraph. It remains to compute the size of $(X \cup Y, \mathcal{B})$. The 2-subsets of $X \cup Y$ that are not contained in any blocks of $\mathcal{B}$ are precisely the 2-subsets of $A_{G_{s-2}}$ and $A_{G_{s-1}}$ and the 2-subsets of elements of $\mathcal{H}_{G_i}$ for $i \in [s]$, except for the 2-subsets of $H \setminus \{\infty_1\}$. Since Y appears precisely s times among these 2-subsets, the total number of distinct 2-subsets of $X \cup Y$ that are not contained in any blocks of $\mathcal{B}$ is $\sum_{i=1}^{s-1}(g_i/2 + 1) + (g_s - 3)/2 + (10 - 6) - 1 + 12 - (s - 1) = \frac{1}{2}(n + 27)$, where $n = 2 + \sum_{i=1}^{s} g_i$. Hence $|\mathcal{B}| = \frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 27))$, as required. □

COROLLARY 5.15. *For all* $n \equiv 2 \pmod{12}$, $n \geq 50$, *there exists a* 2-$(n, 4, 1)$ *packing of size* $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 24))$ *with a leave containing* $2 \circ K_4$.

Proof. Apply Lemma 5.13 with $\{4\}$-GDDs of type $12^{(n-2)/12}$ and type 2^7, which exist by Theorem 2.6. □

COROLLARY 5.16. *For* $n = 29$ *and for all* $n \equiv 5 \pmod{12}$, $n \geq 101$, *there exists a* 2-$(n, 4, 1)$ *packing of size* $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 27))$ *with a leave containing* $2 \circ K_4$.

Proof. Apply Lemma 5.14 with $\{4\}$-GDDs of type $12^{(n-29)/12}27^1$, which exists by Theorem 2.10, $\{4\}$-GDDs of type 2^7, which exists by Theorem 2.6, and $\{4\}$-GDDs of type $2^{12}5^1$, which exists by Theorem 2.7. □

COROLLARY 5.17. *For* $n = 20$ *and for all* $n \equiv 8 \pmod{12}$, $n \geq 68$, *there exists a* 2-$(n, 4, 1)$ *packing of size* $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 24))$ *with a leave containing* $2 \circ K_4$.

Proof. Apply Lemma 5.13 with $\{4\}$-GDDs of type $12^{(n-20)/12}18^1$, which exists by Theorem 2.10, and $\{4\}$-GDDs of types 2^7 and 2^{10}, which exists by Theorem 2.6. □

COROLLARY 5.18. *For* $n = 23$ *and for all* $n \equiv 11 \pmod{12}$, $n \geq 83$, *there exists a* 2-$(n, 4, 1)$ *packing of size* $\frac{1}{6}(\binom{n}{2} - \frac{1}{2}(n + 27))$ *with a leave containing* $2 \circ K_4$.

Proof. Apply Lemma 5.14 with $\{4\}$-GDDs of type $12^{(n-23)/12}21^1$, which exists by Theorem 2.10, $\{4\}$-GDDs of type 2^7, which exists by Theorem 2.6, and $\{4\}$-GDDs of type $2^9 5^1$, which exists by Theorem 2.7. □

5.4. The case $n \equiv 0$, 3, 6, or 9 (mod 12). The leave $L = (X, \mathcal{E})$ must have vertices all of degree 2 (mod 3). Furthermore, $|\mathcal{E}| \equiv 0 \pmod 6$ when $n \equiv 0$ or 9 (mod 12), and $|\mathcal{E}| \equiv 3 \pmod 6$ when $n \equiv 3$ or 6 (mod 12).

If L contains $K_5 - e$ or $2 \circ K_4$, then L must have at least six vertices each of degree at least five and the remaining vertices each of degree at least two. Hence, L must have at least $n + 9$ edges when $n \equiv 6$ or 9 (mod 12) and at least $n + 12$ edges when $n \equiv 0$ or 3 (mod 12). Consequently, for $G \in \{K_5 - e, 2 \circ K_4\}$, we have

$$m(n, 4, G) \leq \begin{cases} \frac{1}{6}(\binom{n}{2} - (n + 9)) & \text{if } n \equiv 6 \text{ or } 9 \pmod{12}, \\ \frac{1}{6}(\binom{n}{2} - (n + 12)) & \text{if } n \equiv 0 \text{ or } 3 \pmod{12}. \end{cases}$$

These bounds can again be met with the following constructions.

LEMMA 5.19. *For* $n = 6$ *and for all* $n \equiv 6$ *or* 9 (mod 12), $n \geq 21$ *there exists a* 2-$(n, 4, 1)$ *packing of size* $\frac{1}{6}(\binom{n}{2} - (n + 9))$ *with a leave containing* G, *where* $G \in \{K_5 - e, 2 \circ K_4\}$.

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $3^{(n-6)/3}6^1$, which exists by Theorem 2.8. Then $(X, \mathcal{A})$ is a 2-$(n, 4, 1)$ packing with a leave containing K_6, and hence $K_5 - e$ and $2 \circ K_4$. The size of $(X, \mathcal{A})$ is easily verified: $|\mathcal{A}| = \frac{1}{6}(\binom{n}{2} - \frac{n-6}{3}\binom{3}{2} - \binom{6}{2}) = \frac{1}{6}(\binom{n}{2} - (n + 9))$. □

LEMMA 5.20. *There exists a* 2-$(15, 4, 1)$ *packing of size* 13 *with a leave containing* G, *where* $G \in \{K_5 - e, 2 \circ K_4\}$.

Proof. The 13 blocks of a 2-$(15, 4, 1)$ packing with a leave containing $K_5 - e$ are

{2,6,13,14}, {3,6,9,10}, {4,7,9,13}, {4,5,6,12}, {1,6,11,15},
{3,7,11,14}, {2,7,8,15}, {1,8,9,14}, {3,12,13,15}, {2,9,11,12},
{1,7,10,12}, {5,10,14,15}, {5,8,11,13}.

The 13 blocks of a 2-$(15, 4, 1)$ packing with a leave containing $2 \circ K_4$ are

{1,8,12,13}, {6,8,11,14}, {4,6,9,15}, {3,7,8,9}, {2,8,10,15},
{2,9,13,14}, {4,5,7,14}, {1,6,7,10}, {1,5,11,15}, {2,7,11,12},
{4,10,11,13}, {3,12,14,15}, {5,9,10,12}. ☐

LEMMA 5.21. *For all $n \equiv 0$ or 3 (mod 12), $n \geq 48$, there exists a 2-$(n, 4, 1)$ packing of size $\frac{1}{6}(\binom{n}{2} - (n+12))$ with a leave containing G, where $G \in \{K_5 - e, 2 \circ K_4\}$.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{4\}$-GDD of type $3^{(n-15)/3}15^1$, which exists by Theorem 2.8. Let Y be the group of cardinality 15 in $\mathcal{G}$ and $(Y, \mathcal{B})$ be a 2-$(15, 4, 1)$ packing of size 13 with a leave containing G, which exists by Lemma 5.20. Then $(X, \mathcal{A} \cup \mathcal{B})$ is a 2-$(n, 4, 1)$ packing with a leave containing G. The size of $(X, \mathcal{A} \cup \mathcal{B})$ is easily verified: $|\mathcal{A} \cup \mathcal{B}| = \frac{1}{6}(\binom{n}{2} - \frac{n-12}{3}\binom{3}{2} - 2\binom{6}{2}) + 13 = \frac{1}{6}(\binom{n}{2} - (n + 12))$. ☐

5.5. Remaining small orders. The values of n for which $m(n, 4, K_5 - e)$ and $m(n, 4, 2 \circ K_4)$ remain undetermined are as follows:

	Unsettled n												
$m(n, 4, K_5 - e)$	8	9	10	11	12	13	16	17	18	19	20	24	25
	26	27	28	32	36	37	38	39	73	76	85		
$m(n, 4, 2 \circ K_4)$	8	9	10	11	12	13	14	16	17	18	19	24	25
	26	27	28	32	35	36	37	38	39	41	44	47	53
	56	59	65	71	73	76	77	85	89				

For $n = 19$, we have the following tighter upper bound.

LEMMA 5.22. *For $G \in \{K_5 - e, 2 \circ K_4\}$, we have $m(19, 4, G) \leq 24$.*

Proof. Suppose we have a 2-$(19, 4, 1)$ packing of size 25 with a leave containing G, and then we can add a K_4 in G to this packing, giving a 2-$(19, 4, 1)$ packing of size 26. This is a contradiction, since $D(19, 4, 2) = 25$. ☐

For values of $n < 16$, it is possible to determine $m(n, 4, G)$, $G \in \{K_5 - e, 2 \circ K_4\}$, via exhaustive search. Let H be a specific subgraph of K_n isomorphic to G. We form a graph Γ_n whose vertex set is the set of all K_4's of $K_n - H$, and two vertices in Γ_n are adjacent if and only if the corresponding K_4's are edge-disjoint. Then $m(n, 4, G)$ is equal to the size of a maximum clique in Γ_n. We used Cliquer, an implementation of Östergård's exact algorithm for maximum cliques [12], to determine the size of maximum cliques in Γ_n, for $n \leq 15$.

When $n \geq 16$, it is infeasible to use Cliquer, so we resort to a stochastic local search heuristic to construct packings of the required size directly. The results of our computation are summarized in Table 5.1, while the blocks of the actual packings are listed in Appendices A and B.

5.6. Piecing things together. The results in previous subsections can be summarized as follows.

TABLE 5.1
Values of $m(n,4,K_5-e)$ and $m(n,4,2\circ K_4)$ for some small values of n. A blank entry indicates an unknown value.

	n												
n	8	9	10	11	12	13	16	17	18	19	20	24	25
$m(n,4,K_5-e)$	1	2	3	4	6	9			21	24	28	40	
n	26	27	28	32	36	37	38	39	73	76	85		
$m(n,4,K_5-e)$	50	52			97								
n	8	9	10	11	12	13	14	16	17	18	19	24	25
$m(n,4,2\circ K_4)$	1	2	3	4	6	9	11			21	24	40	
n	26	27	28	32	35	36	37	38	39	41	44	47	53
$m(n,4,2\circ K_4)$		52											
n	56	59	65	71	73	76	77	85	89				
$m(n,4,2\circ K_4)$													

THEOREM 5.23. *For all $n\geq 5$, we have $m(n,4,K_5-e)=\frac{1}{6}(\binom{n}{2}-f(n))$, where*

$$f(n)=\begin{cases} 18 & \text{if } n\equiv 1 \text{ or } 4 \pmod{12},\ n\neq 13,\\ 15 & \text{if } n\equiv 7 \text{ or } 10 \pmod{12},\ n\notin\{10,19\},\\ (n+24)/2 & \text{if } n\equiv 2 \text{ or } 8 \pmod{12},\ n\neq 8,\\ (n+15)/2 & \text{if } n\equiv 5 \text{ or } 11 \pmod{12},\ n\neq 11,\\ n+9 & \text{if } n\equiv 6 \text{ or } 9 \pmod{12},\ n\neq 9,\\ n+12 & \text{if } n\equiv 0 \text{ or } 3 \pmod{12},\ n\neq 12,\\ 22 & \text{if } n=8,\\ 24 & \text{if } n=9,\\ 27 & \text{if } n=10,\\ 31 & \text{if } n=11,\\ 30 & \text{if } n=12,\\ 24 & \text{if } n=13,\\ 27 & \text{if } n=19,\end{cases}$$

except possibly for $n\in\{16,17,25,28,32,37,38,39,73,76,85\}$.

THEOREM 5.24. *For all $n\geq 6$, we have $m(n,4,2\circ K_4)=\frac{1}{6}(\binom{n}{2}-f(n))$, where*

$$f(n)=\begin{cases} 18 & \text{if } n\equiv 1 \text{ or } 4 \pmod{12},\ n\neq 13,\\ 21 & \text{if } n\equiv 7 \text{ or } 10 \pmod{12},\ n\notin\{10,19\},\\ (n+24)/2 & \text{if } n\equiv 2 \text{ or } 8 \pmod{12},\ n\notin\{8,14\},\\ (n+27)/2 & \text{if } n\equiv 5 \text{ or } 11 \pmod{12},\ n\neq 11,\\ n+9 & \text{if } n\equiv 6 \text{ or } 9 \pmod{12},\ n\neq 9,\\ n+12 & \text{if } n\equiv 0 \text{ or } 3 \pmod{12},\ n\neq 12,\\ 22 & \text{if } n=8,\\ 24 & \text{if } n=9,\\ 27 & \text{if } n=10,\\ 31 & \text{if } n=11,\\ 30 & \text{if } n=12,\\ 24 & \text{if } n=13,\\ 25 & \text{if } n=14,\\ 27 & \text{if } n=19,\end{cases}$$

except possibly for $n \in \{16, 17, 25, 26, 28, 32, 35, 36, 37, 38, 39, 41, 44, 47, 53, 56, 59, 65, 71, 73, 76, 77, 85, 89\}$.

6. Conclusion. Theorems 4.5, 5.23, and 5.24 can be expressed more succinctly in terms of $D(n, 3, 2)$ and $D(n, 4, 2)$ as follows.

THEOREM 6.1. *For all* $n \geq 4$,

$$m(n, 3, K_4 - e) + 2 = \begin{cases} D(n, 3, 2) & \text{if } n \equiv 0,\ 2, \text{ or } 5 \pmod 6, \\ D(n, 3, 2) - 1 & \text{if } n \equiv 1,\ 3, \text{ or } 4 \pmod 6. \end{cases}$$

THEOREM 6.2. *For all* $n \geq 5$,

$$m(n, 4, K_5 - e) + 2 = \begin{cases} D(n, 4, 2) + 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 9,\ 10, \text{ or } 11 \pmod{12}, \\ & n \notin \{9, 10, 11\}, \\ D(n, 4, 2) & \text{if } n \equiv 0,\ 2,\ 3, \text{ or } 8 \pmod{12},\ n \notin \{8, 12\}, \\ D(n, 4, 2) - 1 & \text{if } n \equiv 1 \text{ or } 4 \pmod{12},\ n \neq 13, \\ n - 5 & \text{if } n \in \{8, 9, 10, 11\}, \\ 8 & \text{if } n = 12, \\ 11 & \text{if } n = 13, \end{cases}$$

except possibly for $n \in \{16, 17, 25, 28, 32, 37, 38, 39, 73, 76, 85\}$.

THEOREM 6.3. *For all* $n \geq 6$,

$$m(n, 4, 2 \circ K_4) + 2 = \begin{cases} D(n, 4, 2) + 1 & \text{if } n \equiv 6 \text{ or } 9 \pmod{12},\ n \neq 9, \\ D(n, 4, 2) & \text{if } n \equiv 0,\ 2,\ 3,\ 5,\ 7,\ 8,\ 10, \text{ or } 11 \pmod{12}, \\ & n \notin \{8, 10, 11, 12, 14\}, \\ D(n, 4, 2) - 1 & \text{if } n \equiv 1 \text{ or } 4 \pmod{12},\ n \neq 13, \\ n - 5 & \text{if } n \in \{8, 9, 10, 11\}, \\ 8 & \text{if } n = 12, \\ 11 & \text{if } n = 13, \\ 13 & \text{if } n = 14, \end{cases}$$

except possibly for $n \in \{16, 17, 25, 26, 28, 32, 35, 36, 37, 38, 39, 41, 44, 47, 53, 56, 59, 65, 71, 73, 76, 77, 85, 89\}$.

These have the following consequences.

COROLLARY 6.4. *For all* $n \geq 4$, $T(n, \mathcal{F}(3), 2) = D(n, 3, 2)$.

COROLLARY 6.5. *For all* $n \geq 6$,

$$T(n, \mathcal{F}(4), 2) = \begin{cases} D(n, 4, 2) + 1 & \text{if } n \equiv 5,\ 6,\ 7,\ 9,\ 10, \text{ or } 11 \pmod{12}, \\ & n \notin \{9, 10, 11\}, \\ D(n, 4, 2) & \text{if } n \equiv 0,\ 1,\ 2,\ 3,\ 4, \text{ or } 8 \pmod{12}, \\ & n \notin \{8, 12, 13\}, \\ n - 5 & \text{if } n \in \{8, 9, 10, 11\}, \\ 8 & \text{if } n = 12, \\ 11 & \text{if } n = 13, \end{cases}$$

except possibly for $n \in \{16, 17, 25, 28, 32, 37, 38, 39, 73, 76, 85\}$.

Appendix A. Some maximum 2-$(n,4,1)$ packings with a leave containing $K_5 - e$.

In each case, the edges of the $K_5 - e$ in the leave are $\binom{[5]}{2} \setminus \{\{4,5\}\}$.

A.1. The blocks of a maximum 2-$(10,4,1)$ packing with a leave containing $K_5 - e$. $\{4,5,6,7\}$, $\{3,7,8,9\}$, $\{1,6,8,10\}$.

A.2. The blocks of a maximum 2-$(18,4,1)$ packing with a leave containing $K_5 - e$.

{4,8,12,16}, {3,6,7,8}, {3,11,13,16}, {2,9,15,16}, {10,11,12,14},
{2,7,11,17}, {4,9,13,14}, {1,6,9,17}, {5,13,17,18}, {3,14,15,17},
{2,8,14,18}, {4,7,10,15}, {2,6,10,13}, {1,8,11,15}, {4,6,11,18},
{5,8,9,10}, {1,10,16,18}, {5,7,14,16}, {3,9,12,18}, {1,7,12,13},
{5,6,12,15}.

A.3. The blocks of a maximum 2-$(19,4,1)$ packing with a leave containing $K_5 - e$.

{8,14,17,18}, {2,9,13,14}, {3,7,12,14}, {1,10,14,19}, {4,5,10,18},
{4,6,14,16}, {6,11,18,19}, {4,11,13,17}, {3,8,15,19}, {5,12,13,19},
{1,9,12,18}, {3,13,16,18}, {2,7,15,18}, {3,9,10,17}, {4,7,9,19},
{2,16,17,19}, {5,6,7,17}, {2,6,8,12}, {10,12,15,16}, {7,8,10,13},
{5,8,9,16}, {5,11,14,15}, {1,7,11,16}, {1,6,13,15}.

A.4. The blocks of a maximum 2-$(20,4,1)$ packing with a leave containing $K_5 - e$.

{4,6,16,18}, {3,12,16,20}, {1,10,11,15}, {9,12,14,19}, {2,7,10,12},
{6,7,15,19}, {9,10,17,18}, {4,9,11,13}, {4,12,15,17}, {4,5,10,19},
{1,8,12,18}, {3,13,18,19}, {5,8,14,17}, {1,16,17,19}, {1,7,13,14},
{2,6,13,17}, {11,14,18,20}, {2,8,11,19}, {5,6,11,12}, {5,13,15,20},
{3,8,9,15}, {8,10,13,16}, {3,7,11,17}, {2,14,15,16}, {3,6,10,14},
{5,7,9,16}, {4,7,8,20}, {1,6,9,20}.

A.5. The blocks of a maximum 2-$(24,4,1)$ packing with a leave containing $K_5 - e$.

{12,14,15,18}, {3,6,16,18}, {6,9,10,13}, {5,9,15,22}, {3,9,11,21},
{4,8,15,19}, {1,18,21,22}, {12,16,17,19}, {11,12,22,23}, {4,9,23,24},
{4,5,6,12}, {3,10,12,24}, {5,8,21,24}, {6,14,17,21}, {1,8,12,13},
{6,19,22,24}, {4,16,20,21}, {2,18,19,23}, {1,7,17,23}, {3,17,20,22},
{1,11,16,24}, {2,13,14,16}, {2,7,10,21}, {5,7,14,20}, {8,10,17,18},
{13,18,20,24}, {2,9,12,20}, {7,8,16,22}, {3,7,13,19}, {2,15,17,24},
{5,11,13,17}, {13,15,21,23}, {10,11,19,20}, {1,9,14,19}, {4,7,11,18},
{1,6,15,20}, {3,8,14,23}, {2,6,8,11}, {5,10,16,23}, {4,10,14,22}.

A.6. The blocks of a maximum 2-$(26, 4, 1)$ packing with a leave containing $K_5 - e$.

{4,17,22,24}, {3,11,17,20}, {5,7,18,22}, {4,16,18,23}, {1,7,19,25},
{14,21,22,23}, {1,10,18,26}, {2,11,21,26}, {3,6,7,23}, {11,14,16,19},
{12,20,24,26}, {4,7,14,26}, {3,9,16,22}, {6,10,15,16}, {3,10,12,19},
{7,8,15,17}, {4,9,13,19}, {5,12,13,21}, {15,19,22,26}, {5,19,23,24},
{4,12,15,25}, {3,15,18,21}, {8,9,21,25}, {6,12,17,18}, {5,8,16,26},
{2,7,9,12}, {9,17,23,26}, {1,8,20,22}, {5,9,11,15}, {7,10,21,24},
{1,13,14,15}, {6,19,20,21}, {7,13,16,20}, {10,11,22,25}, {2,6,13,22},
{2,16,24,25}, {9,14,18,20}, {2,8,18,19}, {1,6,9,24}, {4,6,8,11},
{5,6,14,25}, {8,10,13,23}, {11,13,18,24}, {2,10,14,17}, {3,13,25,26},
{3,8,14,24}, {2,15,20,23}, {1,11,12,23}, {4,5,10,20}, {1,16,17,21}.

A.7. The blocks of a maximum 2-$(27, 4, 1)$ packing with a leave containing $K_5 - e$.

{2,7,16,21}, {7,20,26,27}, {5,17,25,27}, {5,15,21,23}, {5,6,11,22},
{13,21,22,27}, {3,8,11,26}, {6,15,17,24}, {4,5,7,19}, {1,6,18,27},
{3,18,21,24}, {2,11,12,13}, {9,13,16,23}, {10,11,14,15}, {3,14,23,27},
{4,8,15,18}, {14,19,22,24}, {1,10,19,23}, {3,12,16,20}, {2,8,23,24},
{5,8,9,20}, {4,12,14,21}, {4,9,11,27}, {3,6,10,25}, {8,14,16,17},
{2,15,19,27}, {9,12,15,22}, {3,7,13,15}, {1,8,12,25}, {3,9,17,19},
{19,20,21,25}, {2,6,14,20}, {6,8,13,19}, {7,11,24,25}, {1,11,17,21},
{4,10,17,20}, {9,10,21,26}, {10,16,24,27}, {4,16,22,25}, {7,12,17,18},
{1,7,9,14}, {2,17,22,26}, {11,16,18,19}, {5,12,24,26}, {1,15,16,26},
{5,10,13,18}, {1,13,20,24}, {18,20,22,23}, {2,9,18,25}, {4,6,23,26},
{13,14,25,26}, {7,8,10,22}.

A.8. The blocks of a maximum 2-$(36, 4, 1)$ packing with a leave containing $K_5 - e$.

{7,10,17,35}, {11,15,26,36}, {6,16,25,29}, {1,12,24,28}, {3,13,34,35},
{30,31,35,36}, {21,23,28,34}, {1,14,19,35}, {8,9,28,32}, {15,18,21,25},
{3,18,26,27}, {1,8,25,30}, {3,10,23,29}, {6,28,30,33}, {15,19,23,24},
{4,14,17,34}, {7,13,26,28}, {10,19,22,36}, {6,11,12,34}, {1,7,11,29},
{5,13,17,25}, {14,24,26,31}, {13,19,27,29}, {1,20,23,26}, {2,22,31,34},
{14,23,25,36}, {5,16,24,33}, {4,18,29,33}, {4,21,26,32}, {8,22,26,29},
{9,11,22,25}, {12,18,20,32}, {2,11,20,21}, {11,13,31,32}, {10,14,30,32},
{3,9,33,36}, {3,11,24,30}, {24,29,32,36}, {7,18,24,34}, {7,19,21,31},
{3,7,12,25}, {2,8,13,24}, {2,7,14,16}, {5,7,8,20}, {10,11,16,28},
{5,6,18,31}, {8,11,14,18}, {3,17,19,32}, {10,20,25,31}, {4,5,11,19},
{16,18,19,30}, {16,20,34,36}, {3,6,15,20}, {4,8,10,12}, {6,9,13,14},
{9,17,20,24}, {13,20,22,33}, {4,6,7,36}, {1,13,18,36}, {5,26,30,34},
{1,6,22,32}, {16,21,27,35}, {12,13,21,30}, {2,9,18,35}, {12,17,29,31},
{8,17,21,36}, {7,9,23,30}, {20,28,29,35}, {2,15,29,30}, {4,20,27,30},
{1,15,16,17}, {1,9,27,31}, {4,15,28,31}, {12,14,15,33}, {9,12,19,26},
{25,26,33,35}, {6,10,24,27}, {3,14,21,22}, {2,23,32,33}, {5,14,27,28},
{5,12,22,23}, {25,27,32,34}, {3,8,16,31}, {4,13,16,23}, {8,19,33,34},
{2,6,17,26}, {5,15,32,35}, {2,19,25,28}, {9,10,15,34}, {6,8,23,35},
{1,10,21,33}, {7,15,22,27}, {4,22,24,35}, {11,17,27,33}, {2,12,27,36},
{5,9,21,29}, {17,18,22,28}.

Appendix B. Some maximum 2-$(n, 4, 1)$ packings with a leave containing $2 \circ K_4$.

B.1. The blocks of a maximum 2-$(18, 4, 1)$ packing with a leave containing $2 \circ K_4$.

{3,9,10,12}, {1,7,9,16}, {4,7,17,18}, {1,5,13,17}, {5,9,11,14},
{1,6,14,18}, {4,6,8,9}, {2,7,10,14}, {3,14,16,17}, {5,8,10,18},
{1,8,12,15}, {6,10,15,17}, {3,7,8,13}, {3,11,15,18}, {6,7,11,12},
{2,12,16,18}, {10,11,13,16}, {2,8,11,17}, {2,9,13,15}, {4,5,15,16},
{4,12,13,14}.

B.2. The blocks of a maximum 2-$(19, 4, 1)$ packing with a leave containing $2 \circ K_4$.

{3,8,9,16}, {5,12,16,18}, {11,13,15,16}, {1,12,13,19}, {6,8,10,11},
{1,10,14,16}, {6,12,15,17}, {6,7,9,13}, {4,13,14,18}, {1,9,15,18},
{5,9,14,19}, {4,6,16,19}, {4,7,8,12}, {5,8,13,17}, {2,8,14,15},
{3,10,17,18}, {4,5,10,15}, {2,9,10,12}, {1,5,7,11}, {2,7,16,17},
{4,9,11,17}, {3,7,15,19}, {3,11,12,14}, {2,11,18,19}.

B.3. The blocks of a maximum 2-$(24, 4, 1)$ packing with a leave containing $2 \circ K_4$.

{4,13,14,21}, {3,12,16,20}, {3,15,21,22}, {3,17,18,23}, {7,13,15,23},
{4,12,19,22}, {1,9,15,19}, {4,7,8,18}, {6,10,15,18}, {9,11,17,21},
{3,10,11,13}, {5,9,14,22}, {1,11,14,18}, {1,6,8,21}, {6,7,14,20},
{2,13,19,24}, {10,19,20,21}, {5,11,12,15}, {8,11,16,24}, {2,8,10,17},
{2,16,21,23}, {5,8,13,20}, {4,6,9,16}, {1,7,10,16}, {2,7,11,22},
{2,9,18,20}, {14,15,16,17}, {8,9,12,23}, {10,12,14,24}, {3,7,9,24},
{13,16,18,22}, {6,17,22,24}, {1,12,13,17}, {5,7,17,19}, {3,8,14,19},
{4,15,20,24}, {1,20,22,23}, {4,5,10,23}, {6,11,19,23}, {5,18,21,24}.

B.4. The blocks of a maximum 2-$(27, 4, 1)$ packing with a leave containing $2 \circ K_4$.

{6,12,17,21}, {1,5,10,19}, {4,8,10,27}, {4,14,21,22}, {19,21,24,25},
{2,11,23,26}, {3,10,20,24}, {3,9,15,21}, {12,20,26,27}, {8,11,12,25},
{10,15,18,26}, {3,8,19,26}, {4,11,13,20}, {9,22,24,26}, {3,7,22,27},
{1,15,22,23}, {5,14,24,27}, {3,12,14,23}, {9,12,13,19}, {2,9,17,27},
{3,13,18,25}, {4,7,12,15}, {6,14,16,20}, {5,7,9,11}, {6,15,25,27},
{10,17,22,25}, {5,20,23,25}, {2,10,12,16}, {4,6,18,19}, {5,12,18,22},
{3,11,16,17}, {6,8,13,22}, {1,13,16,27}, {2,13,15,24}, {5,8,15,17},
{5,16,21,26}, {13,14,17,26}, {7,10,13,21}, {2,19,20,22}, {1,6,11,24},
{7,17,18,20}, {1,8,20,21}, {1,7,25,26}, {11,14,15,19}, {4,9,16,25},
{7,16,19,23}, {8,16,18,24}, {11,18,21,27}, {4,17,23,24}, {1,9,14,18},
{2,7,8,14}, {6,9,10,23}.

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