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Author(s)	Cheng, Nian-Sheng
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EXPONENTIAL FORMULA FOR BEDLOAD TRANSPORT

Nian-Sheng Cheng[1]

Abstract: An exponential formula that does not involve the concept of the critical shear stress is derived in this study for computing bedload transport rates. The formula represents well various experimental data sets ranging from the weak transport to high shear conditions. Comparisons of the present study are also made with many previous bedload formulas commonly cited in the literature.

Keywords: Bedload, sediment transport, transport rate, shear stress, critical shear stress, weak transport, high shear.

INTRODUCTION

Most of the formulas available in the literature for computing bedload transport rates can be expressed in terms of shear stress excess or its equivalents. Typical examples are those proposed by Meyer-Peter and Mueller (1948) and Yalin (1977), respectively, which include the term of $(\Theta - \Theta_C)$, where $\Theta = u_*^2/[(\rho_S/\rho - 1)gD]$ = the dimensionless shear stress, u_* = shear velocity, ρ_S = density of particle, ρ = density of fluid, g = gravitational acceleration, D = diameter of particle, and Θ_C is its corresponding critical value at the threshold condition of bed particles. Another example is the theoretical expression derived by Bagnold (1973) who reported that the transport rate was proportional to $\left(\sqrt{\Theta} - \sqrt{\Theta_C}\right)$. Obviously, all the formulas of this kind can be used to predict bedload transport rates only for $\Theta > \Theta_C$ or the conditions of moderate to high shear stresses.

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[1]Asst. Professor, Environmental Engineering Research Centre, School of Civil and Structural Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798. cnscheng@ntu.edu.sg

On the other hand, for practical purposes, sediment transport rates can also be formulated to be proportional to Θ^n, where the exponent n was found to vary from 1.5 to 18 (Lavelle and Mofjeld 1987). Generally, the n-value is large for the weak transport, but being reduced for the moderate to high shear conditions. For example, Paintal (1971) found the Einstein's dimensionless transport rate, Φ, to be related to Θ as follows

$$\Phi = 5.56 \times 10^{18} \Theta^{16} \tag{1}$$

for $\Theta = 0.007 \sim 0.07$, whereas Lavelle and Mofjeld (1987) reported that n had a value of 5.1 for $\Theta = 0.13 \sim 1.3$. For high shear (say, $\Theta > 1$), a common value of n = 1.5 can be derived by simplifying several bedload formulas for the limiting condition of $\Theta \gg \Theta_C$ (Chien and Wan 1999). In comparison with the formulas based on the bed shear excess, the relationship of $\Phi \sim \Theta^n$ is especially preferable for low transport rates. Due to the limitation of the concept of the critical shear stress, a shear-excess based formula that works well for $\Theta > \Theta_C$ may not be applicable for weak transport, which usually occurs under the condition of $\Theta < \Theta_C$.

There is no doubt that the concept of critical shear stress is helpful in some practical situations. However, experimental observations of sediment transport such as those conducted by Paintal (1971) indicated that there is no shear stress below which no single grain moves. At very low shear stresses, the possibility of movement becomes very small but never equals zero, so one has to wait for a longer time to observe the movement. For example, a low sediment discharge that was approximately five orders of magnitude less than the so-called threshold condition could be measured only when the sediment flux was allowed to accumulate for a period up to 70 hours (Paintal 1971). More relevant information in this respect can be found in Lavelle and Mofjeld (1987), and also in Buffington (1999) who stated that many stress-transport relationships are power functions with transport rates approaching zero only when shear stress goes to zero.

This study was motivated by the fact that various relationships of $\Phi \sim \Theta^n$ with different n-values can be applied for the conditions from low to high shear stresses. The objective is to explore the possibility of extending some bedload formulas, which work well for the moderate or high shear conditions, to the situation of weak transport for which the previous relationships are usually empirical. An exponential formula, which is not subject to uncertainties inherent in the critical shear stress, is therefore developed in this study for computing the bedload transport rates for the conditions from low to high shear. Comparisons are then made between the relationship obtained in the present study and several well-known data sets available in the literature in addition to many commonly used bedload formulas.

DERIVATION

It is known that for large bed shear stresses, the bedload transport rate, q_b, is proportional to the product of the bed shear stress, τ_b, and shear velocity, u_*, i.e., $q_b \sim \tau_b u_*$ (Yalin 1977). The product, $\tau_b u_*$, represents energy of the flow near the bed, which is closely related to motion of the near-bed particles. With the characteristics of fluid and particles, q_b can be normalised as the Einstein's dimensionless transport rate,

$$\Phi = \frac{q_b}{D\sqrt{(s-1)gD}} \tag{2}$$

where q_b = volumetric transport rate, and $\tau_b u_*$ is normalised as

$$\Omega = \frac{\tau_b u_*}{\rho[(s-1)gD]^{3/2}} \tag{3}$$

Therefore, the proportional relationship of $q_b \sim \tau_b u_*$ can simply be expressed as

$$\Phi = C\Omega \tag{4}$$

where C = constant. This linear relationship is consistent with many bedload transport formulas for the case of high shear, even though its physics remains unclear. For example, the formulas developed, respectively, by Meyer-Peter and Mueller (1948), Englund and Fredsoe (1976), and Yalin (1977) can be readily simplified in the form of (4) for the limiting condition of $\Theta \gg \Theta_C$. Further information can be found in Yalin (1977) and Chien and Wan (1999). A summary of various simplified formulas for $\Theta \gg \Theta_C$ is given in Table 1.

As shown in Table 1, the C-value varies from 8 to 19.3. This is because these formulas have been derived largely based on the experimental results for the moderate shear. In this study, the experimental data obtained by Wilson (1966) for sand transported at high shear ($\Theta > 1$) are used to compute the C-value. In Wilson's (1966) experiments, high-shear flows were generated with a pressurized pipe system, and a moving bed layer with no bed forms, which was composed of sand (specific gravity =2.67, diameter = 0.7 mm), was thus obtained. The computation shows that the average C-value is equal to approximately 13, as demonstrated in Fig. 1. The data points are plotted as Φ versus Ω on the linear scales rather than the logarithmic coordinates. It is noted that the C-value obtained is larger than 12.1, which was chosen by Wilson (1966) as a coefficient in his modified Meyer-Peter-Mueller formula.

For the case of moderate shear, the Φ-Ω relationship is not linear. This is shown in Fig. 2, where the relationship is plotted with the data points reproduced from Chien and Wan (1999). The data included were originally collected by Gilbert (1914) and Meyer-Peter and Mueller (1948), respectively. Also superimposed in this figure is the linear relationship (4) for the high shear condition. Fig. 2 shows that Φ increases rapidly with increasing Ω for $\Omega = 0.01~0.1$ with (4) as an asymptotic for larger Ω-values. In other words, the slope of the trend-line of the experimental data, when plotted on the logarithmic scale, decreases with increasing Ω for small Ω-values, and approaches 1.0 if Ω is further increased. Therefore, as a first approximation, it can be assumed that

$$\frac{d\ln\Phi}{d\ln\Omega} = 1 + \frac{\alpha}{\Omega} \tag{5}$$

Integrating (5) with respect to Ω leads to the following exponential function,

$$\Phi = \beta\,\Omega\,exp\left(-\frac{\alpha}{\Omega}\right) \tag{6}$$

where α, β = constants. Eq. (6) shows that Φ is approximately equal to $\beta\Omega$ if Ω is large, implying $\beta = C = 13$ as compared to (4). The α-value can be evaluated by comparing (6) to the experimental data of Gilbert (1914) and Meyer-Peter and Mueller (1948). This yields that $\alpha \approx 0.05$. Furthermore, since $\Omega = \Theta^{1.5}$, (6) can be re-written in the form of $\Phi \sim \Theta$:

$$\Phi = 13\,\Theta^{1.5}\,exp\left(-\frac{0.05}{\Theta^{1.5}}\right) \tag{7}$$

COMPARISON WITH PREVIOUS FORMULAS FOR MODERATE TO HIGH SHEAR CONDITIONS

Fig. 3 shows that (7) represents the data points well, as expected, and also agrees with the formulas of Meyer-Peter and Mueller (1948) and Einstein (1950) for the condition of moderate shear. For the purposes of clarify, the same data sets are also plotted in Fig. 4 for comparing (7) with the formulas proposed by Bagnold (1973) and Yalin (1977). Fig. 4 displays that (7) is relatively closer to the Bagnold's (1973) formula than to that of Yalin (1977) except for the high shear condition. In Figs. 3 and 4, Chien and Wan (1999)'s approach is adopted for presenting the four previous formulas consistently in the form of Ψ (= $1/\Theta$) against Φ. This includes that the dimensionless critical shear stress is set as that implied by the formula of Meyer-Peter and Mueller (1948), i.e., $\Theta_C = 0.047$, and the dynamic friction factor associated with Bagnold's (1973) formula is taken as 0.63.

COMPARISON WITH EMPIRICAL RELATIONSHIPS FOR WEAK TRANSPORT

As discussed previously, most of the bedload formulas cannot be used to predict the weak transport under the condition of low shear. This is because they are often expressed in terms of bed shear excess such as $(\Theta - \Theta_C)$ or $\left(\sqrt{\Theta} - \sqrt{\Theta_C}\right)$. For the weak transport, the bed shear stress is very close to or even lower than the critical shear stress. For example, the Meyer-Peter and Mueller formula predicts negative transport rates for $\Theta < 0.047$. Therefore, for the weak transport, the present study can only be compared to relationships which are not associated with the threshold condition. Such relationships available in the literature include (1) given by Paintal (1971), and the following relationship suggested by Einstein (1942) for $\Theta < 0.2$:

$$\Phi = 2.15F \, exp\left(-\frac{0.391}{\Theta}\right) \tag{8}$$

where $F = 0.816$ if the viscous effect is ignored. As shown in Fig. 5, (7) is very close to the two empirical relationships, (1) and (8), for $\Phi = 10^{-8} \sim 10^{-4}$. It is noted that the two empirical relationships are not applicable for higher transport rates, and thus deviate obviously from (7) for $\Phi > 0.01$. For further comparison purposes, the experimental data obtained by Paintal (1971) and Taylor and Vanoni (1972) are also plotted as the symbols in the figure. Paintal (1971) conducted his experiments in a titling flume using three kinds of sand with diameters of 2.5 mm, 7.95 mm and 22.5 mm especially for collecting the bedload transport data at low shear conditions. Paintal's (1971) data were obtained mainly under the hydrodynamically rough flow conditions, whereas Taylor and Vanoni's (1972) experimental measurements were limited to small Reynolds numbers ($u_*D/v = 2 \sim 200$) and thus could be affected by the viscosity of fluid. However, as shown in Fig. 5, the two data sets do not differ significantly

regardless of possible effects of the Reynolds number and can generally be represented by the two empirical relationships, (1) and (8), and the exponential bedload formula derived in this study, (7).

CONCLUSIONS

A simple exponential formula is derived for computing bedload transport rates for the conditions of low to high shear stresses. It can represent well the well-known experimental data sets including those obtained by Gilbert (1914), Meyer-Peter and Mueller (1948), Wilson (1966), and Paintal (1971), respectively. For the moderate shear, the formula is very close to those proposed previously by Einstein (1950) and Meyer-Peter and Mueller (1948), respectively, and for the weak transport, it agrees well with the empirical relationships obtained by Einstein (1942) and Paintal (1971), respectively.

APPENDIX I: REFERENCES

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APPENDIX II: NOTATION

The following symbols are used in this paper:

a   = coefficient;

D   = diameter of particle;

F   = coefficient;

g   = gravitational acceleration;

k   = dynamic friction factor;

m   = coefficient;

n   = exponent;

q_b = volumetric bedload transport rate;

s   = Θ/Θ_C - 1;

u_* = shear velocity;

w   = settling velocity;

α = constant;

β = constant;

Θ = $u_*^2/[(s-1)gD]$ = dimensionless shear stress;

Θ_C = dimensionless critical shear stress;

ν = viscosity;

ρ = density of fluid;

ρ_S = density of particle;

τ_b = bed shear stress;

Φ $= \dfrac{q_b}{D\sqrt{(s-1)gD}}$ = Einstein's dimensionless transport rate;

Ψ = $1/\Theta$; and

$$\Omega = \frac{\tau_b u_*}{\rho\left[(s-1)gD\right]^{3/2}} \, .$$

Captions for Figures

Fig. 1. Linear relationship of Φ and Ω for high shear conditions

Fig. 2. Transport rates increase rapidly with increasing Ω for small Ω-values

Fig. 3. Comparisons with formulas derived by Meyer-Peter and Mueller (1948) and Einstein (1950)

Fig. 4. Comparisons with formulas derived by Bagnold (1973) and Yalin (1977)

Fig. 5. Comparisons with experimental data and empirical relationships for weak transport

Table 1 Simplified Bedload Formulas for the Condition of High Shear.

Authors	$\Phi = f(\Theta)$	Simplified formulas	Notes
Meyer-Peter and Mueller (1948)	$\Phi = 8(\Theta - \Theta_C)$	$\Phi = 8\Omega$	$\Theta_C = 0.047$
Bagnold (1973)	$\Phi = \Theta\left(\sqrt{\Theta} - \sqrt{\Theta_c}\right)\dfrac{1}{k}\left[5.75\,log\left(30.2\dfrac{mD}{K_S}\right) - \dfrac{w}{u_*}\right]$	$\Phi = (13.2{\sim}19.3)\Omega$	$k = 0.63$, $K_S/D = 1$, $m = 1.4(\Theta/\Theta_C)^{0.3}$, $w/u_* = 4.5(\Theta_C/\Theta)^{0.5}$ (for $D > 0.7$mm).
Englund and Fredsoe (1976)	$\Phi = 9.3k^{-1}\left(\Theta - \Theta_C\right)\left(\sqrt{\Theta} - 0.7\sqrt{\Theta_C}\right)$	$\Phi = 11.6\Omega$	$k = 0.8$
Yalin (1977)	$\Phi = 0.635s\sqrt{\Theta}\left[1 - \dfrac{1}{as}\,ln(1 + as)\right]$	$\Phi = 13.5\Omega$	$\rho_S/\rho = 2.65$, $s = \Theta/\Theta_C - 1$, $a = 1.66\sqrt{\Theta_C}$