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LARGE SIEVE INEQUALITY WITH CHARACTERS FOR POWERFUL MODULI

STEPHAN BAIER AND LIANGYI ZHAO

Abstract. In this paper we aim to generalize the results in [1],[2],[19] and develop a general formula for large sieve with characters to powerful moduli that will be an improvement to the result in [19].

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1. Introduction

Throughout this paper, we reserve the symbols $c_i$ ($i = 1, 2, ...$) for absolute positive constants. Large sieve was an idea originated by J. V. Linnik [10] in 1941 while studying the distribution of quadratic non-residues. Refinements of this idea were made by many. In this paper, we develop a large sieve inequality for powerful moduli. More in particular, we aim to have an estimate for the following sum

$$
\sum_{q \leq Q} \sum_{\substack{a^k \equiv 1 \pmod{q} \atop (a,q)=1}} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q} n \right) \right|^2,
$$

where $k \geq 2$ is a natural number. In the sequel, let

$$
Z := \sum_{n=M+1}^{M+N} |a_n|^2.
$$

With $k = 1$ in (1.1), it is

$$
\ll (Q^2 + N) Z.
$$

This is in fact the consequence of a more general result first introduced by H. Davenport and H. Halberstam [7] in which the Farey fractions in the outer sums of (1.1) can be replaced by any set of well-spaced points. Applying the said more general result, (1.1) is bounded above by

$$
\ll (Q^{k+1} + QN) Z, \text{ and } \ll (Q^{2k} + N) Z
$$

(see [19]). Literature abound on the subject of the classical large sieve. See [3],[6],[7],[8],[10],[12],[13] and [14]. In [19] it was proved that the sum (1.1) can be estimated by

$$
\ll \left( Q^{k+1} + \left( NQ^{1-1/\kappa} + N^{1-1/\kappa} Q^{1+k/\kappa} \right) N^\varepsilon \right) Z,
$$

where $\kappa := 2^{k-1}$ and the implied constant depends on $\varepsilon$ and $k$. Furthermore, when appropriate, some of the constants $c_i$’s and the implied constants in $\ll$ in the remainder of this paper will depend on $\varepsilon$ or both $\varepsilon$ and $k$. In [1] and [2] this bound was improved for $k = 2$. Extending the elementary method in [1] to higher power moduli, we here establish the following bound for (1.1).
**Theorem 1:** We have

\[
\sum_{q \leq Q} \sum_{\substack{a=1 \atop (a,q)=1}}^{q^\theta} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q^n} \right) \right|^2 \ll (\log \log 10NQ)^{k+1} (Q^{k+1} + N + N^{1/2+\varepsilon} Q^k) Z.
\]

For \( k \geq 3 \) Theorem 1 improves the classical bounds (1.3) as well as Zhao's bound (1.4) in the range \( N^{1/(2k)+\varepsilon} \ll Q \ll N^{(\kappa-2)/(2k-1)\kappa-2k}-\varepsilon \). In particular, for \( k = 3 \) we obtain an improvement in the range \( N^{1/6+\varepsilon} \ll Q \ll N^{1/5-\varepsilon} \). We note that for a large \( k \) the exponent \( (\kappa - 2)/(2(k-1)\kappa - 2k) \) is close to \( 1/(2(k-1)) \).

Extending the Fourier analytic methods in [2], [19], we establish another bound for cubic moduli which improves the bounds (1.3), (1.4) in the range \( N^{7/20+\varepsilon} \ll Q \ll N^{1/3-\varepsilon} \).

**Theorem 2:** Suppose that \( 1 \leq Q \leq N^{1/2} \). Then we have

\[
\sum_{q \leq Q} \sum_{\substack{a=1 \atop (a,q)=1}}^{q^\theta} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q^n} \right) \right|^2 \ll \begin{cases} 
N^\varepsilon(Q^4 + N^{9/10}Q^{6/5}) Z, & \text{if } N^{7/24} \leq Q \leq N^{1/2} , \\
NQ^{\varepsilon/7} Z , & \text{if } 1 \leq Q < N^{7/24} .
\end{cases}
\]

Unfortunately, our Fourier analytic method does not yield any improvement if \( k \geq 4 \).

2. PROOF OF THEOREM 1

Let \( S \) be the set of \( k \)-th powers of natural numbers. Let \( Q_0 \geq \sqrt{N} \). Set \( S(Q_0) = S \cap (Q_0, 2Q_0] \).

We first note, by classical large sieve, setting \( Q = \sqrt{N} \) in (1.2),

\[
\sum_{q \leq \sqrt{N}} \sum_{\substack{a=1 \atop (a,q)=1}}^{q^\theta} \left| \sum_{n=M+1}^{M+N} a_n e \left( \frac{a}{q^n} \right) \right|^2 \leq 2NZ.
\]

Let

\[
S_t(Q_0) = \{ q \in \mathbb{N} : tq \in S(Q_0) \}.
\]

Let \( t = p_1^{n_1} \cdots p_n^{n_n} \) be the prime decomposition of \( t \). Furthermore, let

\[
u_k := \left[ \frac{v_k}{k} \right],
\]

where for \( x \in \mathbb{R}, \lfloor x \rfloor = \min \{ k \in \mathbb{Z} : k \geq x \} \) is the ceiling of \( x \). Moreover, set

\[
f_t = p_1^{n_1} \cdots p_n^{n_n}.
\]

Therefore, for all \( q_0^k = q \in S_t \), \( q \) is divisible by \( t \) if and only if \( q_0 \) is divisible by \( f_t \). Therefore, we have

\[
S_t(Q_0) = \{ q_1^{k} g_t : Q_0^{1/k} / f_t < q_1 \leq (2Q_0)^{1/k} / f_t \},
\]

where

\[
g_t := \frac{f_t}{t}.
\]

Moreover we note that

\[
S_t(Q_0) \subset (Q_0/t, 2Q_0/t]
\]

and that

\[
|S_t(Q_0)| \leq \frac{(2Q_0)^{1/k}}{f_t}.
\]

We set for \( m \in \mathbb{N}, l \in \mathbb{Z} \) with \( (m, l) = 1 \)

\[
A_t(u, m, l) = \max_{Q_0^{1/k} \leq y \leq 2Q_0/t} |\{ q \in S_t(Q_0) \cap (y, y+u] : q \equiv l \mod m \}|.
\]
Let $\delta_l(m,l)$ be the number of solutions $x$ to the congruence
\[ x^k g_l \equiv l \mod m. \]

We now use Theorem 2 in [1] with $Q_0 \geq \sqrt{N}$:

**Theorem 3:** Assume that for all $t \in \mathbb{N}$, $m \in \mathbb{N}$, $l \in \mathbb{Z}$, $u \in \mathbb{R}$ with $t \leq \sqrt{N}$, $m \leq \sqrt{N}/t$, $(m,l) = 1$, $mQ_0/\sqrt{N} \leq u \leq Q_0/t$ the conditions
\[
A_t(u,m,l) \leq C \left( 1 + \frac{|S_t(Q_0)|/m}{Q_0/t} \cdot u \right) \delta_l(m,l),
\]
\[
\sum_{l=1}^{m} \delta_l(m,l) \leq m,
\]
\[
\delta_l(m,l) \leq X
\]
hold for some suitable positive numbers $C$ and $X$. Then,
\[
\sum_{q \in \mathcal{S}(Q_0)} \left( \sum_{a=1}^{q} \sum_{n=M+1}^{M+N} a_n e\left( \frac{a_n}{q} \right) \right)^2 \leq c_0 C(\min\{Q_0, X, N\} + Q_0) \left( \sqrt{N} \log \log 10N + \max_{r \leq \sqrt{N}} \sum_{t \mid r} |S_t(Q_0)| \right) Z.
\]

First, we have to check the validity of the conditions (2.4), (2.5) and (2.6). Conditions (2.4) and (2.5) are obviously satisfied with $C$ absolute. We further suppose that $(g_l, m) = 1$ for otherwise $\delta_l(m,l) = 0$ since $(m,l) = 1$. Therefore, we must estimate the number of solutions to
\[
x^k \equiv g_l \mod m,
\]
where $g_l$ is the multiplicative inverse of $g_l$ modulo $m$. By the virtue of the Chinese remainder theorem, it suffices to estimate the number of solutions to (2.8) with $m$ as a prime power, say $m = p^e$, for $p \in \mathbb{P}$ and $e \in \mathbb{N}$. Note that the function
\[
\sigma_k : (\mathbb{Z}/p^e \mathbb{Z})^* \longrightarrow (\mathbb{Z}/p^e \mathbb{Z})^* : x \longrightarrow x^k
\]
is an endomorphism. Hence it is enough to estimate the size of its kernel $\ker(\sigma_k)$. If $k = \prod_{i=1}^{h} \pi_i^{a_i}$ is the prime decomposition of $k$, then
\[
\sigma_k = \prod_{i=1}^{h} \sigma_{\pi_i^{a_i}}.
\]
Thus,
\[
|\ker \sigma_k| \leq \prod_{i=1}^{h} |\ker \sigma_{\pi_i^{a_i}}|^{a_i}.
\]

Hence, it suffices to estimate the size of $|\ker \sigma_p|$ for prime numbers $\pi$.

For $p \in \mathbb{P}$,
\[
x^\pi - 1 \equiv 0 \mod p
\]
has at most $\pi$ solutions. By elementary result (see [15], for example), a solution, a mod $p^e$ with $e \geq 1$, of the congruence
\[
x^\pi - 1 \equiv 0 \mod p^e
\]
lifts to more than one solution to
\[
x^\pi - 1 \equiv 0 \mod p^{e+1}
\]
only when $p|\pi a^{\pi-1}$ and $p^{e+1}|a^{\pi-1} - 1$. If $p \neq \pi$, $p|\pi a^{\pi-1}$ implies $p|a$, but it is not possible that $p^{e+1}|a^{\pi-1} - 1$ as $(a^{\pi-1}, a) = 1$. Thus, in this case (2.10) has at most $\pi$ solutions for all $e \geq 1$. In the following, we consider the case $p = \pi$. 
By Fermat’s little theorem, there exists only one solution of the congruence
\[ x^\pi - 1 \equiv 0 \pmod{\pi}, \]
namely \( 1 \mod{\pi} \). This solution lifts to exactly \( \pi \) solutions to
\[ x^\pi - 1 \equiv 0 \pmod{\pi^2}, \]
namely
\[ 1, 1 + \pi, 1 + 2\pi, \ldots, 1 + (\pi - 1)\pi \mod{\pi^2}. \]
More generally, if \( a \mod{\pi^e} \) is a solution to
\begin{equation}
(2.11) \quad x^\pi - 1 \equiv 0 \pmod{\pi^e},
\end{equation}
then, if \( a \) lifts to solutions to
\[ x^\pi - 1 \equiv 0 \pmod{\pi^{e+1}}, \]
they are of the form
\begin{equation}
(2.12) \quad a, a + \pi^e, a + 2\pi^e, \ldots, a + (\pi - 1)\pi^e \mod{\pi^{e+1}}.
\end{equation}
Assume there are \( j_1, j_2 \in \{0, \ldots, \pi - 1\} \), \( j_1 \neq j_2 \) such that both \( a + j_1\pi^e \) and \( a + j_2\pi^e \) lift to solutions modulo \( \pi^{e+2} \). Then \( \pi^{e+2}[(a + j_1\pi^e)^\pi - 1] \) and \( \pi^{e+2}[(a + j_2\pi^e)^\pi - 1] \) are divisible by \( \pi^{e+2} \). If \( e \geq 2 \), this implies \( a \equiv 0 \pmod{\pi} \), but then \( a \) cannot be a solution to (2.11). Therefore, if \( e \geq 2 \), only one of the solutions (2.12) lifts to a solution modulo \( \pi^{e+2} \). From this we infer that the number of solutions to (2.11) never exceeds \( \pi^2 \), i.e.
\[ |\ker \sigma| \leq \pi^2. \]
Combining this with (2.9), we get
\[ |\ker \sigma_k| \leq k^2. \]
Therefore, by the Chinese remainder theorem, we obtain
\[ \delta_k(m, l) \leq k^{2\omega(m)}, \]
where \( \omega(m) \) is the number of distinct prime divisors of \( m \). Since \( 2^{\omega(m)} \) is the number of square-free divisors of \( m \), we have
\[ k^{2\omega(m)} \leq \tau(m)^{2\log_2 k} \ll m^\varepsilon, \]
where \( \tau(m) \) is the number of divisors of \( m \). Thus, if \( m \leq \sqrt{N} \), (2.6) holds with
\[ X \ll N^\varepsilon. \]
Now, by Theorem 3,
\begin{equation}
(2.13) \quad \sum_{q \in \mathbb{Q}(Q_0)} \sum_{(a, q) = 1}^q \left| \sum_{n=M+1}^{M+N} a_n e\left( \frac{a_n}{q} \right) \right|^2
\end{equation}
is majorized by
\[ \ll (\min\{Q_0 N^\varepsilon, N\} + Q_0) \left( \sqrt{N} \log \log(10N) + \max_{r \leq \sqrt{N}} \sum_{q | r} \frac{Q_0^{1/k} f_t^{-1}}{f_t} \right) Z. \]
The function
\[ G(r) = \sum_{t | r} \frac{1}{f_t} \]
is clearly multiplicative. If \( r \) is a prime power \( p^e \), then
\[ G(r) \leq 1 + k \left( \frac{1}{p} + \frac{1}{p^2} + \ldots \right) = 1 + \frac{k}{p-1} \leq \left( 1 + \frac{1}{p-1} \right)^k = \left( \frac{p^\varepsilon}{\varphi(p^e)} \right)^k. \]
Hence, for all \( r \in \mathbb{N} \) we have
\[
G(r) \leq \left( \frac{r}{\varphi(r)} \right)^{k} \ll (\log \log 10r)^k.
\]
Hence (2.13) is
\[
\ll (\log \log 10NQ_0)^{k+1} (\sqrt{N} + Q_0^{1/k})(\min \{Q_0N^r, N \} + Q_0).
\]
The above is always majorized by
\[
\ll (\log \log 10NQ_0)^{k+1} \left(Q_0^{1+1/k} + N^{1/2+\varepsilon}Q_0 \right).
\]
Summing over all relevant dyadic intervals and combining with (2.1), we see that (1.1) is majorized by
\[
\ll (\log \log 10NQ)^{k+1} (Q^{k+1} + N + N^{1/2+\varepsilon}Q^k)Z.
\]
Therefore, our result follows. \( \Box \)

3. Proof of Theorem 2

3.1. Reduction to Farey fractions in short intervals. As in [1], [2], our starting point is the following general large sieve inequality.

**Lemma 1:** Let \((\alpha_r)_{r \in \mathbb{N}}\) be a sequence of real numbers. Suppose that \(0 < \Delta \leq 1/2\) and \(R \in \mathbb{N}\). Put
\[
K(\Delta) := \max_{\alpha \in \mathbb{R}} \sum_{r=1}^{R} \frac{1}{\|x - \alpha\| \leq \Delta}
\]
where \(\|x\|\) denotes the distance of a real \(x\) to its closest integer. Then
\[
\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq c_1 K(\Delta)(N + \Delta^{-1})Z.
\]

In the sequel, we suppose that \(S\) is the set of cubes of natural numbers and that \(\alpha_1, ..., \alpha_R\) is the sequence of Farey fractions \(a/q\) with \(q \in S(Q_0), 1 \leq a \leq q\) and \((a,q) = 1\), where \(Q_0 \geq 1\). We further suppose that \(\alpha \in \mathbb{R}\) and \(0 < \Delta \leq 1/2\). Put
\[
I(\alpha) := [\alpha - \Delta, \alpha + \Delta] \quad \text{and} \quad P(\alpha) := \sum_{q \in S \cap [Q_0, 2Q_0]} 1_{\|a/q - \alpha\| \leq 1_{I(\alpha)}}.
\]
Then we have
\[
K(\Delta) = \max_{\alpha \in \mathbb{R}} P(\alpha).
\]
Therefore, the proof of Theorem 2 reduces to estimating \(P(\alpha)\).

As in [1] and [2], we begin with an idea of D. Wolke [18]. Let \(\tau\) be a positive number with
\[
1 \leq \tau \leq \frac{1}{\sqrt{\Delta}}.
\]
In [1] and [2] we put \(\tau := 1/\sqrt{\Delta}\), but in fact our method works for all \(\tau\) satisfying (3.2). We will later fix \(\tau\) in an optimal manner. In the said earlier papers, \(\tau = 1/\sqrt{\Delta}\) was the optimal choice.

By Dirichlet’s approximation theorem, \(\alpha\) can be written in the form
\[
\alpha = \frac{b}{r} + z,
\]
where
\[
r \leq \tau, \quad (b,r) = 1, \quad |z| \leq \frac{1}{r\tau}.
\]
Thus, it suffices to estimate \(P(b/r + z)\) for all \(b,r, z\) satisfying (3.3).
We further note that we can restrict ourselves to the case when
\begin{equation}
(3.4) \quad z \geq \Delta.
\end{equation}
If \(|z| < \Delta\), then
\[ P(\alpha) \leq P\left(\frac{b}{r} - \Delta\right) + P\left(\frac{b}{r} + \Delta\right). \]
Furthermore, we have
\[ \Delta \leq \frac{1}{r^2} \leq \frac{1}{r\tau}. \]
Therefore this case can be reduced to the case \(|z| = \Delta\). Moreover, as \(P(\alpha) = P(-\alpha)\), we can choose \(z\) positive. So we can assume \((3.4)\), without any loss of generality.

Summarizing the above observations, we deduce

**Lemma 2:** We have
\begin{equation}
(3.5) \quad K(\Delta) \leq 2 \max_{r \in \mathbb{N}} \max_{0 \leq b \leq 1} \Delta \leq 1/\tau \quad \max_{\Delta \leq x \leq \Delta + 1/\tau} P\left(\frac{b}{r} + x\right).
\end{equation}

3.2. estimation of \(P(b/r + z) - first way\). We now prove a first estimate for \(P(b/r + z)\) by using some results in [1]. In the sequel, we suppose that the conditions \((3.2)\), \((3.3)\) and \((3.4)\) are satisfied.

By inequality \((41)\) in [1], we have
\begin{equation}
(3.6) \quad P\left(\frac{b}{r} + z\right) \leq 1 + 6 \sum_{t|r} \sum_{0 < m \leq 4rzQ_0/t} \max_{(m, r/t) = 1} A_t\left(\frac{\Delta Q_0}{t}, \frac{r}{t}, -\delta m\right),
\end{equation}
where \(A_t(u, m, l)\) is defined as in \((2.3)\) and \(\delta\) is the multiplicative inverse of \(b\) modulo \(r\). By the results of section 2, for \(S\) the set of cubes, the conditions \((2.4)\), \((2.5)\) and \((2.6)\) with \(X = \Delta^{-\varepsilon}\) are satisfied for all \(t \in \mathbb{N}, m \in \mathbb{N}, l \in \mathbb{Z}, u \in \mathbb{R}\) with \(t \leq \tau, m \leq \tau/t, (m, l) = 1, mQ_0/\tau \leq u \leq Q_0/\tau\). Conditions \((2.4)\) and \((2.6)\) imply
\begin{equation}
(3.7) \quad \sum_{0 < m \leq 4rzQ_0/t} \max_{(m, r/t) = 1} A_t\left(\frac{\Delta Q_0}{t}, \frac{r}{t}, -\delta m\right) \leq C \left(1 + \frac{\Delta t|S_t(Q_0)|}{r^2}\right) \frac{4rzQ_0X}{t}
\end{equation}
From \((3.6)\), \((3.7)\) and
\[
\sum_{t|r} \frac{1}{t} \leq \prod_{p|t} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \ldots\right) = \prod_{p|t} \frac{p}{p-1} = \frac{r}{\varphi(r)} \leq c_2 \log \log 10r,
\]
we derive
\begin{equation}
(3.8) \quad P\left(\frac{b}{r} + z\right) \leq 1 + c_3 Q_0 X \left(rz \log \log 10r + \Delta \sum_{t|r} |S_t(Q_0)|\right).
\end{equation}
Furthermore, by \((2.2)\) and \((2.14)\), we have
\[
\sum_{t|r} |S_t(Q_0)| \ll (\log \log 10r)^3 Q_0^{1/3}.
\]
Thus, from \((3.8)\) and the fact that \(r \leq \tau = \Delta^{-1/2}\), we obtain

**Proposition 1:** Let \(S\) be the set of cubes of natural numbers. Suppose that the conditions \((3.2)\), \((3.3)\) and \((3.4)\) are satisfied. Then we have
\begin{equation}
(3.9) \quad P\left(\frac{b}{r} + z\right) \leq 1 + c_4 \Delta^{-\varepsilon} \left(Q_0^{4/3} \Delta + Q_0 rz\right).
\end{equation}
3.3. Estimation of $P(b/r + z)$ - second way. We now prove a second estimate for $P(b/r + z)$ by extending the Fourier analytic methods in [2], [19] to cubic moduli. The following bound for $P(b/r + z)$ can be proved in the same manner as Lemma 2 in [2].

**Lemma 3:** Let $S$ be the set of cubes of natural numbers. Suppose that
\begin{equation}
\frac{Q_0 \Delta}{z} \leq \delta \leq Q_0.
\end{equation}
Then,
\begin{equation}
P\left(\frac{b}{r} + z\right) \leq c_6 \left(1 + \frac{1}{\delta} \int_{Q_0}^{x_{Q_0}} \Pi(\delta, y) \, dy\right),
\end{equation}
where $I(\delta, y) = [y^{1/3} - c_9 \delta/Q_0^{1/3}, y^{1/3} + c_9 \delta/Q_0^{1/3}]$, $J(\delta, y) = [(y - 4\delta) rz, (y + 4\delta) rz]$ and
\begin{equation}
\Pi(\delta, y) = \sum_{q \in I(\delta, y)} \sum_{m \equiv bq^3 \mod r} 1.
\end{equation}

We shall prove the following

**Proposition 2:** Let $S$ be the set of cubes of natural numbers. Suppose that the conditions (3.2), (3.3) and (4.4) are satisfied. Then we have
\begin{equation}
P\left(\frac{b}{r} + z\right) \leq c_7 \Delta^{-c} \left(Q_0^{1/3} \Delta + Q_0^{1/3} \Delta r^{-1/3} z^{-1} + \Delta^{-1/2} (rz)^{1/2}\right).
\end{equation}

To derive Proposition 2 from Lemma 3, we need the following standard results from Fourier analysis.

**Lemma 4:** (Poisson summation formula, [5]) Let $f(x)$ be a complex-valued function on the real numbers that is piecewise continuous with only finitely many discontinuities and for all real numbers $a$ satisfies
\begin{equation}
f(a) = \frac{1}{2} \left(\lim_{x \to a^-} f(x) + \lim_{x \to a^+} f(x)\right).
\end{equation}
Moreover, suppose that $f(x) \leq c_9 (1 + |x|)^{-c}$ for some $c > 1$. Then,
\begin{equation}
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n), \quad \text{where } \hat{f}(x) := \int_{-\infty}^{\infty} f(y)e(xy) \, dy,
\end{equation}
the Fourier transform of $f(x)$.

**Lemma 5:** (see [19], for example) For $x \in \mathbb{R} \setminus \{0\}$ define
\begin{equation}
\phi(x) := \left(\frac{\sin \pi x}{2x}\right)^2, \quad \text{and } \phi(0) := \lim_{x \to 0} \phi(x) = \frac{\pi^2}{4}.
\end{equation}
Then $\phi(x) \geq 1$ for $|x| \leq 1/2$, and the Fourier transform of the function $\phi(x)$ is
\begin{equation}
\hat{\phi}(s) = \frac{\pi^2}{4} \max\{1 - |s|, 0\}.
\end{equation}

**Lemma 6:** (see Lemma 3.1. in [9]) Let $F : [a, b] \to \mathbb{R}$ be twice differentiable. Assume that $|F'(x)| \geq u > 0$ for all $x \in [a, b]$. Then,
\begin{equation}
\left|\int_a^b e^{iF(x)} \, dx\right| \leq \frac{c_9}{u}.
\end{equation}
Lemma 7: (see Lemma 4.3.1. in [4]) Let $F : [a, b] \to \mathbb{R}$ be twice continuously differentiable. Assume that $|F''(x)| \geq u > 0$ for all $x \in [a, b]$. Then,

$$\left| \int_a^b e^{iF(x)} \, dx \right| \leq \frac{c_1 u}{\sqrt{u}}$$

We shall also need the following estimates for cubic exponential sums.

Lemma 8: (see [11], [17]) Let $c \in \mathbb{N}$, $k, l \in \mathbb{Z}$ with $(k, c) = 1$. Then,

$$\sum_{d=1}^c e \left( \frac{kd^3 + ld}{c} \right) \leq c_1 c^{1/2+\varepsilon}(l, c).$$

Furthermore,

$$\sum_{d=1}^c e \left( \frac{kd^3}{c} \right) \leq c_1 c^{2/3}.$$

Proof of Proposition 2: We put

$$\delta := \frac{Q_0 \Delta}{z}.$$  

By Lemma 5, (3.12) can be estimated by

$$\Pi(\delta, y) \leq \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - y^{1/3}}{2c_0 \delta / Q_0^{2/3}} \right) \sum_{m \equiv -bq^3 \mod r} \phi \left( \frac{m - yrz}{8\delta rz} \right).$$  

Using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.15) into

$$\sum_{m \equiv -bq^3 \mod r} \phi \left( \frac{m - yrz}{8\delta rz} \right) = 8\delta z \sum_{j \in \mathbb{Z}} e \left( \frac{jbq^3}{r} + jyz \right) \hat{\phi}(8j\delta z).$$

Therefore, we get for the double sum on the right-hand side of (3.15)

$$\sum_{q \in \mathbb{Z}} \phi \left( \frac{q - y^{1/3}}{2c_0 \delta / Q_0^{2/3}} \right) \sum_{m \equiv -bq^3 \mod r} \phi \left( \frac{m - yrz}{8\delta rz} \right)$$

$$= 8\delta z \sum_{j \in \mathbb{Z}} e(jyz) \hat{\phi}(8j\delta z) \sum_{d=1}^{\hat{r}} e \left( \frac{jbq^3}{\hat{r}} \right) \sum_{k \equiv d \mod \hat{r}} \phi \left( \frac{k - y^{1/3}}{2c_0 \delta / Q_0^{2/3}} \right),$$

where $\hat{r} := r/(r, j)$ and $\hat{j} := j/(r, j)$. Again using Lemma 4 after a linear change of variables, we transform the inner sum on the right-hand side of (3.16) into

$$\sum_{k \equiv d \mod \hat{r}} \phi \left( \frac{k - y^{1/3}}{2c_0 \delta / Q_0^{2/3}} \right) = \frac{2c_0 \delta}{\hat{r}Q_0^{2/3}} \sum_{l \in \mathbb{Z}} \left( l \cdot \frac{d - y^{1/3}}{\hat{r}} \right) \hat{\phi} \left( \frac{2c_0 \delta l}{\hat{r}Q_0^{2/3}} \right).$$
From (3.16) and (3.17), we obtain
\[
\frac{1}{\delta} \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - y^{1/3}}{2c_0 \delta / \mathbb{Q}_0^{2/3}} \right) \sum_{m \in \mathbb{Z}} \phi \left( \frac{m - yr z}{8\delta r z} \right) dy
\]
\[
\leq \frac{16c_0 \delta z}{\mathbb{Q}_0^{2/3}} \sum_{j \in \mathbb{Z}} \tilde{\phi}(8j \delta z) \frac{r}{\mathbb{Q}_0^{2/3}} \sum_{l \in \mathbb{Z}} \phi \left( \frac{2c_0 \delta l}{l \mathbb{Q}_0^{2/3}} \right) \left| \sum_{d=1}^r e \left( j \frac{bd^2 + ld}{r} \right) \right| \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) dy.
\]  

Applying the Lemmas 5 and 8 to the right-hand side of (3.18), and taking \( r \leq 1/\sqrt{\Delta} \) by (3.2) and (3.3) into account, we deduce
\[
\frac{1}{\delta} \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - y^{1/3}}{c_0 \delta / \mathbb{Q}_0^{2/3}} \right) \sum_{m \in \mathbb{Z}} \phi \left( \frac{m - yr z}{8\delta r z} \right) dy
\]
\[
\leq \frac{c_{12} \delta z \Delta^{-\varepsilon}}{\mathbb{Q}_0^{2/3}} \left( \sum_{|l| \leq 1/(8\delta z)} \frac{1}{\sqrt{r}} \sum_{|l| \leq l \mathbb{Q}_0^{2/3}} (l, \tilde{r}) \left| \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) dy \right| \right)
\]
\[
+ \sum_{|j| \leq 1/(8\delta z)} \frac{1}{\sqrt{r}} \left| \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} e (jyz) dy \right| \right).
\]

If \( j \neq 0 \), then
\[
\left| \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} e (jyz) dy \right| \leq \frac{1}{|j|}.
\]

If \( j = 0 \) and \( l \neq 0 \), then
\[
\left| \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) dy \right| \leq \frac{c_{13} \mathbb{Q}_0^{2/3}}{|l|},
\]
by Lemma 6 (take into account that \( \tilde{r} = 1 \) if \( j = 0 \)). If \( j \neq 0 \) and \( l \neq 0 \), then Lemma 7 yields
\[
\left| \int_{\mathbb{Q}_0}^{2\mathbb{Q}_0} e \left( jyz - l \cdot \frac{y^{1/3}}{r} \right) dy \right| \leq \frac{c_{14} \sqrt{r} \mathbb{Q}_0^{5/6}}{\sqrt{|l|}}.
\]

Therefore, the right-hand side of (3.19) is majorized by
\[
\leq c_{15} \delta \Delta^{-\varepsilon} \left( z \mathbb{Q}_0^{1/3} + \frac{1}{\mathbb{Q}_0^{2/3}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r}} + z \frac{1}{l} \sum_{1 \leq j \leq \mathbb{Q}_0^{2/3} / (2c_0 \delta)} \sum_{1 \leq l \leq l \mathbb{Q}_0^{2/3} / (2c_0 \delta)} \frac{(l, \tilde{r})}{\sqrt{l}} \right).
\]

Now, we estimate the sums in the last line of (3.20). Using (3.2), (3.3) and (3.14), we obtain
\[
\sum_{1 \leq l \leq \mathbb{Q}_0^{2/3} / (2c_0 \delta)} \frac{1}{l} \leq c_{16} \Delta^{-\varepsilon}.
\]

Using the definition of \( \tilde{r} \), (3.2), (3.3) and (3.14), we obtain
\[
\sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r}} = \frac{1}{\sqrt{r}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{|l|} \leq \frac{c_{17} \Delta^{-\varepsilon}}{\sqrt{r}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{|l|} \sum_{l \in \mathbb{Z}} t^{-2/3} \leq \frac{c_{18} \Delta^{-2\varepsilon r - 1/3}}{\sqrt{r}}.
\]
For $A \geq 1$, we have
\[ \left. \sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq t \leq A} \frac{1}{\sqrt{t}} \leq \sum_{1 \leq j \leq A/t} \frac{1}{\sqrt{t}} < \sqrt{A} \sum_{1 \leq j \leq A/t} \frac{1}{\sqrt{t}} \leq 2 \sqrt{A}. \]
Therefore,
\[ \left. \sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq t \leq A} \frac{1}{\sqrt{t}} \leq \frac{c_{19} \Delta^{-\varepsilon} Q_0^{1/3}}{\sqrt{\delta z}} \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{\bar{r}}. \]
Using the definition of $\bar{r}$, we obtain
\[ \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{\bar{r}} = \sqrt{\bar{r}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{\sqrt{t}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{t^{5/2}} \leq \frac{c_{20} \sqrt{\bar{r}}}{\delta z}. \]
Combining Lemma 3 and (3.19-3.24), we obtain
\[ P \left( \frac{b}{r} + z \right) \leq c_7 \Delta^{-3\varepsilon} \left( 1 + \delta_2 Q_0^{1/3} + \delta_3 Q_0^{2/3} r^{-1/3} + \delta^{-1/2} Q_0^{1/2} \sqrt{\bar{r}} \right). \]
From (3.14) and (3.25), we infer the desired estimate. Note that the first term in the right-hand side of (3.25) can be absorbed into the last term on the right-hand side of (3.13) by (3.4).

### 3.4. Final proof of Theorem 2.

Combining Propositions 1,2 and (3.3), we obtain
\[ P \left( \frac{b}{r} + z \right) \leq c_{21} \Delta^{-\varepsilon} \left( Q_0^{4/3} \Delta + \min \left\{ Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1} \right\} + \Delta^{-1/2} r^{-1/2} \right). \]
If
\[ z \leq \Delta^{1/2} Q_0^{-1/3} r^{-2/3}, \]
then
\[ \min \left\{ Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1} \right\} = Q_0 r z \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}. \]
If
\[ z > \Delta^{1/2} Q_0^{-1/3} r^{-2/3}, \]
then
\[ \min \left\{ Q_0 r z, Q_0^{1/3} \Delta r^{-1/3} z^{-1} \right\} = Q_0^{1/3} \Delta r^{-1/3} z^{-1} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}. \]
From the above inequalities and (3.3), we deduce
\[ \min \left\{ Q_0 r z, Q_0^{1/3} \Delta r^{-1/2} z^{-1} \right\} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3} \leq Q_0^{2/3} \Delta^{1/2} r^{1/3}. \]
Combining (3.26) and (3.27), we get
\[ P \left( \frac{b}{r} + z \right) \leq c_{22} \Delta^{-\varepsilon} \left( Q_0^{4/3} \Delta^{\varepsilon} + Q_0^{2/3} \Delta^{1/2} r^{1/3} + \Delta^{-1/2} r^{-1/2} \right). \]
Now we choose
\[ \tau := \begin{cases} N^{6/5} Q_0^{-4/5}, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\ Q_0^{1/7}, & \text{if } 1 \leq Q_0 < N^{7/8}, \end{cases} \quad \text{and} \quad \Delta := \begin{cases} N^{-1}, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\ Q_0^{-8/7}, & \text{if } 1 \leq Q_0 < N^{7/8}. \end{cases} \]
Then the condition (3.2) is satisfied in each case, and from (3.28) and Lemmas 1.2, we obtain
\[ \sum_{Q_0^{1/3} \leq t \leq (2Q_0)^{1/3}} \sum_{a=1}^{a^2} \left| S \left( \frac{a}{t^3} \right) \right|^2 \leq \begin{cases} N^\varepsilon \left( Q_0^{4/3} + N^{9/10} Q_0^{2/5} \right) Z, & \text{if } N^{7/8} \leq Q_0 \leq N^{3/2}, \\ N Q_0^{2/7 + \varepsilon} Z, & \text{if } 1 \leq Q_0 < N^{7/8}. \end{cases} \]
We can divide the interval $[1, Q]$ into $O(\log Q)$ subintervals of the form $[Q_0^{1/3}, (2Q_0)^{1/3}]$, where $1 \leq Q_0 \leq Q^3$. Hence, the result of Theorem 2 follows from (3.29). □
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