<table>
<thead>
<tr>
<th>Title</th>
<th>A coherent image source method for flat waveguides with locally reacting boundaries</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Min, Hequn.; Qiu, Xiaojun.; Li, Ningrong.</td>
</tr>
<tr>
<td>Date</td>
<td>2010</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10220/7076">http://hdl.handle.net/10220/7076</a></td>
</tr>
<tr>
<td>Rights</td>
<td>© 2010 Australian Acoustical Society. This paper was published in Proceedings of 20th International Congress on Acoustics, ICA 2010 and is made available as an electronic reprint (preprint) with permission of Australian Acoustical Society. The paper can be found at the following official URL: <a href="http://www.acoustics.asn.au/conference_proceedings/ICA2010/cdrom-ICA2010/papers/p386.pdf">http://www.acoustics.asn.au/conference_proceedings/ICA2010/cdrom-ICA2010/papers/p386.pdf</a>. One print or electronic copy may be made for personal use only. Systematic or multiple reproduction, distribution to multiple locations via electronic or other means, duplication of any material in this paper for a fee or for commercial purposes, or modification of the content of the paper is prohibited and is subject to penalties under law.</td>
</tr>
</tbody>
</table>
A coherent image source method for flat waveguides with locally reacting boundaries

Hequn Min, Xiaojun Qiu and Ningrong Li

Key Laboratory of Modern Acoustics and Institute of Acoustics, Nanjing University, Nanjing, CHINA

PACS: 43.55.Ka, 43.55.Br

ABSTRACT

An image source method is presented for coherently evaluating the sound field from a point source in flat waveguides with two infinite and parallel locally reacting boundaries, where one is sound absorbing and the other is reflective. The method starts from formulating sound reflections into integrals by plane wave expansion, and the inherent intractability in solving these integrals in such spaces is avoided by introducing a physically plausible assumption that wave fronts remain the same before and after the reflection on a near-rigid boundary. By comparisons to the classical wave theory and the existing coherent ray-based methods, it is shown that the proposed method is considerably accurate to predict the sound propagation in flat waveguides with a sound absorbing ceiling and a reflective floor over a broad frequency range and for various source/receiver geometries, even if at large distances from the source compared to the waveguide height while the existing methods are shown to be erroneous.

INTRODUCTION

Speech privacy and noise control problems are often encountered in open-plan offices and appear serious in large ones [1]. These rooms usually have heights much smaller than the lateral dimensions, reflective floors, and suspended ceilings lined with sound absorbing material. A simplified model for such rooms can be a flat waveguide with two infinite parallel boundaries (floor and ceiling) with locally reactive material.

There have been many methods for predicting the sound field inside a flat waveguide. Based on the incoherent image source method, Kuttruff [2] proposed analytical formulas to predict the sound energy distribution in flat waveguides in 1980s. With similar incoherent ray-based methods, many other researchers have studied sound fields in similar spaces, such as large fitted factories [3-6], workshops [7,8], and dining rooms [9]. These models neglected the interference effect among direct sound and multiple reflections, and in some applications such as that for predicting sound fields of speech or narrow band noise, the models appear erroneous and cannot provide meaningful prediction [10-12].

There are some studies on the coherent ray-based method for the sound field in bounded spaces. Dance et al. [10] have developed an interference model to predict the sound pressure in industrial enclosures by considering the sound propagating phase shift and coherent summation of different reflected waves. Wang et al. [13] employed the image source method to coherently calculate the total field from a point source in open-plan offices by using the plane wave reflection coefficients for the reflections of spherical sound radiation of the image sources. Brekhovskikh [14] described the sound field in a flat-layered homogeneous media as sum of direct sound and multiple reflections from image sources, where each reflection is formulated as a form of plane wave expansion integral. In light of Brekhovskikh’s work and based on the solution for the spherical wave reflection on an infinite plane, Gensane and Santon [15] proposed a generalized solution to effectively model the successive reflections of spherical sound radiation from a point source in bounded spaces having more than two planes. According to the concepts of Gensane and Santon, Lemire and Nicolas [11] replaced the reflection coefficient with a more accurate solution [16] of the spherical wave reflection on one infinite plane to numerically investigate the sound field in flat waveguides. Westwood [17] has proposed a ray-based method for sound fields in flat waveguides with a penetrable bottom boundary and an idealized sound-soft top boundary to model the shallow water ocean environment.

However these previous coherent ray-based methods either require spaces bounded with sufficiently hard boundaries at high frequencies, or appear erroneous when source/receiver distance being large compared to the height in the flat waveguide with a sound absorbing ceiling and a reflective floor, which is a common situation in large fitted open-plan offices. In this paper, research is presented toward a simple but accurate coherent model to predict the sound propagation in flat waveguides with a sound absorbing ceiling and a reflective floor, particularly at large distances from the source compared to the space height.
THEORETICAL METHOD

Formulation of the problem

Figure 1(a) shows a vertical section of a flat waveguide, where the infinite ceiling and floor planes are locally reactive with uniform normalized specific admittance of $\beta_c$ and $\beta_f$, respectively. The height of the space is $h$, a point source is located at $(0, z_s)$ and a receiver is at $(r, z)$ with $r$ being the horizontal distance from the source to receiver. The time-dependent factor $e^{i\omega t}$ is suppressed for simplicity throughout this paper.

In the absence of the ceiling, the reflection of spherical wave incidence can be obtained by the superposition of all the elementary plane wave reflections on the floor plane as [14]

$$P_r = \sum n \frac{jk}{8\pi^2} \int_0^{\pi} \int_0^{2\pi} e^{ikr} V_n(\theta) \sin \theta \, d\theta \, d\varphi, \quad (2)$$

where $P_r$ denotes the field of the first reflection on the floor plane shown in Fig. 1(b), and $R_n$ is the corresponding distance vector from the image source $S_n$ to the receiver. $V_n(\theta)$ represents the plane wave reflection coefficient of the floor at incidence angle $\theta$ and is given by

$$V_n(\theta) = \frac{\cos \theta - \beta_f}{\cos \theta + \beta_f}. \quad (3)$$

With the ceiling, the total field can be constructed with the rays from an infinite number of image sources [14] with an expression of

$$P_n = \sum P_i, \quad (4)$$

in which $n = 0, \pm1, \pm2, \ldots$, and $P_i$ represents the ray contribution from the $n$th image source shown in Fig. 1(b). Particularly $P_0$ denotes direct sound from the real source. Similar to Eq. (2), the ray contribution from the path with $n_f$ floor reflections and $n_c$ ceiling reflections can be expressed with an integral [14]

$$P_n = \frac{jk}{8\pi^2} \int_0^{\pi} \int_0^{2\pi} e^{ikR_n V_n(\theta) \sin \theta \, d\theta \, d\varphi,} \quad (5)$$

where $R_n = (r \cos \varphi, r \sin \varphi, R_c \cos \theta, \ldots)$ is the distance vector from the $n$th image source $S_n$ to the receiver, $\theta$ and $\varphi$ denote the azimuth angles of the direction of each plane ray propagation in vertical and horizontal planes, respectively.

$$V_n(\theta) = \frac{\cos \theta - \beta_f}{\cos \theta + \beta_f}. \quad (6)$$

The orders $n_f$ and $n_c$ can be determined from the order $n$ with a rule of

$$n_f = \left\lfloor \frac{n}{2} \right\rfloor + \frac{1}{2} \text{sign}(n) \text{rem}[|n|, 2], \quad (7a)$$

and

$$n_c = \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{2} \text{sign}(n) \text{rem}[|n|, 2], \quad (7b)$$

where $\text{sign}()$ denotes the signum function, and $\text{rem}([n], 2)$ represents the remainder of $|n|$ after division by 2.

Defining an overall coefficient $V(\theta)$ to replace the term $[V_n(\theta)]^{n_f}[V_n(\theta)]^{n_c}$, $P_n$ in Eq. (5) is rewritten as

$$P_n = \frac{jk}{8\pi^2} \int_0^{\pi} \int_0^{2\pi} e^{ikR_n V(\theta) \sin \theta \, d\theta \, d\varphi,} \quad (8)$$

and then can be further transformed into
\[ P = \frac{jk}{8\pi^2} \int_{\phi_2}^{\phi_1} e^{i\theta - i\varphi} d\varphi \int_{R_0}^{\infty} e^{-i\varphi} V(\theta) \sin \theta d\theta. \quad (9) \]

By using [14,18]

\[ \int e^{-i\varphi} d\varphi = 2\pi J_1(kr \sin \theta), \quad (10) \]

where \( J_n(.) \) represents the Bessel function of zero order, Eq. (9) can be simplified as

\[ P = \frac{jk}{4\pi^2} \int_{\phi_2}^{\phi_1} e^{i\theta - i\varphi} V(\theta) J_1(kr \sin \theta) \sin \theta d\theta. \quad (11) \]

Using the identities for a complex number \( u \) that \( J_n(u) = [H_n'(u) + H_n''(u)]/2 \) and \( H_n'(-u) = -H_n'(u) \), Eq. (11) can be written as

\[ P = \frac{jk}{8\pi^2} \int_{\phi_2}^{\phi_1} e^{i\theta - i\varphi} V(\theta) \sin \theta \cdot J_1(kr \sin \theta) e^{i\theta - i\varphi} d\theta, \quad (12) \]

in which \( H_n'(.) \) and \( H_n''(.) \) are the Hankel functions of first and second kind with zero order. Now the total field from a point source in a flat waveguide is explicitly formulated by Equation (4) accompanied with the integral Equation (12).

**The coherent image source method**

As shown in Fig. 1(b), \( P_0 \) is the direct sound \( e^{i\alpha r}/4\pi R \), and \( P_1 \) and \( P_2 \) is the field of single reflections whose integral expressions with Eq. (12) can be evaluated with the exact (integral) solution provided by Nobile et al. [19],

\[ P_0 = \frac{e^{i\alpha r}}{4\pi R} \left[ 1 - \frac{4jkBR}{\beta + \cos \theta} \int I(k, R, \beta, \theta) \right], \quad (13a) \]

\[ P_1 = \frac{e^{i\alpha r}}{4\pi R} \left[ 1 - \frac{4jkBR}{\beta + \cos \theta} \int I(k, R, \beta, \theta) \right], \quad (13b) \]

where \( I(.) \) is an integral function given by

\[ I(k, R, \beta, \theta) = \int \frac{e^{-i\alpha r/2\beta} \cos \beta \theta - \cos \beta \theta - (1 - \beta^2) \sin \beta \theta}{\sqrt{1 - t^2/H - 2Bt/H}} dt, \quad (14) \]

and \( \beta \) is the normal specific admittance of the plane that the reflection takes place on.

\[ B = -j \sqrt{1 + \beta \cos \theta - (1 - \beta^2) \sin \theta}, \quad (15a) \]

\[ H = 1 + \beta \cos \theta + (1 - \beta^2) \sin \theta. \quad (15b) \]

For the ray with multiple reflections ( \( n \geq 2 \) ) on boundaries, though the exact solution of integral expression of \( P_n \) in Eq. (12) is generally not possible, it is feasible to analytically approximate such integral with the method of steepest descents for larger \( kr \) [14,16]. A second order approximate solution provided by Brekhovskikh [14] in term of asymptotic series with accuracy to terms of 1/(kr)^3 is

\[ P_n = e^{i\alpha r} V(\theta) - e^{i\alpha r} \int \frac{1}{2kR} \left[ V'(\theta) \cot \theta + V''(\theta) \right] + \ldots, \quad (16) \]

where \( V'(\theta) \) and \( V''(\theta) \) represent the first and second derivative of \( V(\theta) \) at \( \theta_n \), and \( V(\theta_n) = [V'(\theta)]^2 - [V''(\theta)]^2 \). Although Eq. (16) can be used in flat waveguides directly, its convergence needs to be analyzed before the application in the circumstance concerned in this paper. After substituting the expressions of \( V(\theta) \) and \( V'(\theta) \) in Eqs. (3) and (6) into Eq. (16), a common factor of \( \left( \cos \theta_n + \beta \gamma(\cos \theta_n + \beta \gamma) \right) \) can be found in all the series terms in Eq. (16).

In the current circumstance, the floor is sound reflective, which leads that \( |\beta| \to 0 \). If the receivers are far from the source compared to the waveguide height, which is a common situation in large open-plan offices, the incidence angle \( \theta \) of rays at the first few orders will approach \( \pi/2 \) as shown in Fig. 1(b). This causes that \( \cos \theta_n + \beta \gamma \to 0 \) and then limits the convergence radius of the asymptotic series in Eq. (16). The second order approximation in Eq. (16) is not sufficiently accurate for the current problem theoretically.

To explicitly obtain the higher orders of the asymptotic series in Eq. (16), exponential complexity will be encountered as the order increases [14,15] and such extension appears not sensible as the common factor \( \left( \cos \theta_n + \beta \gamma(\cos \theta_n + \beta \gamma) \right) \to \infty \) [16]. The steepest descent method modified by subtraction of the pole has been employed to remedy the similarly worsen accuracy of the asymptotic series for sound propagation along a single reflective boundary [16,20]. The modified method is based on the Laurent series expansion of the integrand in Eq. (2) to avoid the singularities from the poles. Nonetheless such strategy becomes difficult in the cases of flat waveguides with two boundaries, because the poles in this case come from the denominator \( \cos \theta_n + \beta \gamma(\cos \theta_n + \beta \gamma) \) of the integrand in Eq. (12) and mostly are high order ones. It is intractable to explicitly obtain the residues of this integrand at these higher poles for Laurent series expansion.

Analysis on the physics of the problem might be helpful to get over the mathematical intractability above. The solution in Eq. (13a) or (13b) for the singly reflected field from a point source on an infinite plane can be rewritten for generality with a form of image source method as

\[ P_n = P_s \cdot Q_n, \quad (17) \]

where \( P_s \) is the field of single reflection of spherical wave from a point source on an infinite plane and \( P_s \) is the direct field at receiver from the image source due to the single reflection. \( Q_n \) is the single reflection coefficient for spherical wave incidence, and can be determined from Eqs. (13a) or (13b) by

\[ Q_n = 1 - \frac{4jkBR}{\beta + \cos \theta_n} I(k, R, \beta, \theta_n), \quad (18) \]

where \( R_s \) is the propagation distance of the ray and interpretations of other parameters are in accord with those in Eqs. (13a), (13b), and (14).

It is revealed that the reflected wave of the spherical sound incidence from an image source remains almost spherical along a hard enough boundary [21,22]. The different part of
wave fronts of $P_\infty$ or $P_n$, which may be regarded as the spatial distribution of wave energy propagation in time, can always be described as a function of the incident angle $\theta$ from the image source to different spatial locations (receivers). From Eq. (17), the phenomenon that the wave front of $P_n$ is almost the same with that of $P_\infty$, requires a reflection coefficient $Q_n$ that is weakly dependent on $\theta$. This indicates that the coefficient $Q_n$ can be assumed to be uniform for different part of incident wave fronts in the reflection on a hard boundary. Thus according to Eq. (17), each reflection of a ray on a reflective boundary that is denoted by RB, regardless of the reflection being the first or the successive one in the ray propagation from the spherical radiation, can be heuristically approximated by

$$P_n \approx P_\infty \cdot Q_n(IS,R \mid RB),$$

where $P_n$ denotes the ray field at receiver after a reflection on RB, and $P_\infty$ represents the expected ray field at the receiver if the boundary RB is rigid, which is just the perfectly reflected field on this boundary. $Q_n(IS,R \mid RB)$ is used to replace the single reflection coefficient $Q_n$ for reflections on the reflective boundary RB and corresponds to the ray from the image source IS to receiver R, which is determined by Eq. (18) also.

In the current circumstance, the floor boundary is reflective. Thus before and after each reflection on it, the wave fronts can be assumed to remain the same. The ray field alterations after each reflection can be quantified by a weighting factor $Q_n(IS,R \mid FB)$ that depends on the floor boundary (FB) and the ray geometry according to Eq. (19). Thus in the propagation of a ray with reflection order $n$, the ray propagation of the ray with reflection order $n$ should be once weighted by the reflection coefficient $Q_n(IS,R \mid FB)$. Therefore after the ray field being weighted for $n$ times due to “transmission” through the floor and its images in propagation, the field contribution of the ray, $P_n$, can be approximated as

$$P_n = \left[ Q_n(IS,R \mid FB) \right] \frac{2\pi}{4\pi - \beta} \int_0^{2\pi} |V(\theta)| \sin \theta \cdot H_n(kr \sin \theta)e^{-\alpha n_4,\theta} d\theta,$$

(20)

where $Q_n(IS,R \mid FB)$ can be determined from Eq. (18). Compared to Eq. (12), the integral of $P_n$ in Eq. (20) has been simplified, and now the integrand involves only the reflection coefficient on the ceiling.

The integral in Eq. (20) describes the field of the ray after $n$ times reflections on the absorbent ceiling. This can be evaluated by the second order approximation in Eq. (16) with ensured accuracy because $|\beta|$ does not approach zero and for the asymptotic series there is no singularity similar to that from $(\cos \theta + \beta) \rightarrow 0$ as $\theta \rightarrow \pi/2$. So $P_n$ can be further approximated for larger $kr$ by

$$P_n = \left[ Q_n(IS,R \mid FB) \right] \frac{2\pi}{4\pi - \beta} \int_0^{2\pi} |V(\theta)| \frac{2n_4}{2IR} \left( V(\theta \mid CB,n) \cos \theta + V(\theta \mid CB,n) \right) e^{-\alpha n_4,\theta} d\theta,$$

(21)

where $V(\theta \mid CB,n)$ represents a total plane wave reflection coefficient to take account of the successive $n$ times plane wave reflections on the absorbent ceiling boundary (CB) at an incident angle $\theta$, which equals $|V(\theta)|$ in this case. $V(\theta \mid CB,n)$ and $V(\theta \mid CB,n)$ are the first and second derivatives of $V(\theta \mid CB,n)$ at $\theta$, respectively. The factor $e^{-\alpha n_4,\theta}$ is the direct sound field at the receiver from the nth image source, while the factors in front of it act like a combined coherent reflection coefficient to evaluate the overall influence from all the successive reflections during the ray propagation. Hence $P_n$ can be expressed as

$$P_n = Q_n \frac{e^{-\alpha n_4}}{4\pi\sqrt{R}},$$

(22)

where $Q_n$ denotes the combined reflection coefficient corresponding to the ray with reflection order $n$ and

$$Q_n \approx \left[ Q_n(IS,R \mid FB) \right] \frac{e^{-\alpha n_4,\theta}}{4\pi\sqrt{R}},$$

(23)

in which $Q_n(IS,R \mid CB,n)$ represents a total reflection coefficient taking account of all the successive reflections on an absorber boundary for a ray with spherical radiation, where $n$ times successive reflections have taken place on the absorbent ceiling during the propagation of the ray from $S_n$ to $R$. The coefficient $Q_n(IS,R \mid CB,n)$ can be approximated by

$$Q_n(IS,R \mid CB,n) = V(\theta \mid CB,n) = \frac{2n_4}{2IR} \left( V(\theta \mid CB,n) \cos \theta + V(\theta \mid CB,n) \right),$$

(24)

Eqs. (22) and (23) are the main contribution of this paper, which in company with Eq. (4) delivers a coherent image source method to approximately solve the total field from a point source in flat waveguides with a sound absorbing ceiling and a reflective floor. On the basis of asymptotic approximations by the steepest descent method for large $kr$ and a physically plausible assumption that wave fronts remains the same before and after each reflection on a sufficiently hard boundary, the solution has no other limitations on the source/receiver geometry in principle. In the next section, numerical simulations will be carried out to validate the model and compare it to the existing coherent ray-based methods.

**NUMERICAL RESULTS AND DISCUSSIONS**

The wave theory [23,24] is used as a reference to validate the ray-based methods being used for predicting the sound fields in flat waveguides concerned in this paper. The equations in the wave theory to solve the sound field from a point source in flat waveguides are detailed in the appendix. In simulations, the analytical solution of Eq. (A6) in Appendix is available only if the boundaries are rigid. For absorbent boundaries with complex admittance, Equation (A6) becomes complex and must be solved numerically [23-25]. For the image source methods, the locations of image sources can be determined through Fig. 1(b). The maximum order $l$ is determined when the accumulated total field amplitude differs less than 0.1% from that accumulated up to the order $l=10$.

The Delany and Bazley model [26] is employed to evaluate the boundary admittances in simulations, which are given by

**Proceedings of 20th International Congress on Acoustics, ICA 2010**
\[ \beta = \left[ 1 + 0.057\left( \rho_f / \sigma \right)^{10} - 0.087\left( \rho_f / \sigma \right)^{1000} \right], \] 

where \( \beta \) is the normalized specific boundary admittance, \( \rho_f = 1.293 \text{ kg m}^{-3} \) denotes the air density at the room temperature, and \( f \) is the actuating frequency of sound wave. Parameter \( \sigma \) is the flow resistivity of boundary material with a unit of cgs (1 cgs = 1 kPa s m\(^{-3}\)). Although the Delany and Bazley model is semi-empirical and sometimes may be far from ideal to represent the realistic circumstance in open-plan offices, it is used in this paper due to its simpleness. Fig. 2 shows the normal incident absorption coefficient of materials with flow resistivities used hereafter versus frequency.

![Figure 2. Spectra of normal incident absorption coefficient of materials with various flow resistivities \( \sigma \) in simulations.](image)

Two numerical cases are investigated to access the sound field prediction performance in flat waveguides by the proposed method and the existing coherent ray-based methods, such as the method of Brekhovskikh [14], the method of Lemire and Nicolas [11], and the method of Gensane and Santon [15]. In the simulations, the method of Lemire and Nicolas is modified by replacing the approximate spherical wave reflection coefficient, the term \( R_c + (1 - R_c) e^{i \omega} \) in Eq. (7) of Ref. 11, with the exact integral one in Eq. (18) posed by Nobile et al. [19] for better accuracy [11,19]. The ceiling of the flat waveguide is assumed to be highly sound absorbing in the first case and to be moderately absorbent in the second, while the floor keeps reflective and the waveguide height is always 2 m in both cases.

In the first case, the ceiling material has a flow resistivity of 30 cgs (similar to highly sound absorbing wool), and the flow resistivity of the floor material is assumed to be 10000 cgs (similar to typical terrazzo tile or high-density wood). The predictions of the sound pressure level (SPL) are firstly investigated versus the horizontal distance between the source and receiver at 1000 Hz, where heights of the source and receivers are chosen to be 0.25 m and the normal incident absorption coefficients of the ceiling and floor are 0.85 and 0.03 respectively. The corresponding results with the proposed method, the wave theory, and the exiting ray-based methods are compared in Fig. 3. It is shown that the results from the proposed method and those from the wave theory almost overlap over the range of \( r/h \) from 1 to 50. The method of Gensane and Santon [15] provides accurate results only when the horizontal distance between the source and receiver is smaller than two times of the waveguide height, and the modified method of Lemire and Nicolas [11] is accurate with \( r/h \) below 3. Although the method of Brekhovskikh [14], which is the second order approximate solution in Eq. (16) for the flat waveguide, predicts much better than the latter two existing image source methods, large discrepancies are found between the results from it and those from the wave theory after \( r \) becomes larger than 5 times of \( h \). This indicates that when receivers are far away from the source compared to the heights of flat waveguides, the existing coherent ray-based methods are hard to predict the sound propagation accurately.

![Figure 3. Predictions of the sound pressure levels (SPL) at frequency 1000 Hz versus \( r \) in a flat waveguide with \( h \) of 2 m, where \( z_c = z = 0.25 \text{ m} \). The flow resistivities of the ceiling and floor are 30 and 10000 cgs respectively. Five methods are compared: proposed method (the solid line), wave theory (the points), the ray method from Brekhovskikh (Ref. 14) (the red dashed line), the modified method of Lemire and Nicolas (Ref. 11) (the dash-dotted line), and the method of Gensane and Santon (Ref. 15) (the magenta dotted line).](image)

Computations have been carried out in this case to further validate the proposed method in the frequencies range from 100 Hz to 2000 Hz, where receivers are chosen with a set of horizontal distances from the source of 5, 10, and 20 times of the waveguide height. Figures 4 and 5 show the results corresponding to the situations where the source and receivers are close to the sound reflective floor \( (z_c, z = 0.25 \text{ m}) \) or the absorbent ceiling \( (z_c, z = 1.8 \text{ m}) \) respectively, so as to simulate the propagation of noise from the floor-standing electrical appliances or the ceiling-mounted ventilation devices in fitted open-plan offices. The situation for sound propagations at the height of a person seated \( (z_c, z = 1.2 \text{ m}) \) is also investigated and the corresponding results are presented in Fig. 6. As shown from Figs. 4 to 6, the prediction of SPLs with the proposed method always agree well with the wave theory over a broad frequency range, except small differences observed at frequencies below 200 Hz, which may be explained by two reasons. Firstly the accuracy of Eq. (16) is ensured in theory for large \( kr \). Secondly at these low frequencies, the magnitude of \( \beta \) becomes small from the evaluation by the Delany and Bazley model, and this leads to large \( \cos(\theta_c + \beta) \) in the common factor of the series in Eq. (24) as well as the large \( \cos(\theta_c + \beta) \) for the series in Eq. (16) as
\( \theta \to \pi / 2 \), which can distinctly lower the degree of approximation for \( Q_{\text{eq}}(S,R|c,l,n) \).

In the second numerical case, the ceiling is modelled with material having a flow resistivity of 150 cgs, and for the floor the flow resistivity of material remains to be 10000 cgs. The corresponding SPL versus \( r/h \) curves predicted with the proposed method, the wave theory and the existing ray-based methods are compared in Fig. 7, where \( z_i = z = 0.25 \) m and the frequency is 1000 Hz still for examination while the normal incident absorption coefficient of the ceiling becomes 0.5. The agreement between the proposed method and the wave theory is remarkable with \( r/h \) ranging from 1 to 50 again. The method of Brekhovskikh [14] is shown with errors also for \( r/h \) larger that 5 while the performance from the methods of Lemire et al. [11] and Gensane et al. [15] seem better by comparison with that in Fig. 4 of the first case, which might be explained by the less wave fronts distortion after each reflection on the ceiling as it being less absorbent. The spectra of SPL from 100 Hz to 2000 Hz predicted with the proposed method and the wave theory are shown in Figs. 8 to 10, where the source/receiver geometries are based on the same relevant considerations as in Figs. 4 to 6 respectively. It is clear that the predictions from the proposed method are in good agreement with the reference method over a broad frequency range again, except small differences at some low frequencies. Similar results were obtained by selecting other flow resistivities for ceiling material or other source/receiver geometries, which are not presented here for brevity.
The computational time from the calculation with the proposed method and the numerical evaluation of the wave theory is also compared. In evaluation with the wave theory, the computational time increases exponentially with the frequency, because more roots of eigenvalues in Eq. (A6) are needed to be included for convergence with the wave theory at higher frequencies in the flat waveguide. At the highest frequency of 2000 Hz, it takes over 45 seconds for a personal computer with a 2.4 GHz Intel Q6600 processor to execute the numerical evaluation of the wave theory. However less than one second is needed for the same computer to implement the calculation with the proposed method at a single frequency and the computational time is frequency independent. It is shown that for evaluating the sound field in a flat waveguide with impedance boundaries, the proposed method is simple and much faster than the wave theory for an equivalent accuracy degree.

From the above results for sound fields in the flat waveguide with a sound absorbing ceiling and a reflective floor, it is clear that the proposed coherent image source method is valid and can provide more accurate predictions than the existing coherent ray-based methods for various source/receiver geometries. Although the method is derived on the basis of assumptions of $kr$ being large, it is found from the numerical investigations that the proposed method can remain accurate at somewhat low frequencies. Moreover, the numerical results validate the idea proposed in Eq. (23) of separating the reflections on the reflective boundaries from those on absorbent ones in applying reflection coefficients in the image source method. This idea is expected to be useful for developing an accurate and versatile coherent image source method for sound fields in realistic enclosures.

**CONCLUSION**

In the present study, an accurate coherent image source method has been derived to predict the sound fields in the flat waveguide with a sound absorbing ceiling and a reflective floor, which can be a simplified model for the fitted open-plan offices. Based on the theory of expanding spherical wave into plane wave integrals, the method provides a physically meaningful prediction avoiding the intractability in analytically approximating the integrals in such flat waveguides, by introducing an assumption that wave fronts remain the same before and after each reflection on a reflective boundary.

The method has been numerically validated with comparison to the wave theory for various source/receiver geometries in flat waveguides over a broad frequency range. It is shown that the proposed method is valid and accurate to predict the sound propagation, even if at large distances from the source compared to the waveguide height while the existing coherent ray-based methods are inaccurate. It is also shown that the propose method is much more time efficient than the wave theory for the flat waveguide with impedance boundaries. The proposed method is useful for the sound field prediction in fitted open-plan offices.

**ACKNOWLEDGEMENTS**

The authors are grateful to Weisong Chen, Hongjie Pu and Yan Chen for the valuable discussions. Project 10804049 supported by NSFC.

**APPENDIX**

Based on the wave theory in Refs. 23 and 24, the sound field from a point source in flat waveguides can be solved as a summation of modes along directions $z$ and $r$ in Fig. 1(a). It is expressed as [11,15]

$$P_r = \frac{jA}{4} \sum_{m} \psi_m(z) \cdot \psi_m(z) \cdot H_m(K, r), \quad (A1)$$

where $A$ is the power amplitude of the point source and assumed to be unit. $z_s$ and $z$ are heights of the source and receiving points respectively and $r$ denotes the source/receiver horizontal distance. The term $\| \psi_m \|$ represents the norm of the eigenfunction $\psi_m(z)$.

$$\| \psi_m \| = \int \left[ \psi_m(z) \right]^2 dz \quad (A2)$$

The eigenfunction $\psi_m(z)$ can be expressed as

$$\psi_m(z) = e^{ikz} + r e^{-ikz}, \quad (A3)$$
where \( k_z \) denotes the eigenvalue in \( z \) direction and \( r_c \) is a complex constant. Correspondingly the wave number in \( r \) direction in Eq. (A1), \( K_r \), is determined from \( k_z \) by

\[
K_r = \sqrt{k_z^2 - k_r^2},
\]

(A4)

where the complex square root is taken to yield a result with a positive real part and \( k \) is the wave number. The eigenvalue \( k_z \) and constant \( r_c \) are obtained from the boundary condition equations along the pair of planes in the posed problem, which are

\[
\begin{align*}
\frac{\partial \psi_{z}(h)}{\partial z} - jk \psi_{z}(h) &= 0, & \text{on the ceiling boundary} \\
\frac{\partial \psi_{z}(0)}{\partial z} + jk \psi_{z}(0) &= 0, & \text{on the floor boundary}
\end{align*}
\]

(A5)

Hence the eigenvalue equations can be obtained as

\[
k_z + k_r \beta_n \quad k_z - k_r \beta_n = e^{i \omega n},
\]

(A6)

and

\[
r_c = \frac{k_z + k_r \beta_n}{k_z - k_r \beta_n}.
\]

(A7)

REFERENCES