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CYCLE SYSTEMS IN THE COMPLETE BIPARTITE GRAPH PLUS A ONE-FACTOR

LIQUN PU†, HAO SHEN‡, JUN MA‡, AND SAN LING§

Abstract. Let $K_{n,n}$ denote the complete bipartite graph with $n$ vertices in each partite set and $K_{n,n}+I$ denote $K_{n,n}$ with a one-factor added. It is proved in this paper that there exists an $m$-cycle system of $K_{n,n}+I$ if and only if $n \equiv 1 \pmod{2}$, $m \equiv 0 \pmod{2}$, $4 \leq m \leq 2n$, and $n(n+1) \equiv 0 \pmod{m}$.

Key words. complete bipartite graph, one-factor, cycle system

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1. Introduction. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. An $m$-cycle system of $G$ is a collection $T$ of $m$-cycles such that each edge of $G$ is contained in a unique $m$-cycle of $T$.

It is easy to get the necessary conditions for the existence of an $m$-cycle system of $G$:

\[
\begin{align*}
3 \leq m \leq |V(G)|; \\
|E(G)| &\equiv 0 \pmod{m}; \\
d(u) &\equiv 0 \pmod{2} \text{ for each } u \in V(G),
\end{align*}
\]

where $d(u)$ denotes the number of edges incident with $u$ in $G$.

Let $K_n$ denote the complete graph of order $n$, and let $K_{x,y}$ denote a complete bipartite graph with partite sets of sizes $x$ and $y$. For $G = K_n$ or $K_{n,n}$, let $G+I$ denote $G$ with a one-factor added and $G-I$ denote $G$ with a one-factor removed. The existence of $m$-cycle systems has been studied extensively, and the following results are known.

THEOREM 1.1 (see [1, 6]). Let $m$ and $n$ be positive integers. Then there exists an $m$-cycle system of $K_n$ if and only if $n \equiv 1 \pmod{2}$, $3 \leq m \leq n$, and $n(n-1) \equiv 0 \pmod{2m}$.

THEOREM 1.2 (see [7]). Let $m \equiv 0 \pmod{2}$ and $m \geq 4$. Then there exists an $m$-cycle system of $K_{x,y}$ if and only if $x,y \geq \frac{1}{2}m$, $x \equiv y \equiv 0 \pmod{2}$, and $xy \equiv 0 \pmod{m}$.

THEOREM 1.3 (see [5]). Let $n$ be an even integer and $m$ be an integer in the range $3 \leq m \leq n$. Then there exists an $m$-cycle system of $K_n+I$ if and only if $m$ divides $\frac{n^2}{2}$.
Let $m$ and $n$ be positive integers. Then there exists an $m$-cycle system of $K_n$ if and only if $n \equiv 0 \pmod{m}$.

THEOREM 1.5 (see [2, 4]). Let $m$ and $n$ be positive integers. Then there exists an $m$-cycle system of $K_{n,n} - I$ if and only if $n \equiv 1 \pmod{2}$, $m \equiv 0 \pmod{2}$, $4 \leq m \leq 2n$, and $n(n - 1) \equiv 0 \pmod{m}$.

In this paper, we study the existence and construction of $m$-cycle systems for the bipartite graph $K_{n,n} + I$. Since in $K_{n,n} + I$ there are $2n$ vertices, $n^2 + n$ edges, $d(u) = n + 1$ for each vertex $u$, and $m$ must be even, we have the following necessary conditions for the existence of an $m$-cycle system of $K_{n,n} + I$.

**Lemma 1.6.** If there exists an $m$-cycle system of $K_{n,n} + I$, then

\[ \begin{align*}
&n \equiv 1 \pmod{2}, \\
&m \equiv 0 \pmod{2} \text{ and } 4 \leq m \leq 2n, \\
&n(n + 1) \equiv 0 \pmod{m}.
\end{align*} \]

The purpose of this paper is to prove that these conditions are also sufficient for the existence of an $m$-cycle system of $K_{n,n} + I$. This is an extension of the result in [4].

**2. Construction techniques.** A cycle on $m$ vertices is denoted by $C_m$. A $C_n$ in a graph with $n$ vertices is called a Hamilton cycle. If there exists an $m$-cycle system of $G$, then $G$ is $C_m$-decomposable and is denoted by $C_m(G)$.

In this section, we will provide some construction techniques for $m$-cycle systems of $K_{n,n} + I$. For our first construction, we need the following result.

**Lemma 2.1 (see [3]).** Let $n$ be an integer, $n \geq 3$. Then there exists an $n$-cycle system of $K_n$ if and only if $n \equiv 1 \pmod{2}$.

When $m$ is even, we can construct $m$-cycle systems of $K_{\frac{1}{2} m, \frac{1}{2} m} + I$ by applying $\frac{1}{2} m$-cycle systems of $K_{\frac{1}{2} m}$.

**Theorem 2.2.** Let $m$ be a positive integer such that $m \equiv 2 \pmod{4}$ and $m \geq 6$. Then $C_m[K_{\frac{1}{2} m, \frac{1}{2} m} + I]$.

**Proof.** Let $V(K_{\frac{1}{2} m, \frac{1}{2} m}) = \{u_0, v_0, \ldots, u_{\frac{1}{2} m - 1}, v_{\frac{1}{2} m - 1}\}$.

For $C = U_{j0, v_{j0}} \cup U_{j1, v_{j1}} \cup U_{j2, v_{j2}} \cup U_{j3, v_{j3}} \cup \ldots \cup U_{j_{\frac{1}{2} m - 1}, v_{j_{\frac{1}{2} m - 1}}}$, let

\[ C' = \left( w_{j0}, w_{j1}, w_{j2}, w_{j3}, \ldots, w_{j_{\frac{1}{2} m - 1}} \right) \in T, \]

and

\[ C'^{1*} = \left( u_{j0}, v_{j0}, u_{j1}, v_{j1}, u_{j2}, v_{j2}, u_{j3}, v_{j3}, \ldots, u_{j_{\frac{1}{2} m - 1}}, v_{j_{\frac{1}{2} m - 1}} \right) \]

and

\[ C'^{2*} = \left( v_{j0}, u_{j0}, v_{j1}, u_{j1}, v_{j2}, u_{j2}, v_{j3}, u_{j3}, \ldots, v_{j_{\frac{1}{2} m - 1}}, u_{j_{\frac{1}{2} m - 1}} \right). \]

For each $C = \left( w_{i0}, w_{i1}, w_{i2}, w_{i3}, \ldots, w_{i_{\frac{1}{2} m - 1}} \right) \in T \setminus \{C'\}$, let

\[ C^* = \left( u_{i0}, v_{i0}, u_{i1}, v_{i1}, u_{i2}, v_{i2}, u_{i3}, v_{i3}, \ldots, u_{i_{\frac{1}{2} m - 1}}, v_{i_{\frac{1}{2} m - 1}} \right). \]
Let \( T^* = \{ C^* \mid C \in T \setminus \{ C^\prime \} \} \cup \{ C^{\prime 1*}, C^{\prime 2*} \} \) and \( I = \{ u_i v_i \mid 0 \leq i \leq \frac{1}{2} m - 1 \} \). Then \( T^* \) is an \( m \)-cycle system of \( K_{2m, \frac{1}{2} m} + I \). \( \square \)

Now for a positive integer \( n \), let \( D \subseteq \mathbb{Z}_n \) and let \( X(n; D) \) be a graph with vertex set \( V(X(n; D)) = \{ i_j \mid i \in \mathbb{Z}_n, j \in \mathbb{Z}_2 \} \) and edge set \( E(X(n; D)) = \{ \{ i_0, (i + d)_1 \} \mid d \in D \} \). Clearly, \( K_{n,n} = X(n; Z_n) \). The elements of \( D \) are called \((0,1)\)-mixed differences. We say that \( \{ i_0, (i + d)_1 \} \) is an edge of difference \( d \).

Suppose that \( C = ((i_1)_0, (i_2)_1, \ldots, (i_m)_0, (i_1)_1) \) is a \( C_m \) in \( X(n; D) \). For \( x \in \mathbb{Z}_n \), let \( C + x = ((i_1 + x)_0, (i_2 + x)_1, \ldots, (i_m + x)_0, (i_1 + x)_1) \). Obviously, \( C + x \) is still a \( C_m \). Let \( (C) = \{ C + x \mid x \in \mathbb{Z}_n \} \). Here, \( (C) \) is called the orbit generated by \( C \), and \( C \) is called a base cycle of \( (C) \).

In our proof, we denote the union of multisets by \( \cup \), for example, \( \{1, 1, 2\} \cup \{2, 3\} = \{1, 1, 2, 2, 3\} \).

We use the difference method to give constructions of \( m \)-cycle systems of \( X(n; D) \) which we need in this paper.

**Lemma 2.3.** For an even integer \( m \), \( m \geq 4 \), \( C_m | K_{m-1,m-1} + I \), where \( I \) is a one-factor of \( K_{m-1,m-1} \).

**Proof.** We view \( K_{m-1,m-1} \) as \( X(m - 1; Z_{m-1}) \) and \( I = \{ \{ i_0, i_1 \} \mid i \in Z_{m-1} \} \). Let \( d_r \in Z_{m-1} \cup \{ 0 \} \) and

\[
d_r+1 = \begin{cases} 
  r & \text{if } 0 \leq r \leq \frac{1}{2} m - 1, \\
  0 & \text{if } r = \frac{1}{2} m, \\
  r - 1 & \text{if } \frac{1}{2} m + 1 \leq r \leq m - 1.
\end{cases}
\]

Let \( e_r = \sum_{i=1}^r (-1)^{i+1} d_i \) for \( 1 \leq r \leq m \). Then

\[
e_r = e_{r-1} + (-1)^{r+1} d_r.
\]

When \( m \equiv 0 \pmod{4} \),

\[
e_i = \begin{cases} 
  -\frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 1 \leq i \leq \frac{1}{2} m; \\
  \frac{i-1}{2} & \text{if } i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{1}{2} m; \\
  -\frac{m+1-i}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{1}{2} m + 1 \leq i \leq m; \\
  -(m - 1 - \frac{m-i}{2}) & \text{if } i \equiv 0 \pmod{2}, \frac{1}{2} m + 1 \leq i \leq m.
\end{cases}
\]

When \( m \equiv 2 \pmod{4} \),

\[
e_i = \begin{cases} 
  -\frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 1 \leq i \leq \frac{1}{2} m; \\
  \frac{i-1}{2} & \text{if } i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{1}{2} m; \\
  \frac{m-i}{2} & \text{if } i \equiv 0 \pmod{2}, \frac{1}{2} m + 1 \leq i \leq m; \\
  m - 1 - \frac{m+1-i}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{1}{2} m + 1 \leq i \leq m.
\end{cases}
\]

That is,

\[
e_i = e_{m+1-i \pmod{m-1}} (\pmod{m-1}) \text{ for } 1 \leq i \leq \frac{1}{2} m.
\]

When \( m \equiv 0 \pmod{4} \), let \( C \) be the following closed trail:

\[
((e_1)_0, (e_2)_1, (e_3)_0, \ldots, (e_{\frac{1}{2} m - 1})_1, (e_{\frac{1}{2} m - 1})_0, (e_{\frac{1}{2} m})_1, (e_{\frac{1}{2} m + 1})_0, \ldots, (e_{m-1})_0, (e_m)_1).
\]
By (1), $C$ can also be written as

$$((e_1)_0, (e_2)_1, \ldots, (e_{\frac{1}{2}m-1})_0, (e_{\frac{1}{2}m})_1, (e_{\frac{1}{2}m})_0, (e_{\frac{1}{2}m-1})_1, \ldots, (e_2)_0, (e_1)_1).$$

The differences used in $C$ are $d_1, d_2, \ldots, d_m.$

Since

$$0 = e_1 < e_3 < \cdots < e_{\frac{1}{2}m-1} = \frac{1}{4}m - 1$$

and

$$m - 2 = m - 1 + e_2 > m - 1 + e_4 > \cdots > m - 1 + e_{\frac{1}{2}m} = \frac{3}{4}m - 1 > 0,$$

it follows that the vertices of $C$ are distinct so that $C$ is an $m$-cycle.

When $m \equiv 0 \pmod{4}$, let $C$ be the following closed trail:

$$((e_1)_0, (e_2)_1, (e_3)_0, \ldots, (e_{\frac{1}{2}m-1})_1, (e_{\frac{1}{2}m})_0, (e_{\frac{1}{2}m+1})_1, (e_{\frac{1}{2}m+2})_0, \ldots, (e_{m-1})_0, (e_{m})_1).$$

By (1), $C$ can also be written as

$$((e_1)_0, (e_2)_1, (e_3)_0, \ldots, (e_{\frac{1}{2}m-1})_1, (e_{\frac{1}{2}m})_0, (e_{\frac{1}{2}m})_1, (e_{\frac{1}{2}m-1})_0, (e_{\frac{1}{2}m-2})_1, \ldots, (e_2)_0, (e_1)_1).$$

The differences used in $C$ are $d_1, d_2, \ldots, d_m.$

As before, it is easy to check that the vertices of $C$ are distinct so that $C$ is an $m$-cycle. Let $T = (C).$ Then $T$ is an $m$-cycle system of $K_{m-1,m-1} + I$ and $C_m|K_{m-1,m-1} + I.$

3. Cycle decomposition of $K_{n,n} + I$ with $\frac{1}{2}m < n < \frac{3}{2}m.$ The main purpose of this section is to prove Theorem 3.4, which considers cycle decomposition of $K_{n,n} + I$ with $\frac{1}{2}m < n < \frac{3}{2}m.$ Lemmas 3.1, 3.2, and 3.3 will be needed in the proof of Theorem 3.4. The following notation will appear in the three lemmas.

For any integer $x$, let

$$\epsilon(x) = \begin{cases} 0 & \text{if } x \equiv 0 \pmod{2}, \\ 1 & \text{if } x \equiv 1 \pmod{2}. \end{cases}$$

**Lemma 3.1.** Let $m$ and $n$ be positive integers with $m \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{2}$, and $\frac{1}{2}m < n < \frac{3}{2}m.$ Let $g = \gcd(m, n) > 1$ and $n = s\frac{m}{g} - 1.$ Let $D = \{2, 3, \ldots, \frac{m}{g}, n + \frac{m}{g} + 1\}$. Then $C_m|X(n; D)$.

**Proof.** Let $V(X(n; D)) = \{i_j | i \in Z_n, j \in Z_2\}.$ Let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ i + 1 & \text{if } 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} + \frac{m}{g} + 1 & \text{if } i = \frac{m}{g}. \end{cases}$$

For $1 \leq i \leq \frac{m}{g},$ let

$$e_i = \begin{cases} 0 & \text{if } i = 0 \pmod{2}, 0 \leq i \leq \frac{m}{g} - 2, \\ -i + \frac{3}{2} & \text{if } i = 1 \pmod{2}, 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Then

$$e_i = \begin{cases} \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{m}{g} - 2, \\ -i + \frac{3}{2} & \text{if } i \equiv 1 \pmod{2}, 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases} \end{cases}$$
Let $P$ be the trail of length $\frac{m}{g}$ given by

$$P = (e_0)_{0}, (e_1)_{1}, (e_2)_{0}, (e_3)_{1}, \ldots, (e_{\frac{m}{2}}-2)_{0}, (e_{\frac{m}{2}}-1)_{1}, (e_{\frac{m}{2}})_{0}.$$  

The differences used in $P$ are $d_1, d_2, d_3, \ldots, d_{\frac{m}{2}}$.

Since

$$0 = e_0 < e_2 < e_4 < \cdots < e_{\frac{m}{2}} = \frac{n}{g}$$

and

$$s \frac{m}{g} - 3 = n + e_1 > n + e_3 > n + e_5 > \cdots > n + e_{\frac{m}{2}} - 1 = s \frac{m}{g} - \frac{m}{2g} - 2,$$

the vertices of $P$ are distinct so that $P$ is a path. Moreover, the first and last vertices of $P$ are the only ones which are congruent modulo $\frac{n}{g}$. It follows that

$$C = P \cup \left( P + \frac{n}{g} \right) \cup \left( P + \frac{2n}{g} \right) \cup \cdots \cup \left( P + \frac{(g-1)n}{g} \right)$$

is a $C_m$.

In $C$, each difference in $D$ occurs exactly $g$ times, and $\{i_0, (i+d)_{1}\}$ incident with edges of difference $d$ are all congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that $T$ is an $m$-cycle system of $X(n; D)$ and $C_m|X(n; D)$.

**Lemma 3.2.** Let $m$ and $n$ be positive integers with $m \equiv 0 \pmod{4}$, $n \equiv 1 \pmod{2}$, and $\frac{1}{2} m < n < \frac{3}{2} m$. Let $g = \gcd (m, n) > 1$ and $n = \frac{s m}{g} - 1$. Let $D_l = \{(l - 1) \frac{m}{g} + 1, (l - 1) \frac{m}{g} + 2, \ldots, l \frac{m}{g} - 1, l \frac{m}{g}, (l - 1) \frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(l)\}\ \{l - 2) \frac{m}{g} + \frac{m}{2g} + \varepsilon(l) + 1\}$ for $2 \leq l \leq s$. Then $C_m|X(n; D_l)$.

**Proof.** Let $V(X(n; D_l)) = \{i, j \in Z_n, j \in Z_2\}$. For $l \equiv 0 \pmod{2}$, let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ (l - 1) \frac{m}{g} + i & \text{if } 1 \leq i < \frac{n}{g} - \frac{m}{2g} + 1, \\ (l - 1) \frac{m}{g} + i + 1 & \text{if } \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ (l - 1) \frac{m}{g} + \frac{m}{2g} + \frac{m}{2g} & \text{if } i = \frac{m}{g}. \end{cases}$$

For $1 \leq i \leq \frac{m}{g}$, let

$$e_i = \begin{cases} e_0 = 0, \\ e_{i-1} + (-1)^d_i. \end{cases}$$

Then

$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ -(l - 1) \frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ \frac{i}{2} + 1 & \text{if } i \equiv 0 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ -(l - 1) \frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let $P$ be the trail of length $\frac{m}{g}$ given by

$$P = (e_0)_{0}, (e_1)_{1}, (e_2)_{0}, (e_3)_{1}, \ldots, (e_{\frac{m}{2}}-2)_{0}, (e_{\frac{m}{2}}-1)_{1}, (e_{\frac{m}{2}})_{0}.$$
The differences used in $P$ are $d_1, d_2, d_3, \ldots, d_m$.

Since
$$0 = e_0 < e_2 < e_4 < \cdots < e_m = \frac{n}{g}$$
and
$$(s - l + 1)\frac{m}{g} - 2 = n + e_1 > n + e_3 > n + e_5 > \cdots > n + e_m = (s - l + 1)\frac{m}{g} - \frac{m}{2g} - 2,$$
the vertices of $P$ are distinct so that $P$ is a path. Moreover, the first and last vertices are the only ones which are congruent modulo $\frac{n}{g}$. It follows that
$$C = P \cup \left( P + \frac{n}{g} \right) \cup \left( P + \frac{2n}{g} \right) \cup \cdots \cup \left( P + \frac{(g - 1)n}{g} \right)$$
is a $C_m$.

In $C$, each difference in $D$ occurs exactly $g$ times, and $\{i_0, (i + d)_{1}\}$ incident with edges of difference $d$ are congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that $T$ is an $m$-cycle system of $X(n; D_l)$ and $C_m \mid X(n; D_l)$ for $l$ even.

For $l \equiv 1 \pmod{2}$, let
$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ (l - 1)\frac{m}{g} + i & \text{if } 1 \leq i < \frac{n}{g} - \frac{m}{2g}, \\ (l - 1)\frac{m}{g} + i + 1 & \text{if } \frac{n}{g} - \frac{m}{2g} \leq i \leq \frac{n}{g} - 1, \\ (l - 1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 & \text{if } i = \frac{m}{g}. \end{cases}$$

Then
$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i < \frac{n}{g} - \frac{m}{2g} - 1, \\ -(l - 1)\frac{m}{g} - \frac{i + 1}{2} & \text{if } i \equiv 1 \pmod{2}, 0 \leq i < \frac{n}{g} - \frac{m}{2g} - 1, \\ \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, \frac{n}{g} - \frac{m}{2g} \leq i \leq \frac{m}{g} - 1, \\ -(l - 1)\frac{m}{g} - \frac{i + 3}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{n}{g} - \frac{m}{2g} \leq i \leq \frac{m}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let $P$ be the trail of length $\frac{m}{g}$ given by
$$P = (e_0)_0, (e_1)_1, (e_2)_0, (e_3)_1, \ldots, (e_{m-2})_0, (e_{m-1})_1, (e_m)_0.$$

The differences used in $P$ are $d_1, d_2, d_3, \ldots, d_m$.

Since
$$0 = e_0 < e_2 < e_4 < \cdots < e_m = \frac{n}{g}$$
and
$$(s - l + 1)\frac{m}{g} - 2 = n + e_1 > n + e_3 > n + e_5 > \cdots > n + e_m = (s - l + 1)\frac{m}{g} - \frac{m}{2g} - 2,$$
the vertices of $P$ are distinct so that $P$ is a path. Moreover, the first and last vertices are the only ones which are congruent modulo $\frac{n}{g}$. It follows that
$$C = P \cup \left( P + \frac{n}{g} \right) \cup \left( P + \frac{2n}{g} \right) \cup \cdots \cup \left( P + \frac{(g - 1)n}{g} \right)$$
is a $C_m$.

In $C$, each difference in $D$ occurs exactly $g$ times, and \{i_0, (i + d)_1\} incident with edges of difference $d$ are congruent modulo $\frac{m}{g}$. Let $T = (C)$. It follows that $T$ is an $m$-cycle system of $X(n; D_l)$ and $C_m | X(n; D_l)$ for $l$ odd. \hfill \Box

**Lemma 3.3.** Let $m$ and $n$ be positive integers with $m \equiv 2 \pmod{4}$, $n \equiv 1 \pmod{2}$, and $\frac{1}{2}m < n < \frac{3}{2}m$. Let $g = \gcd (m, n) > 1$ and $n = s\frac{m}{g} - 1$. Let $D_l = \{(l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \ldots, l\frac{m}{g} - 1, l\frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1\} \setminus \{(l-2)\frac{m}{g} + \frac{m}{2g} + \frac{n}{g} + 1\}$ for $2 \leq l \leq s$. Then $C_m | X(n; D_l)$.

**Proof.** Let $V(X(n; D)) = \{i| i \in Z_n, j \in Z_2\}$ and let

$$d_i = \begin{cases} 0 & \text{if } i = 0, \\ (l-1)\frac{m}{g} + i & \text{if } 1 \leq i < \frac{n}{g} - \frac{m}{2g} + 1, \\ (l-1)\frac{m}{g} + i + 1 & \text{if } \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{n}{g} - 1, \\ (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 & \text{if } i = \frac{m}{g}. \end{cases}$$

For $1 \leq i \leq \frac{m}{g}$, let

$$e_i = \begin{cases} e_0 = 0, \\ e_i = e_{i-1} + (-1)^i d_i. \end{cases}$$

Then

$$e_i = \begin{cases} \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ -(l-1)\frac{m}{g} - \frac{i+1}{2} & \text{if } i \equiv 1 \pmod{2}, 0 \leq i \leq \frac{n}{g} - \frac{m}{2g}, \\ \frac{i}{2} & \text{if } i \equiv 0 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{n}{g} - 1, \\ -(l-1)\frac{m}{g} - \frac{i+3}{2} & \text{if } i \equiv 1 \pmod{2}, \frac{n}{g} - \frac{m}{2g} + 1 \leq i \leq \frac{n}{g} - 1, \\ \frac{n}{g} & \text{if } i = \frac{m}{g}. \end{cases}$$

Let $P$ be the trail of length $\frac{m}{g}$ given by

$$P = (e_0), (e_1), (e_2), \ldots, (e_{\frac{m}{g} - 2}), (e_{\frac{m}{g} - 1}), (e_{\frac{m}{g}}).$$

The differences used in $P$ are $d_1, d_2, d_3, \ldots, d_{\frac{m}{g}}$.

Since

$$0 = e_0 < e_2 < e_4 < \cdots < e_{\frac{m}{g}} = \frac{n}{g}$$

and

$$(s-l+1)\frac{m}{g} - 2 = n + e_1 > n + e_3 > n + e_5 > \cdots > n + e_{\frac{m}{g} - 1} = (s-l+1)\frac{m}{g} - \frac{m}{2g} - 2,$$

the vertices of $P$ are distinct so that $P$ is a path. Moreover, the first and last vertices are the only ones which are congruent modulo $\frac{n}{g}$. It follows that

$$C = P \cup \left( P + \frac{n}{g} \right) \cup \left( P + \frac{2n}{g} \right) \cup \cdots \cup \left( P + \frac{(g-1)n}{g} \right)$$

is a $C_m$. In $C$, each difference in $D$ occurs exactly $g$ times, and \{i_0, (i + d)_1\} incident with edges of difference $d$ are congruent modulo $\frac{n}{g}$. Let $T = (C)$. It follows that $T$ is an $m$-cycle system of $X(n; D_l)$ and $C_m | X(n; D_l)$. \hfill \Box
With the above preparations, we now prove the following theorem.

**Theorem 3.4.** Let \( m \) be an even integer and \( n \) be an odd integer with \( \frac{1}{2}m < n < \frac{3}{2}m \). Then there exists an \( m \)-cycle system of \( K_{n,m} + I \) if and only if \( m \) divides \( n^2 + n \).

**Proof.** The necessity is similar to that in Lemma 1.6; here we consider only the sufficiency. Let \( g = \gcd (m, n) \). If \( g = 1 \), then since \( n(n+1) \equiv 0 \pmod{m} \) and \( n + 1 < 2m \), we have \( n = m - 1 \). By Lemma 2.3, there exists an \( m \)-cycle system of \( K_{m-1,m-1} + I \).

If \( n \neq m - 1 \), then \( g > 1 \). Since \( n(n+1) \equiv 0 \pmod{m} \), we have \( n + 1 = \frac{m^2}{g} \).

When \( m \equiv 0 \pmod{4} \), let

\[
I = \left\{ i_0, \left( i + (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(s) \right)_1 \mid i \in Z_n \right\}.
\]

We can put an additional difference \((s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(s)\) on \( Z_n \). Then

\[
Z_n \uplus \left\{ (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(s) \right\} = \bigcup_{l=1}^{s-1} D_l \uplus D_s
\]

where

\[
D_1 = \left\{ 2, \ldots, \frac{m}{g} \cdot \frac{n}{g} + \frac{m}{2g} + 1 \right\}
\]

and

\[
D_l = \left\{ (l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \ldots, \frac{m}{g} - 1, \frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(l) \right\}
\]

\[
\setminus \left\{ (l-2)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + \varepsilon(l+1) \right\}
\]

for \( 2 \leq l \leq s \).

By Lemmas 3.1 and 3.2, there exists an \( m \)-cycle system of \( K_{n,n} + I \). This completes this case.

When \( m \equiv 2 \pmod{4} \), let

\[
I = \left\{ i_0, \left( i + (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right)_1 \mid i \in Z_n \right\}.
\]

We can put an additional difference \((s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1\) on \( Z_n \). Then

\[
Z_n \uplus \left\{ (s-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right\} = \bigcup_{l=1}^{s-1} D_l \uplus D_s
\]

where

\[
D_1 = \left\{ 2, \ldots, \frac{m}{g} \cdot \frac{n}{g} + \frac{m}{2g} + 1 \right\}
\]

and

\[
D_l = \left\{ (l-1)\frac{m}{g} + 1, (l-1)\frac{m}{g} + 2, \ldots, \frac{m}{g} - 1, \frac{m}{g}, (l-1)\frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1 \right\}
\]
\[
\left\lceil \left(\begin{array}{c}
(l-2) \frac{m}{g} + \frac{n}{g} + \frac{m}{2g} + 1
\end{array}\right) \right\rceil
\]

for \(2 \leq l \leq s\).

By Lemmas 3.1 and 3.3, there exists an \(m\)-cycle system of \(K_{n,n} + I\). This completes the proof. \(\square\)

4. Main result. Now we are in position to prove the main theorem of this paper.

**Theorem 4.1.** Let \(m\) be an even integer and \(n\) be an odd integer with \(4 \leq m \leq 2n\). Then \(K_{n,n} + I\) can be decomposed into cycles of length \(m\) if and only if \(m\) divides \(n^2 + n\).

**Proof.** The necessity can be found in Lemma 1.6; we need only prove the sufficiency. Let \(n = qm + r\), where \(q\) is a positive integer and \(\frac{1}{2}m \leq r < \frac{3}{2}m\). Let \(V(K_{n,n}) = \{v_1, v_2, \ldots, v_{qm+r}\} \cup \{u_1, u_2, \ldots, u_{qm+r}\}\), \(V_i = \{v_{(i-1)m+j}| 1 \leq j \leq m\}\), and \(U_i = \{u_{(i-1)m+j}| 1 \leq j \leq m\}\) for \(1 \leq i \leq q\). Let \(V_{q+1} = \{v_{qm+j}| 1 \leq j \leq r\}\), \(U_{q+1} = \{u_{qm+j}| 1 \leq j \leq r\}\), and \(I = \{u_iv_i| 1 \leq i \leq n\}\).

Let \(H_{i,i}\) be a subgraph of \(K_{m-1,m-1} + I\) induced by \((V_i \setminus \{v_{(i-1)m+1}\}) \cup (U_i \setminus \{u_{(i-1)m+1}\})\) for \(1 \leq i \leq q\). Then \(H_{i,i} = K_{m-1,m-1} + I_{i,i}\), where
\[
I_{i,i} = \{v_{(i-1)m+r}v_{(i-1)m+r}| 2 \leq r \leq m\}.
\]

By Lemma 2.3, \(C_m|H_{i,i}\) for \(1 \leq i \leq q\). Let \(T_{i,j}\) be the \(m\)-cycle system of \(H_{i,j}\).

Let \(H_{i,j}\) be a subgraph of \(K_{m,m}\) induced by \(V_i \cup U_j\), where \(1 \leq i, j \leq q\) and \(i \neq j\). Then \(H_{i,j} = K_{m,m}\). By Theorem 1.2, \(C_m|H_{i,j}\). Let \(T_{i,j}\) be the \(m\)-cycle system of \(H_{i,j}\).

Let \(H_{i+1,1}\) be a subgraph of \(K_{r+1,m}\) induced by \((V_{q+1} \cup \{v_{(i-1)m+1}\}) \cup U_i\) for \(1 \leq i \leq q\). Then \(H_{i+1,1} = K_{r+1,m}\). By Theorem 1.2, \(C_m|H_{i+1,1}\) for \(1 \leq i \leq q\). Let \(T_{i+1,1}\) be the \(m\)-cycle system of \(H_{i+1,1}\).

Let \(H_{m+r+1}\) be a subgraph of \(K_{m,r+1}\) induced by \(V_i \cup (U_{q+1} \cup \{u_{(i-1)m+1}\})\), where \(1 \leq i \leq q\). Then \(H_{m+r+1} = K_{m,r+1}\). By Theorem 1.2, \(C_m|H_{m+r+1}\) for \(1 \leq i \leq q\). Let \(T_{m+r+1}\) be the \(m\)-cycle system of \(H_{m+r+1}\).

Let \(H_{r,r}\) be a subgraph of \(K_{r+r+1}\) induced by \(V_{q+1} \cup U_{q+1}\). Then \(H_{r,r} = K_{r,r} + I_{r,r}\), where \(I_{r,r} = \{u_{qm+j}v_{qm+j}| 1 \leq j \leq r\}\). By Theorem 1.2, \(C_m|H_{r,r}\). Let \(T_{r,r}\) be the \(m\)-cycle system of \(H_{r,r}\).

Let
\[
T = \bigcup_{1 \leq i,j \leq q} T_{i,j} \bigcup \left( T_{m+r+1} \cup T_{r+1,1} \right) \bigcup T_{r,r}.
\]

Then \(T\) is an \(m\)-cycle system of \(K_{n,n} + I\). This concludes the proof. \(\square\)

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**REFERENCES**


