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# **Analysis of bedload transport in laminar flows**

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## **ABSTRACT**

Bedload transport generally depends on the bed shear stress and Reynolds number. Many studies conducted for the condition of turbulent flows have revealed the dependence of the transport rate on the bed shear stress, while knowledge of the Reynolds number effect on the transport rate is very limited. As an extreme case to reflect the viscous effect on sediment transport, sediment transport in laminar flows is considered in this paper. A stochastic approach is adopted to explore how the transport rate can be associated with characteristics of laminar flows. First, the probability of erosion in the absence of turbulence is assumed to depend only on the randomness of bed particles. The probability is then applied to formulate the sediment transport rate, of which the derivation is made largely based on Einstein's bedload theory. The theoretical result indicates that the dimensionless transport rate for laminar flows is dependent on the dimensionless shear stress and dimensionless particle diameter or the shear Reynolds number. Comparisons are finally made between the derived formula and an empirical correlation available in the literature.

## **Keywords**

Bedload; Sediment transport; Laminar flow; Shear stress; Probability of erosion; Reynolds number; Viscous effect

## 1. INTRODUCTION

Sediment transport in rivers and coastal waters usually occurs under turbulent flow conditions, so it is not surprising that most previous studies on bedload transport rates are limited to cases where the turbulent effect is dominant. As a consequence, almost all the formulas available in the literature for computing bedload transport rates relate the dimensionless bedload transport rate,  $\phi$ , to the dimensionless shear stress or the Shields number,  $\tau_*$ , only [6,18]. Herein,  $\phi = q_V / [D\sqrt{\Delta g D}]$  and  $\tau_* = u_*^2 / (\Delta g D)$ , where  $q_V$  is the volumetric sediment transport rate per unit width;  $D$  the particle diameter;  $g$  the gravitational acceleration;  $u_*$  the shear velocity;  $\Delta$  the  $(\rho_s - \rho) / \rho$ ,  $\rho_s$  the particle density and  $\rho$  the fluid density. However, the bed sediment can be significantly subjected to the viscous effect of fluid, for example, for the case of fine-grained bed sediment. Theoretically, the viscous effect on the near-bed flow structure cannot be ignored if the shear Reynolds number is small, say,  $Re_* = u_* D / \nu < 70$ , where  $\nu$  is the kinematic viscosity of fluid. Moreover, from dimensional analysis, it can be deduced that  $\phi$  should depend on both  $\tau_*$  and  $Re_*$  [9,18]. This implies that the existing formulas may be applicable only for hydraulically rough boundary conditions. Therefore, it should be much more important to investigate how to include the shear Reynolds number in computing bedload transport rates than to develop another relationship between  $\phi$  and  $\tau_*$  [18].

To single out the viscous effect on transport rates, an extreme example to be considered is sediment transport in laminar flows. Such a study may not be directly related to practical problems in hydraulic engineering, but results to be obtained could be extended for predicting sediment transport over hydraulically smooth beds, where the fluid forces are transferred to the bed sediment primarily by the viscous shear [10]. In addition, the current understanding of complex bedform mechanics could be improved by examining the process of bed-form initiation in laminar flows, which is relatively simple but still similar to the case in the presence of turbulence [7]. Other relevant topics of interest may include laminar overland flows [12,14].

In spite of a number of studies on the bedload transport which have been conducted under turbulent conditions, only very few works can be found in the literature which are purposely concerned with transport phenomena in laminar flows. Girgis [9] experimentally investigated sediment transport over a flat bed in the laminar flow for a range of water temperatures (4 °C, 17 °C, 30 °C). The laminar flow with low Reynolds number but high shear stress was generated by placing a flat plate 2mm over a flat sediment bed, which comprised of fine quartz sand with the mean sieve diameter of 0.143mm. Sand samples were taken over the middle part of the flat sand bed, in a strip 137mm wide, in order to eliminate possible errors produced by the side boundaries of the channel. Short sampling times were used in the case of high bed stresses to avoid considerable erosion of the sand bed. Other information related to this experiment can also be found in [10]. By applying the linear regression analysis to non-dimensionalised experimental data, Girgis [9] obtained the following empirical expression:

$$q_* = 1.42\tau_*^{2.98} Re_*^{1.97} \quad (1)$$

where  $q_* = \rho_* q_V / (\Delta \rho D u_*)$  is the normalised transport rate.

The critical condition for the initiation of sediment motion is a preliminary but important component for describing the mechanism of sediment transport. The initial motion under laminar flow conditions has been observed by Yalin and Karahan [16] and Pilotti and Menduni [15]. Yalin and Karahan conducted their experiments in a recirculating flume with a glycerine-water mixture flowing over a sediment bed, which was prepared with sieved river sand of 0.56–2.85mm in diameter. They reported that the critical shear stress corresponding to laminar and turbulent flows could not be determined by a unique relation. However, their experimental results of the critical shear stress obtained under laminar conditions show no clear differences from those for turbulent flows in the case of  $Re_* < 1$ . A similar attempt was recently made by Pilotti and Menduni [15], whose experiments were performed with a water-

glucose solution recirculated in a tilting flume 2m long and 15cm wide. Their findings are very consistent with those reported by Yalin and Karahan [16].

Sediment transport consists of stochastic events, of which random characteristics generally result from those related to sediment particles and those induced by flow turbulence. The random characteristics of the flow are important when the flow is turbulent, while the randomness of bed particles is always present, being independent of flow conditions. A bed particle can be different from others because of its shape, size and position on the bed. For the case of laminar flows, the random characteristics of the sediment bed become a dominant factor affecting sediment transport. This may also be true to some extent for sediment transport over hydraulically smooth beds.

Use of stochastic approaches to investigate sediment transport was pioneered by Einstein [8], being subsequently followed by Paintal [13], He and Han [11], etc. Einstein postulated the probability of particle movement as the probability of lift exceeding the submerged weight of the particle, leading to the transport function being related to characteristics of turbulent flows. In comparison, the randomness of bed particles was additionally examined by Paintal [13] and He and Han [11] by including the effect of the exposure of bed particles.

This study aims to apply the stochastic method to bedload transport in laminar flows. It is assumed that the probability of the bed particle instability in the absence of turbulence is solely due to the randomness associated with the geometric configuration of bed particles.

## **2. DERIVATION**

Einstein's consideration [8] on bedload transport is a ground-breaking framework to statistically describe sediment motion. His bedload function was derived based on several assumptions, which have been assessed critically by several investigators [2,5,11,18]. However, the basic equation is often considered to be acceptable, which

was originally proposed by Einstein to relate the sediment transport rate to the probability of erosion, the diameter of particle and a characteristic time. This relationship can be expressed as

$$q_v = a \frac{p}{1-p} \frac{D^2}{t} \quad (2)$$

where  $q_v$  is the bedload transport rate per unit width measured in volume;  $a$  the coefficient;  $p$  the probability of a particle to be dislodged or probability of erosion;  $D$  the diameter of particles, and  $t$  the time of exchange between bedload and bed material. Eq. (2) was derived for the equilibrium condition of the rate of erosion being equal to the rate of deposition per unit area of bed, so it can be used, in principle, for computing bedload transport rates for both turbulent and laminar flows. However, it should be noted that further substantiation of Eq. (2) for the present study depends on how to evaluate the parameters  $p$  and  $t$  for the condition of laminar flows.

### **2.1. Probability of erosion, $p$**

The probability of erosion of bed particles generally depends on near-bed flow conditions and characteristics of the particle packing on the bed surface. In the presence of intensive turbulence, bed particles may saltate while detaching from their positions of rest. For this case, the probability of erosion can be reasonably defined as that of the hydrodynamic lift force exerted on a particle being greater than its submerged weight. By assuming that the lift force follows a Gaussian distribution, Einstein [8] attained a theoretical expression for the probability of erosion, leading to his well-known bed-load function for equilibrium sediment beds. However, Einstein did not consider explicitly the effect of the bed randomness on the probability of erosion.

For laminar flows, turbulent disturbances disappear and the instability of bed particles is mainly associated with their geometric randomness. This problem has been recently addressed by Cheng et al. [3], who describe the instability of each individual bed particle based on the concept of effective shear stress. The effective

shear stress,  $\tau_e$ , is defined as a minimum shear stress required for a particle to move, varying randomly over the bed solely because of the randomness of bed particles. Therefore, the probability of erosion can be expressed as

$$p = p(\tau_e < \tau) = \int_0^{\tau} f(\tau_e) d\tau_e \quad (3)$$

where  $\tau$  is the bed shear stress exerted by the flow, and  $f(\tau_e)$  the probability density function of  $\tau_e$ . For a flat bed comprised of uniform particles, the variation of  $\tau_e$  can be demonstrated to be a narrow-banded random process with amplitudes equal to the magnitudes of the effective shear stress [3]. This result subsequently enables a theoretical derivation, showing that  $f(\tau_e)$  follows the Rayleigh distribution, i.e.

$$f(\tau_e) = \frac{\pi}{2} \frac{\tau_e}{\overline{\tau_e}^2} \exp \left[ -\frac{\pi}{4} \left( \frac{\tau_e}{\overline{\tau_e}} \right)^2 \right] \quad (4)$$

where  $\overline{\tau_e}$  is the average value of the effective shear stress. Substituting Eq. (4) into Eq. (3) gives

$$p = 1 - \exp \left[ -\frac{\pi}{4} \left( \frac{\tau}{\overline{\tau_e}} \right)^2 \right] \quad (5)$$

Eq. (5) can be used for computing the probability of erosion provided that  $\overline{\tau_e}$  is known. Unfortunately, the information on  $\overline{\tau_e}$  is lacking in the literature. As an approximation,  $\overline{\tau_e}$  may be related to the critical shear stress,  $\tau_c$ , which can be computed using the Shields diagram that describes incipient sediment motion. Their difference is that  $\overline{\tau_e}$  is concerned with the instability of more bed particles than implied for  $\tau_c$ . When  $\tau = \overline{\tau_e}$ , Eq. (5) gives a value of  $p = 54\%$ , indicating that slightly more than half of the particles on the bed are in motion.

In comparison, when  $\tau = \tau_c$ , much fewer particles are assumed to be in motion. This critical condition could be further quantified by taking the probability of erosion to be approximately 0.6% for the case of hydraulically rough beds [2].

Incipient sediment motion in laminar flows has been examined earlier by Yalin and Karahan [16]. Their experimental results indicate that the critical condition seems to deviate from that developed for turbulent flows if the critical shear velocity based Reynolds number  $Re_{*c}(= u_{*c}D/\nu)$  is greater than unity. This finding has been confirmed recently by Pilotti and Menduni [15]. The experimental data from both studies are plotted in Fig. 1 as  $Re_{*c}$ , versus the dimensionless particle diameter,  $D_* = (\Delta g/\nu^2)^{1/3}D$ .

Fig. 1 shows that a power function can be used to fit the data with  $Re_{*c}$ , varying from 0.02 to 48.8, yielding

$$Re_{*c} = 0.382D_*^{1.36} \quad (6)$$

Since  $D_* = Re_{*c}^{2/3}\tau_{*c}^{-1/3}$ , Eq. (6) can be further re-written as

$$\tau_{*c} = 0.120Re_{*c}^{-0.21} \quad (7)$$

In Fig. 2, Eq. (7) is compared to the updated Shields diagram [1,17], showing that the critical condition for incipient sediment motion in laminar flows is not consistent with that applicable for turbulent flows.

Given the difference between the critical shear stress and the effective shear stress, a relation that is similar to Eq. (7) but with a different coefficient may be assumed for evaluating the effective shear stress. This yields

$$\tau_{*c} = \beta Re_{*e}^{-0.21} \quad (8)$$

where  $\tau_{*e} = \bar{\tau}_e/(\rho g \Delta D)$ ,  $Re_{*e} = u_{*e}D/\nu$ ,  $u_{*e} = \sqrt{\bar{\tau}_e/\rho}$  and  $\beta$  is the coefficient. In Fig. 2, Eq. (8) is sketched as a straight dashed line, which is parallel to the solid straight line representing Eq. (7). Both straight lines are associated with different probabilities of erosion. For higher probabilities, the dashed line would be farther



above the Shields curve. Therefore, the  $\beta$ -value is expected to be larger than the coefficient included in Eq. (7). Considering  $Re_{*e} = D_*^{3/2}\tau_{*e}^{1/2}$  Eq. (8) can be rewritten in terms of  $D_*$  :

$$\bar{\tau}_c = \beta^{0.91} \rho g \Delta D D_*^{-0.29} \quad (9)$$

Then, substituting it into Eq. (5) gives

$$p = 1 - \exp \left[ -\frac{\pi}{4\beta^{1.82}} \tau_*^2 D_*^{0.58} \right] \quad (10)$$

Eq. (10) indicates that the probability of erosion is related to the dimensionless shear stress and the dimensionless particle diameter. This result is different from those derived for turbulent flows, where the probability is only a function of the dimensionless shear stress [2,8].

Furthermore, the value of  $\beta$  in Eq. (10) can be estimated by considering the critical condition for initiation of sediment motion. This is because the probability of erosion at the critical condition can be computed using Eq. (10) by taking  $\tau_* = \tau_{*c}$ . Cheng and Chiew [2] demonstrated that this probability is approximately 0.6% in the presence of hydraulically rough beds. If the same critical probability is used for incipient sediment motion in laminar flows, then the coefficient,  $\beta$ , can be determined from Eq. (10), as detailed below.

From Eq. (6),  $\tau_*$  can be related to  $D_*$  as follows

$$\tau_{*c} = 0.147 D_*^{-0.29} \quad (11)$$

Substituting Eq. (11) into Eq. (10) and taking  $p = 0.6\%$ , one gets  $\beta = 1.77$ , and Eq. (10) thus changes to

$$p = 1 - \exp[-0.278\tau_*^2 D_*^{0.58}] \quad (12)$$

Additional computations indicate that the  $\beta$ -value varies from 1.06 to 1.95, if the critical probability is taken between 0.5% and 1.5%. Such uncertainties would lead to slight changes in the coefficient (0.278) used in Eq. (12), but it can be readily shown that they have no effects on the transport rate formula, Eq. (19), where the combined coefficient is finally evaluated with the experimental data.

## 2.2. Characteristic time, $t$

Einstein [8] defined the characteristic time as the time of exchange between bed material and bedload, which is proportional to  $D/w$ , where  $w$  is the settling velocity of the particle. According to Paintal [13], the time can also be interpreted as the time through which the particle moves from its position of rest to occupy a new position of deposit. Obviously, the time of travel for a particle varies at different transport stages. For example, at low transport rates, it can be speculated that the time of travel for a particle is much shorter than the time of rest. This implies that the characteristic time, through which the particle is travelling, would be inversely proportional to the mean probability of movement of the particle. Therefore, the characteristic time should be revised as

$$t = a_1 \frac{D}{wp} \quad (13)$$

On the other hand, because of the dominant viscous effect, the settling velocity can be derived from the Stokes law. As a result, the settling velocity is proportional to  $\Delta g D^2 / \nu$  [4], i.e.

$$w = a_2 \frac{\Delta g D^2}{\nu} \quad (14)$$

where  $a_2$  is the constant. Substituting Eq. (14) into Eq. (13) yields

$$t = \frac{a_1}{a_2} \frac{v}{\Delta g D p} \quad (15)$$

### 2.3. Dimensionless transport rate, $\phi$

Substituting Eqs. (12) and (15) into Eq. (2) leads to

$$q_v = \frac{aa_2}{a_1} \frac{\Delta g D^3}{v} [\exp(0.278\tau_*^2 D_*^{0.58}) + \exp(-0.278\tau_*^2 D_*^{0.58}) - 2] \quad (16)$$

Upon dividing both sides by  $\sqrt{\Delta g D D}$ , Eq. (16) can be revised as an expression for the dimensionless transport rate:

$$\begin{aligned} \phi &= a_3 D_*^{1.5} [\sinh(0.139\tau_*^2 D_*^{0.58})]^2 \quad \text{or} \\ \phi &= a_3 \tau_*^{-0.5} Re_* [\sinh(0.139\tau_*^{1.81} Re_*^{0.39})]^2 \end{aligned} \quad (17)$$

where  $a_3 = 4aa_2/a_1$ . Eq. (17) demonstrates that for laminar flows, the dimensionless transport rate is dependent on  $\tau_*$  and  $D_*$ , or  $\tau_*$  and  $Re_*$ . Note that  $D_*$  or  $Re_*$  is often excluded in many bedload formulas developed under the condition of turbulent flows [6,18].

## 3. COMPARISON WITH EMPIRICAL RESULTS

The coefficient,  $a_3$  included in Eq. (17) can be evaluated by comparing the equation to the experimental data of sediment transport in laminar flows that were reported by Girgis [9]. Fig. 3 shows that Eq. (17) is generally supported by the measurements. The best fit is achieved by taking  $a_3$  to be approximately 41.

In addition, Eq. (17) can be simplified to a power function if  $0.139\tau_*^2 D_*^{0.58}$  is much less than unity. This is because the hyperbolic sine function can be expanded in the following series form:

$$\sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + O(x^7) \quad (18)$$

By ignoring the terms higher than the second order in the series, Eq. (17) reduces to

$$\phi = 0.773D_*^{2.66}\tau_*^4 \quad \text{or} \quad \phi = 0.773Re_*^{1.78}\tau_*^{3.12} \quad (19)$$

It is interesting to note that Eq. (19) is quite similar to the empirical correlation, Eq. (1). The latter can be re-written as

$$\phi = 0.884D_*^{2.96}\tau_*^{4.47} \quad \text{or} \quad \phi = 0.884Re_*^{1.97}\tau_*^{3.48} \quad (20)$$

by noting  $q_* = \rho_s \phi / (\Delta \rho \tau_*^{0.5})$  and taking  $\rho_s / \rho = 2.65$ . Both Eqs. (19) and (20) are plotted in Fig. 4 as  $\phi$  against  $\tau_*$  for  $Re$  varying from 0.1 to 100. It can be seen that the two equations are generally close to each other. This is not surprising because they are validated using the same data sets. Obviously, further research is needed to completely verify the present analysis. On the other hand, it should be noted that the effect of the flow depth is not considered in this study. This effect may be significant if the sediment size is relatively large when comparing with the flow depth, for example, for some shallow overland flows.

#### 4. CONCLUSIONS

Bedload transport in laminar flows is investigated in this study as a typical example for exploring the viscous effect on sediment transport. The stochastic approach that was proposed previously by Einstein is employed here but with newly defined parameters. Firstly, the probability of erosion is assumed to be associated with the irregularity of bed particles in the absence of turbulent fluctuations. The analytic result indicates that the probability of erosion is a function of the dimensionless bed shear stress and the shear velocity based Reynolds number. Secondly, the characteristic time for the bed particle motion in laminar flows is also formulated by

including apparently viscous effect. The theoretically derived equation for the dimensionless transport rate depends on both the dimensionless bed shear stress and Reynolds number. It is also shown that the equation reduces to a power function for relatively low transport rates, which is generally supported by the limited experimental data published in the literature.

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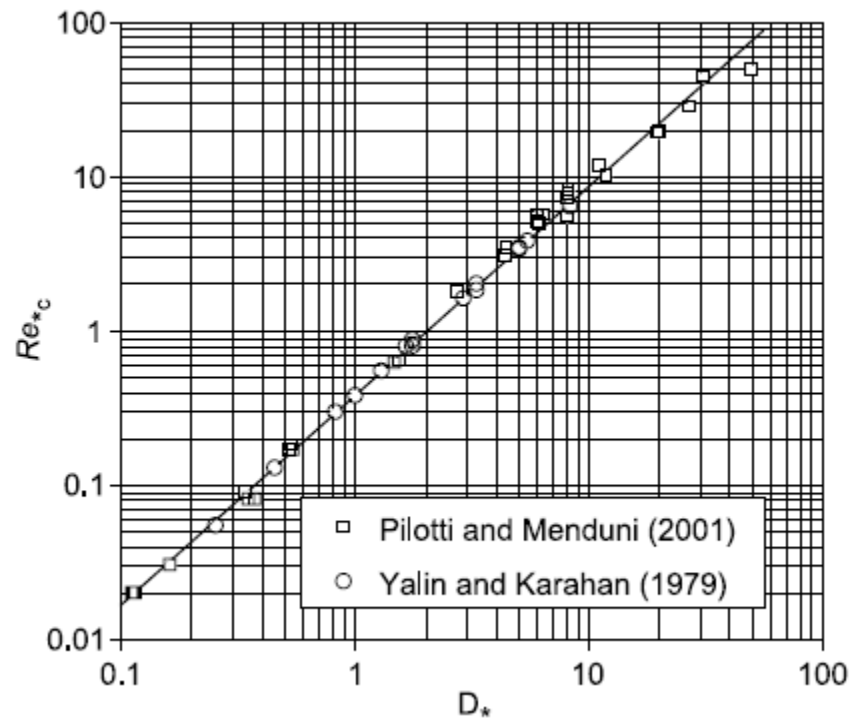


Figure 1

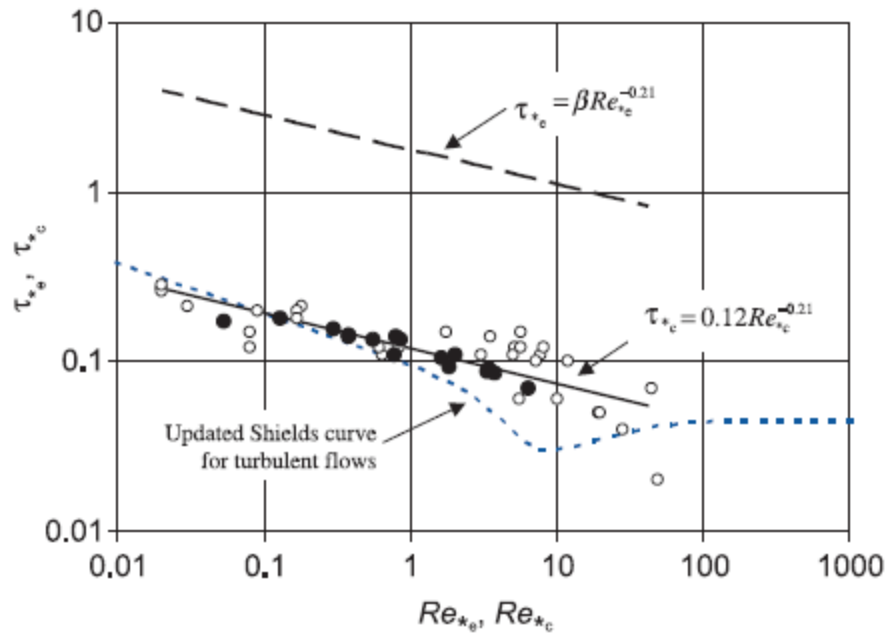


Figure 2

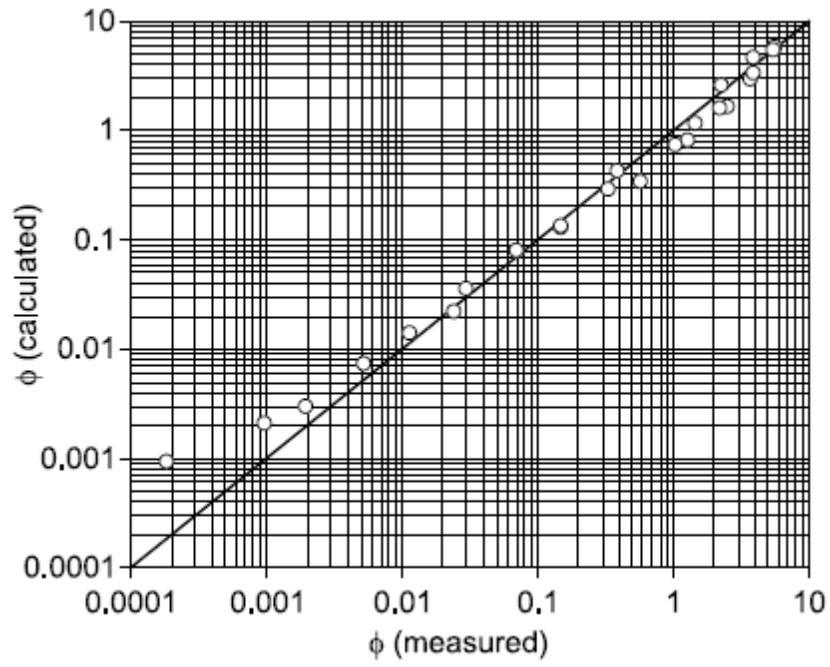


Figure 3

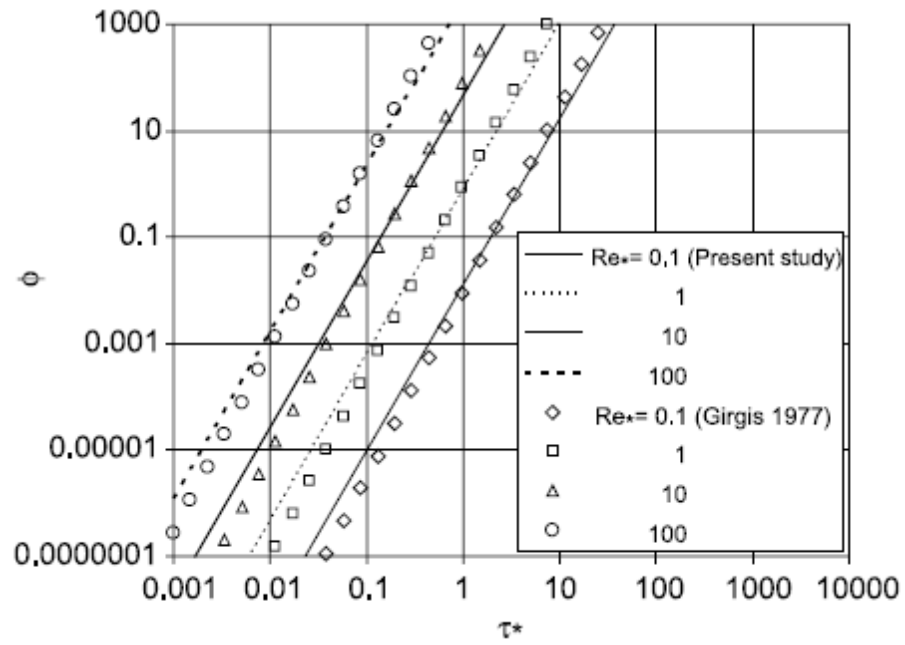


Figure 4