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LARGE SIEVE INEQUALITY WITH CHARACTERS FOR POWERFUL MODULI

STEPHAN BAIER AND LIANGYI ZHAO

Abstract. In this paper we aim to generalize the results in [1],[2],[19] and develop a general formula for large sieve with characters to powerful moduli that will be an improvement to the result in [19].

Mathematics Subject Classification 2000: 11N35, 11L07, 11B57

keywords: large sieve inequality; power moduli.

1. Introduction

Throughout this paper, we reserve the symbols $c_i$ ($i = 1, 2, \ldots$) for absolute positive constants. Large sieve was an idea originated by J. V. Linnik [10] in 1941 while studying the distribution of quadratic non-residues. Refinements of this idea were made by many. In this paper, we develop a large sieve inequality for powerful moduli. More in particular, we aim to have an estimate for the following sum

\[ \sum_{q \leq Q} \sum_{\substack{a \equiv 1 \pmod{q} \atop \gcd(a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q^n} \right) \right|^2, \]

where $k \geq 2$ is a natural number. In the sequel, let

\[ Z := \sum_{n=M+1}^{M+N} |a_n|^2. \]

With $k = 1$ in (1.1), it is

\[ \ll (Q^2 + N) Z. \] \hspace{1cm} (1.2)

This is in fact the consequence of a more general result first introduced by H. Davenport and H. Halberstam [7] in which the Farey fractions in the outer sums of (1.1) can be replaced by any set of well-spaced points. Applying the said more general result, (1.1) is bounded above by

\[ \ll (Q^{k+1} + QN) Z, \text{ and } \ll (Q^{2k} + N) Z \] \hspace{1cm} (1.3)

(see [19]). Literature abound on the subject of the classical large sieve. See [3], [6], [7], [8], [10], [12], [13] and [14]. In [19] it was proved that the sum (1.1) can be estimated by

\[ \ll \left( Q^{k+1} + \left( NQ^{1-1/\kappa} + N^{1-1/\kappa} Q^{1+k/\kappa} \right) N^\varepsilon \right) Z, \] \hspace{1cm} (1.4)

where $\kappa := 2^{k-1}$ and the implied constant depends on $\varepsilon$ and $k$. Furthermore, when appropriate, some of the constants $c_i$’s and the implied constants in $\ll$ in the remainder of this paper will depend on $\varepsilon$ or both $\varepsilon$ and $k$. In [1] and [2] this bound was improved for $k = 2$. Extending the elementary method in [1] to higher power moduli, we here establish the following bound for (1.1).
Theorem 1: We have
\[
\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}} q^n \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{an}{q} \right) \right|^2 \ll (\log \log 10NQ)^{k+1} (Q^{k+1} + N + N^{1/2+\varepsilon} Q^k) Z.
\]

For $k \geq 3$ Theorem 1 improves the classical bounds (1.3) as well as Zhao's bound (1.4) in the range $N^{k/(2k)+\varepsilon} \ll Q \ll N^{(k-2)/(2(k-1)k-2k)}$. In particular, for $k = 3$ we obtain an improvement in the range $N^{1/6+\varepsilon} \ll Q \ll N^{1/5-\varepsilon}$. We note that for a large $k$ the exponent $(k-2)/(2(k-1)k-2k)$ is close to $1/(2(k-1))$.

Extending the Fourier analytic methods in [2], [19], we establish another bound for cubic moduli which improves the bounds (1.3), (1.4) in the range $N^{7/20+\varepsilon} \ll Q \ll N^{1/3-\varepsilon}$.

Theorem 2: Suppose that $1 \leq Q \leq N^{1/2}$. Then we have
\[
\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}} q^n \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{an}{q} \right) \right|^2 \ll \begin{cases} N^\varepsilon (Q^4 + N^9/10Q^{6/5}) Z, & \text{if } N^{7/24} \leq Q \leq N^{1/2}, \\ NQ^{9/7+\varepsilon} Z, & \text{if } 1 \leq Q < N^{7/24}. \end{cases}
\]

Unfortunately, our Fourier analytic method does not yield any improvement if $k \geq 4$.

2. PROOF OF THEOREM 1

Let $S$ be the set of $k$-th powers of natural numbers. Let $Q_0 \geq \sqrt{N}$. Set
\[
S(Q_0) = S \cap (Q_0, 2Q_0].
\]
We first note, by classical large sieve, setting $Q = \sqrt{N}$ in (1.2),
\[
\sum_{q \leq \sqrt{N}} \sum_{a=1}^{q} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{an}{q} \right) \right|^2 \leq 2NZ.
\]

Let
\[
S_t(Q_0) = \{ q \in \mathbb{N} : tq \in S(Q_0) \}.
\]
Let $t = p_1^{v_1} \cdots p_n^{v_n}$ be the prime decomposition of $t$. Furthermore, let
\[
u_t := \left\lfloor \frac{v_t}{k} \right\rfloor,
\]
where for $x \in \mathbb{R}$, $[x] = \min \{ k \in \mathbb{Z} : k \geq x \}$ is the ceiling of $x$. Moreover, set
\[
f_t = p_1^{u_1} \cdots p_n^{u_n}.
\]
Therefore, for all $q_k^t = q \in S$, $q$ is divisible by $t$ if and only if $q_0$ is divisible by $f_t$. Therefore, we have
\[
S_t(Q_0) = \{ q_k^t g_t : Q_0^{1/k} / f_t < q_k^t \leq (2Q_0)^{1/k} / f_t \},
\]
where
\[
g_t := \frac{f_t}{t}.
\]
Moreover we note that
\[
S_t(Q_0) \subset (Q_0/t, 2Q_0/t]
\]
and that
\[
|S_t(Q_0)| \leq \frac{(2Q_0)^{1/k}}{f_t}.
\]
We set for $m \in \mathbb{N}$, $l \in \mathbb{Z}$ with $(m, l) = 1$
\[
A_l(u, m, l) = \max_{Q_0^{1/k} \leq y \leq 2Q_0/t} \{|q \in S_t(Q_0) \cap (y, y + u] : q \equiv l \mod m|\}.
\]
Let $\delta_t(m,l)$ be the number of solutions $x$ to the congruence
\[ x^k g_t \equiv l \mod m. \]
We now use Theorem 2 in [1] with $Q_0 \geq \sqrt{N}$:

**Theorem 3:** Assume that for all $t \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{Z}$, $u \in \mathbb{R}$ with $t \leq \sqrt{N}$, $m \leq \sqrt{N}/t$, $(m,l) = 1$, $mQ_0/\sqrt{N} \leq u \leq Q_0/t$ the conditions
\[
A_t(u,m,l) \leq C \left( 1 + \frac{|S_t(Q_0)|/m}{Q_0/t} \cdot u \right) \delta_t(m,l),
\]
\[
\sum_{l \leq X} \delta_t(m,l) \leq m,
\]
\[
\delta_t(m,l) \leq X
\]
hold for some suitable positive numbers $C$ and $X$. Then,
\[
\sum_{q \in S(Q_0)} \sum_{a,q=1}^{q} \left| \sum_{r=M+1}^{M+N} a_n e \left( \frac{a_n r}{q} \right) \right|^2 \leq c_0 C \left( \min \{Q_0X,N\} + Q_0 \right) \left( \sqrt{N} \log \log 10N + \max_{r \leq \sqrt{N}} \sum_{l \leq r} |S_t(Q_0)| \right) \mathbb{Z}.
\]

First, we have to check the validity of the conditions (2.4), (2.5) and (2.6). Conditions (2.4) and (2.5) are obviously satisfied with $C$ absolute. We further suppose that $(g_t,m) = 1$ for otherwise $\delta_t(m,l) = 0$ since $(m,l) = 1$. Therefore, we must estimate the number of solutions to
\[
x^k \equiv \overline{g_t} \mod m,
\]
where $\overline{g_t}$ is the multiplicative inverse of $g_t$ modulo $m$. By the virtue of the Chinese remainder theorem, it suffices to estimate the number of solutions to (2.8) with $m$ as a prime power, say $m = p^e$, for $p \in \mathbb{P}$ and $e \in \mathbb{N}$. Note that the function
\[ \sigma_k : (\mathbb{Z}/p^e \mathbb{Z})^* \rightarrow (\mathbb{Z}/p^e \mathbb{Z})^* : x \rightarrow x^k \]
is an endomorphism. Hence it is enough to estimate the size of its kernel $\ker(\sigma_k)$. If $k = \pi_1^{a_1} \cdots \pi_h^{a_h}$ is the prime decomposition of $k$, then
\[ \sigma_k = \prod_{i=1}^{h} \sigma_{\pi_i}^{a_i}. \]
Thus,
\[
|\ker \sigma_k| \leq \prod_{i=1}^{h} |\ker \sigma_{\pi_i}|^{a_i}.
\]
Hence, it suffices to estimate the size of $|\ker \sigma_e|$ for prime numbers $\pi$.

For $p \in \mathbb{P},$
\[ x^e - 1 \equiv 0 \mod p \]
has at most $\pi$ solutions. By elementary result (see [15], for example), a solution, a mod $p^e$ with $e \geq 1$, of the congruence
\[
x^e - 1 \equiv 0 \mod p^e
\]
lifts to more than one solution to
\[ x^e - 1 \equiv 0 \mod p^{e+1} \]
only when $p|a^{e-1}$ and $p^{e+1}|a - 1$. If $p \neq \pi$, $p|a^{e-1}$ implies $p|a$, but it is not possible that $p^{e+1}|a - 1$ as $(a - 1,a) = 1$. Thus, in this case (2.10) has at most $\pi$ solutions for all $e \geq 1$. In the following, we consider the case $p = \pi$. 
By Fermat’s little theorem, there exists only one solution of the congruence
\[ x^\pi - 1 \equiv 0 \mod \pi, \]
namely \( 1 \mod \pi \). This solution lifts to exactly \( \pi \) solutions to
\[ x^\pi - 1 \equiv 0 \mod \pi^2, \]
namely
\[ 1, 1 + \pi, 1 + 2\pi, \ldots, 1 + (\pi - 1)\pi \mod \pi^2. \]
More generally, if \( a \mod \pi^e \) is a solution to
(2.11)
\[ x^\pi - 1 \equiv 0 \mod \pi^e, \]
then, if \( a \) lifts to solutions to
\[ x^\pi - 1 \equiv 0 \mod \pi^{e+1}, \]
they are of the form
(2.12)
\[ a, a + \pi^e, a + 2\pi^e, \ldots, a + (\pi - 1)\pi^e \mod \pi^{e+1}. \]
Assume there are \( j_1, j_2 \in \{0, \ldots, \pi - 1\}, j_1 \neq j_2 \) such that both \( a + j_1\pi^e \) and \( a + j_2\pi^e \) lift to solutions modulo \( \pi^{e+2} \). Then \( \pi^{e+2}|(a + j_1\pi^e)^\pi - 1 \) and \( \pi^{e+2}|(a + j_2\pi^e)^\pi - 1 \), hence
\[
(a + j_1\pi^e)^\pi - (a + j_2\pi^e)^\pi = (j_1 - j_2)\pi^e \sum_{i=0}^{\pi-1} (a + j_1\pi^e)^{\pi - 1 - i}(a + j_2\pi^e)^i
\]
is divisible by \( \pi^{e+2} \). If \( e \geq 2 \), this implies \( a \equiv 0 \mod \pi \), but then \( a \) cannot be a solution to (2.11). Therefore, if \( e \geq 2 \), only one of the solutions (2.12) lifts to a solution modulo \( \pi^{e+2} \). From this we infer that the number of solutions to (2.11) never exceeds \( \pi^2 \), i.e.
\[ |\ker \sigma_\pi| \leq \pi^2. \]
Combining this with (2.9), we get
\[ |\ker \sigma_k| \leq k^2. \]
Therefore, by the Chinese remainder theorem, we obtain
\[ \delta_t(m, l) \leq k^{2\omega(m)}, \]
where \( \omega(m) \) is the number of distinct prime divisors of \( m \). Since \( 2^{\omega(m)} \) is the number of square-free divisors of \( m \), we have
\[ k^{2\omega(m)} \leq \tau(m)^2 \log_2 k \ll m^\varepsilon, \]
where \( \tau(m) \) is the number of divisors of \( m \). Thus, if \( m \leq \sqrt{N} \), (2.6) holds with
\[ X \ll N^{\varepsilon}. \]

Now, by Theorem 3,
(2.13)
\[ \sum_{q \in S(Q_0)} \sum_{\substack{a=1 \atop (a, q) = 1}}^{q} \left| \sum_{n=M+1}^{M+N} a_ne \left( \frac{a}{q} \right) \right|^2 \]
is majorized by
\[ \ll \left( \min\{Q_0 N^{\varepsilon}, N\} + Q_0 \right) \left( \sqrt{N \log \log(10N)} + \max_{r \leq \sqrt{N}} \sum_{l \mid r} Q_0^{1/k} f_l^{-1} \right) Z. \]
The function
\[ G(r) = \sum_{t \mid r} \frac{1}{f_t} \]
is clearly multiplicative. If \( r \) is a prime power \( p^s \), then
\[ G(r) \leq 1 + k \left( \frac{1}{p} + \frac{1}{p^2} + \ldots \right) = 1 + \frac{k}{p - 1} \leq \left( 1 + \frac{1}{p^s} \right)^k = \left( \frac{p^s}{\varphi(p^s)} \right)^k. \]
Hence, for all $r \in \mathbb{N}$ we have
\[(2.14) \quad G(r) \leq \left( \frac{r}{\varphi(r)} \right)^k \ll (\log \log 10r)^k.\]
Hence (2.13) is
\[\ll (\log \log 10NQ_0)^{k+1} (\sqrt{N} + Q_0^{1/k}) (\min \{ Q_0N^\varepsilon, N \} + Q_0).\]
The above is always majorized by
\[\ll (\log \log 10NQ_0)^{k+1} \left( Q_0^{1+1/k} + N^{1/2+\varepsilon}Q_0 \right).\]
Summing over all relevant dyadic intervals and combining with (2.1), we see that (1.1) is majorized by
\[\ll (\log \log 10NQ)^{k+1} (QN + N^{1/2+\varepsilon}Q^k)Z.\]
Therefore, our result follows. □

3. PROOF OF THEOREM 2

3.1. Reduction to Farey fractions in short intervals. As in [1], [2], our starting point is the following general large sieve inequality.

**Lemma 1**: Let $(\alpha_r)_{r \in \mathbb{N}}$ be a sequence of real numbers. Suppose that $0 < \Delta \leq 1/2$ and $R \in \mathbb{N}$. Put
\[(3.1) \quad K(\Delta) := \max_{\alpha \in \mathbb{R}} \sum_{r=1}^{R} 1, \quad \text{where} \quad ||x|| \quad \text{denotes the distance of a real} \ x \ \text{to its closest integer. Then}
\quad \sum_{r=1}^{R} |S(\alpha_r)|^2 \leq c_1 K(\Delta)(N + \Delta^{-1})Z.
\]

In the sequel, we suppose that $S$ is the set of cubes of natural numbers and that $\alpha_1, ..., \alpha_R$ is the sequence of Farey fractions $a/q$ with $q \in S(Q_0), 1 \leq a \leq q$ and $(a,q) = 1$, where $Q_0 \geq 1$. We further suppose that $\alpha \in \mathbb{R}$ and $0 < \Delta \leq 1/2$. Put
\[I(\alpha) := [\alpha - \Delta, \alpha + \Delta] \quad \text{and} \quad P(\alpha) := \sum_{\substack{q \in S \cap (Q_0, 2Q_0) \\{ (a,q) = 1 \} \\{ \alpha/q \in I(\alpha) \} \alpha/q \in I(\alpha)}} 1.\]

Then we have
\[K(\Delta) = \max_{\alpha \in \mathbb{R}} P(\alpha).\]
Therefore, the proof of Theorem 2 reduces to estimating $P(\alpha)$.

As in [1] and [2], we begin with an idea of D. Wolke [18]. Let $\tau$ be a positive number with
\[(3.2) \quad 1 \leq \tau \leq \frac{1}{\sqrt{\Delta}}.\]
In [1] and [2] we put $\tau := 1/\sqrt{\Delta}$, but in fact our method works for all $\tau$ satisfying (3.2). We will later fix $\tau$ in an optimal manner. In the said earlier papers, $\tau = 1/\sqrt{\Delta}$ was the optimal choice.

By Dirichlet's approximation theorem, $\alpha$ can be written in the form
\[\alpha = \frac{b}{r} + z,\]
where
\[(3.3) \quad r \leq \tau, \quad (b,r) = 1, \quad |z| \leq \frac{1}{r \tau}.\]
Thus, it suffices to estimate $P(b/r + z)$ for all $b, r, z$ satisfying (3.3).
We further note that we can restrict ourselves to the case when
\[ z \geq \Delta. \]  
(3.4)

If \( |z| < \Delta \), then
\[ P(\alpha) \leq P\left(\frac{b}{r} - \Delta\right) + P\left(\frac{b}{r} + \Delta\right). \]

Furthermore, we have
\[ \Delta \leq \frac{1}{r^2} \leq \frac{1}{r}. \]

Therefore this case can be reduced to the case \( |z| = \Delta \). Moreover, as \( P(\alpha) = P(-\alpha) \), we can choose \( z \) positive.

Summarizing the above observations, we deduce

**Lemma 2:** We have
\[ K(\Delta) \leq 2 \max_{r \in \mathbb{N}} \max_{b \in \mathbb{Z}} \max_{\Delta \leq z \leq 1/(\tau r)} \max_{(b,r) = 1} P\left(\frac{b}{r} + z\right). \]
(3.5)

3.2. **Estimation of** \( P(b/r + z) - \text{first way.} \) We now prove a first estimate for \( P\left(\frac{b}{r} + z\right) \) by using some results in [1]. In the sequel, we suppose that the conditions (3.2), (3.3) and (3.4) are satisfied.

By inequality (41) in [1], we have
\[ P\left(\frac{b}{r} + z\right) \leq 1 + 6 \sum_{t \mid r} \sum_{0 < m \leq 4rzQ_0/t} A_t\left(\frac{\Delta Q_0}{t^2}, \frac{r}{t}, -\theta m\right), \]
(3.6)

where \( A_t(u,m,l) \) is defined as in (2.3) and \( \theta \) is the multiplicative inverse of \( b \) modulo \( r \). By the results of section 2, for \( S \) the set of cubes, the conditions (2.4), (2.5) and (2.6) with \( X = \Delta^{-\varepsilon} \) are satisfied for all \( t \in \mathbb{N}, m \in \mathbb{N}, l \in \mathbb{Z}, u \in \mathbb{R} \) with \( t \leq \tau, m \leq \tau/t, (m,l) = 1, mQ_0/r \leq u \leq Q_0/t \). Conditions (2.4) and (2.6) imply
\[ \sum_{0 < m \leq 4rzQ_0/t} A_t\left(\frac{\Delta Q_0}{t^2}, \frac{r}{t}, -\theta m\right) \leq C\left(1 + \frac{\Delta t|S_t(Q_0)|}{rzQ_0X} \right) \frac{4rzQ_0X}{t} \]
(3.7)

From (3.6), (3.7) and
\[ \sum_{t \mid r} \frac{1}{t} \leq \prod_{p \mid r} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = \prod_{p \mid r} \frac{p}{p - 1} = \frac{r}{\varphi(r)} \leq c_2 \log \log 10r, \]
we derive
\[ P\left(\frac{b}{r} + z\right) \leq 1 + c_3Q_0X\left(rz \log \log 10r + \Delta \sum_{t \mid r} |S_t(Q_0)|\right). \]
(3.8)

Furthermore, by (2.2) and (2.14), we have
\[ \sum_{t \mid r} |S_t(Q_0)| \ll (\log \log 10r)^3Q_0^{1/3}. \]

Thus, from (3.8) and the fact that \( r \leq \tau = \Delta^{-1/2} \), we obtain

**Proposition 1:** Let \( S \) be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (3.4) are satisfied. Then we have
\[ P\left(\frac{b}{r} + z\right) \leq 1 + c_4\Delta^{-\varepsilon}\left(Q_0^{1/3}\Delta + Q_0r^z\right). \]
(3.9)
3.3. Estimation of $P(b/r + z)$ - second way. We now prove a second estimate for $P(b/r + z)$ by extending the Fourier analytic methods in [2], [19] to cubic moduli. The following bound for $P(b/r + z)$ can be proved in the same manner as Lemma 2 in [2].

Lemma 3: Let $S$ be the set of cubes of natural numbers. Suppose that

$$\frac{Q_0 \Delta}{z} \leq \delta \leq Q_0.$$  

Then,

$$P\left(\frac{b}{r} + z\right) \leq c_5 \left(1 + \frac{1}{\delta} \int_{Q_0}^{Q_0} \Pi(\delta, y) \, dy\right),$$  

where $I(\delta, y) = [y^{1/3} - c_0 \delta/Q_0^{2/3}, y^{1/3} + c_0 \delta/Q_0^{2/3}]$, $J(\delta, y) = [(y - 4\delta)rz, (y + 4\delta)rz]$ and

$$\Pi(\delta, y) = \sum_{q \in J(\delta, y)} \sum_{m \in S, \equiv -bq^3 \mod r} 1.$$  

We shall prove the following

Proposition 2: Let $S$ be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (3.4) are satisfied. Then we have

$$P\left(\frac{b}{r} + z\right) \leq c_7 \Delta^{-\epsilon} \left(Q_0^{4/3} \Delta + Q_0^{1/3} \Delta^{-1/3} z^{-1} + \Delta^{-1/2} (rz)^{1/2}\right).$$  

To derive Proposition 2 from Lemma 3, we need the following standard results from Fourier analysis.

Lemma 4: (Poisson summation formula, [5]) Let $f(x)$ be a complex-valued function on the real numbers that is piecewise continuous with only finitely many discontinuities and for all real numbers $a$ satisfies

$$f(a) = \frac{1}{2} \left(\lim_{x \to a^-} f(x) + \lim_{x \to a^+} f(x)\right).$$  

Moreover, suppose that $f(x) \leq c_6 (1 + |x|)^{-c}$ for some $c > 1$. Then,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n), \text{ where } \hat{f}(x) := \int_{-\infty}^{\infty} f(y)e(xy) \, dy,$$

the Fourier transform of $f(x)$.

Lemma 5: (see [19], for example) For $x \in \mathbb{R} \setminus \{0\}$ define

$$\phi(x) := \left(\frac{\sin \pi x}{2x}\right)^2, \text{ and } \phi(0) := \lim_{x \to 0} \phi(x) = \frac{\pi^2}{4}.$$  

Then $\phi(x) \geq 1$ for $|x| \leq 1/2$, and the Fourier transform of the function $\phi(x)$ is

$$\hat{\phi}(s) = \frac{\pi^2}{4} \max\{1 - |s|, 0\}.$$

Lemma 6: (see Lemma 3.1. in [9]) Let $F : [a, b] \to \mathbb{R}$ be twice differentiable. Assume that $|F'(x)| \geq u > 0$ for all $x \in [a, b]$. Then,

$$\left| \int_{a}^{b} e^{i F(x)} \, dx \right| \leq c_9 \frac{a}{u}.$$
Lemma 7: (see Lemma 4.3.1. in [4]) Let $F : [a, b] \to \mathbb{R}$ be twice continuously differentiable. Assume that $|F''(x)| \geq u > 0$ for all $x \in [a, b]$. Then,

$$\left| \int_{a}^{b} e^{itF(x)}dx \right| \leq \frac{c_{10}}{\sqrt{u}}$$

We shall also need the following estimates for cubic exponential sums.

Lemma 8: (see [11], [17]) Let $c \in \mathbb{N}$, $k, l \in \mathbb{Z}$ with $(k, c) = 1$. Then,

$$\sum_{d=1}^{c} e\left(\frac{kd^3 + ld}{c}\right) \leq c_{11}c^{1/2+\varepsilon}(l, c).$$

Furthermore,

$$\sum_{d=1}^{c} e\left(\frac{kd^3}{c}\right) \leq c_{11}c^{2/3}.$$

Proof of Proposition 2: We put

(3.14) \hspace{1cm} \delta := \frac{Q_{0}A}{z}.

By Lemma 5, (3.12) can be estimated by

(3.15) \hspace{1cm} \Pi(\delta, y) \leq \sum_{q \in \mathbb{Z}} \phi\left(\frac{q - y^{1/3}}{2c_{0}\delta/Q_{0}^{2/3}}\right) \sum_{m \equiv -bq^{3} \mod r} \phi\left(\frac{m - y\delta z}{8\delta r z}\right).

Using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.15) into

$$\sum_{m \equiv -bq^{3} \mod r} \phi\left(\frac{m - y\delta z}{8\delta r z}\right) = 8\delta z \sum_{j \in \mathbb{Z}} e\left(\frac{jbq^{3}}{r} + jyz\right) \hat{\phi}(8j\delta z).$$

Therefore, we get for the double sum on the right-hand side of (3.15)

(3.16) \hspace{1cm} \sum_{q \in \mathbb{Z}} \phi\left(\frac{q - y^{1/3}}{2c_{0}\delta/Q_{0}^{2/3}}\right) \sum_{m \equiv -bq^{3} \mod r} \phi\left(\frac{m - y\delta z}{8\delta r z}\right) = 8\delta z \sum_{j \in \mathbb{Z}} e(jyz)\hat{\phi}(8j\delta z) \sum_{d=1}^{\tilde{r}} e\left(\frac{jbq^{3}}{r}\right) \sum_{k \equiv d \mod \tilde{r}} \phi\left(\frac{k - y^{1/3}}{2c_{0}\delta/Q_{0}^{2/3}}\right),

where $\tilde{r} := r/(r,j)$ and $\tilde{j} := j/(r,j)$. Again using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.16) into

(3.17) \hspace{1cm} \sum_{k \equiv d \mod \tilde{r}} \phi\left(\frac{k - y^{1/3}}{2c_{0}\delta/Q_{0}^{2/3}}\right) = \frac{2c_{0}\delta}{\tilde{r}Q_{0}^{2/3}} \sum_{l \in \mathbb{Z}} e\left(l \cdot \frac{d - y^{1/3}}{\tilde{r}}\right) \hat{\phi}\left(\frac{2c_{0}l\delta}{\tilde{r}Q_{0}^{2/3}}\right).
From (3.16) and (3.17), we obtain
\[
\frac{1}{\delta} \int_{Q_0}^{2Q_0} \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - y^{1/3}}{2c_0\delta/Q_0^{2/3}} \right) \sum_{\substack{m \in \mathbb{Z} \atop m \equiv -bq^r \mod r}} \phi \left( \frac{m - yrz}{8drz} \right) \, dy
\]
(3.18)
\[
\leq \frac{16c_0\delta z}{Q_0^{2/3}} \sum_{j \in \mathbb{Z}} \phi \left( \frac{8j\delta z}{r} \right) \sum_{l \in \mathbb{Z}} \phi \left( \frac{2c_0\delta}{rQ_0^{2/3}} \right) \left| \sum_{d=1}^{r} e \left( \frac{jbd^2 + ld}{r} \right) \int_{Q_0}^{2Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) \, dy \right|^2.
\]
Applying the Lemmas 5 and 8 to the right-hand side of (3.18), and taking \( r \leq 1/\sqrt{\Delta} \) by (3.2) and (3.3) into account, we deduce
(3.19)
\[
\frac{1}{\delta} \int_{Q_0}^{2Q_0} \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - y^{1/3}}{c_0\delta/Q_0^{2/3}} \right) \sum_{\substack{m \in \mathbb{Z} \atop m \equiv -bq^r \mod r}} \phi \left( \frac{m - yrz}{8drz} \right) \, dy
\]
\[
\leq \frac{c_{12}\delta z\Delta^{-\varepsilon}}{Q_0^{2/3}} \left( \sum_{|l| \leq 1/(8\delta z)} \frac{1}{\sqrt{\lambda}} \sum_{||l|| \leq (rQ_0^{2/3})/(2\varepsilon d)} (l, \bar{r}) \right) \left| \int_{Q_0}^{2Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) \, dy \right| + \sum_{|l| \leq 1/(8\delta z)} \frac{1}{\sqrt{\lambda}} \left| \int_{Q_0}^{2Q_0} e \left( jyz \right) \, dy \right|.
\]
If \( j \neq 0 \), then
\[
\left| \int_{Q_0}^{2Q_0} e \left( jyz \right) \, dy \right| \leq \frac{1}{|j|z}.
\]
If \( j = 0 \) and \( l \neq 0 \), then
\[
\left| \int_{Q_0}^{2Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) \, dy \right| \leq \frac{c_{13}Q_0^{2/3}}{|l|}
\]
by Lemma 6 (take into account that \( \bar{r} = 1 \) if \( j = 0 \)). If \( j \neq 0 \) and \( l \neq 0 \), then Lemma 7 yields
\[
\left| \int_{Q_0}^{2Q_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) \, dy \right| \leq \frac{c_{14}\sqrt{r}Q_0^{5/6}}{\sqrt{|l|}}.
\]
Therefore, the right-hand side of (3.19) is majorized by
(3.20)
\[
\leq c_{15}\delta \Delta^{-\varepsilon} \left( zQ_0^{1/3} + \frac{1}{Q_0^{2/3}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r}} + z \sum_{1 \leq j \leq Q_0^{2/3}/(2\varepsilon d)} \frac{1}{t} + zQ_0^{1/6} \sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq t \leq rQ_0^{2/3}/(2\varepsilon d)} \frac{(l, \bar{r})}{\sqrt{t}} \right).
\]
Now, we estimate the sums in the last line of (3.20). Using (3.2), (3.3) and (3.14), we obtain
(3.21)
\[
\sum_{1 \leq t \leq Q_0^{2/3}/(2\varepsilon d)} \frac{1}{t} \leq c_{16}\Delta^{-\varepsilon}.
\]
Using the definition of \( \bar{r} \), (3.2), (3.3) and (3.14), we obtain
(3.22)
\[
\sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r}} = \frac{1}{\sqrt{r}} \sum_{t \mid r} \sqrt{t} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{t} \leq \frac{c_{17}\Delta^{-\varepsilon}}{\sqrt{r}} \sum_{t \mid r} \frac{t^{-2/3}}{l} \leq c_{18}\Delta^{-2\varepsilon r^{-1/3}}.
\]
For $A \geq 1$, we have
\[
\sum_{1 \leq j \leq A} \frac{(l, \tilde r)}{\sqrt{t}} \leq \sum_{1 \leq j \leq A/t} \frac{1}{\sqrt{t}} \ll \sqrt{A} \sum_{l \mid r} 1 \ll \hat r \sqrt{A}.
\]
Therefore,
\[
(3.23) \quad \sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{(l, \tilde r)}{\sqrt{t}} \leq c_1 \Delta^{-\varepsilon} Q_0^{1/3} \sqrt{\delta} \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{r}.
\]
Using the definition of $\hat r$, we obtain
\[
(3.24) \quad \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{\hat r} = \sqrt{r} \sum_{1 \leq j \leq 1/(8\delta z)} \sum_1 \frac{1}{\sqrt{t}} \leq \frac{1}{8\delta z} \sum_{l \mid r} \frac{1}{\sqrt{\delta z}} \leq c_2 \hat r \sqrt{r}.
\]
Combining Lemma 3 and (3.19-3.24), we obtain
\[
(3.25) \quad P\left(\frac{b}{r} + z\right) \leq c_3 \Delta^{-\varepsilon} \left(1 + \delta_3 Q_0^{1/3} + \delta Q_0^{-2/3} \tau^{-1/3} + \delta^{-1/2} Q_0^{1/2} \sqrt{r}\right).
\]
From (3.14) and (3.25), we infer the desired estimate. Note that the first term in the right-hand side of (3.25) can be absorbed into the last term on the right-hand side of (3.13) by (3.4). □

3.4. Final proof of Theorem 2. Combining Propositions 1,2 and (3.3), we obtain
\[
(3.26) \quad P\left(\frac{b}{r} + z\right) \leq c_4 \Delta^{-\varepsilon} \left(Q_0^{4/3} + \Delta + \min \left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} + \Delta^{-1/2} \tau^{-1/2}\right).
\]
If
\[
z \leq \Delta^{1/2} Q_0^{-1/3} \tau^{2/3},
\]
then
\[
\min \left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} = Q_0 r z \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}.
\]
If
\[
z > \Delta^{1/2} Q_0^{-1/3} \tau^{2/3},
\]
then
\[
\min \left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} = Q_0^{1/3} \Delta r^{-1/3} z^{-1} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}.
\]
From the above inequalities and (3.3), we deduce
\[
(3.27) \quad \min \left\{Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1}\right\} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}.
\]
Combining (3.26) and (3.27), we get
\[
(3.28) \quad P\left(\frac{b}{r} + z\right) \leq c_5 \Delta^{-\varepsilon} \left(Q_0^{4/3} \Delta^{1/2} \tau^{1/3} + Q_0^{2/3} \Delta^{1/2} r^{1/3} + \Delta^{-1/2} \tau^{-1/2}\right).
\]
Now we choose
\[
\tau := \begin{cases} 
N^{4/5} Q_0^{-4/5}, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\
Q_0^{1/3}, & \text{if } 1 \leq Q_0 < N^{7/8},
\end{cases}
\quad \text{and} \quad \Delta := \begin{cases} 
N^{-1}, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\
Q_0^{-8/7}, & \text{if } 1 \leq Q_0 < N^{7/8}.
\end{cases}
\]
Then the condition (3.2) is satisfied in each case, and from (3.28) and Lemmas 1,2, we obtain
\[
(3.29) \quad \sum_{Q_0^{1/3} \leq l \leq (2Q_0)^{1/3}} \sum_{(a, q) = 1} \left| S \left(\frac{a}{q^3}\right) \right|^2 \ll \begin{cases} 
N^\varepsilon \left(Q_0^{1/3} + N^{9/10} Q_0^{2/5}\right) Z, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\
N Q_0^{2/7 + \varepsilon} Z, & \text{if } 1 \leq Q_0 < N^{7/8}.
\end{cases}
\]
We can divide the interval $[1, Q]$ into $O(\log Q)$ subintervals of the form $[Q_0^{1/3}, (2Q_0)^{1/3}]$, where $1 \leq Q_0 \leq Q^3$. Hence, the result of Theorem 2 follows from (3.29). □
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Acknowledgments

This paper was written when the first-named author held a postdoctoral position at the Harish-Chandra Research Institute at Allahabad (India) and the second-named author was supported by a postdoctoral fellowship at the University of Toronto. The authors wish to thank these institutions for their financial support.

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