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$c$-Extensions of the $F_4(2)$-building

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Abstract

We construct four geometries $E_1, \ldots, E_4$ with the diagram

such that any two elements of type 1 are incident to at most one common element of type 2 and three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 5. The automorphism group $E_i$ of $E_i$ is flag-transitive, isomorphic to $^2E_6(2) \rtimes 2, 3 \cdot ^2E_6(2) \rtimes 2, 2^{26} : F_4(2)$ and $E_6(2) \rtimes 2$, for $i = 1, 2, 3$ and 4. We calculate the suborbit diagram of the collinearity graph of $E_i$ with respect to the action of $E_i$. By considering the elements in $E_i$ fixed by a subgroup $T_i$ of order 3 in $E_i$ we obtain four geometries $T_1, \ldots, T_4$ with the diagram

on which $C_{E_i}(T_i)$ induces flag-transitive action, isomorphic to $U_6(2) \rtimes 2, 3 \cdot U_6(2) \rtimes 2, 2^{14} \cdot \text{Sp}_6(2)$ and $L_6(2) \rtimes 2$ for $i = 1, 2, 3$ and 4. Next, by considering the elements fixed by a subgroup $S_i$ of order 7 in $E_i$ we obtain four geometries with the diagram

on which $C_{E_i}(S_i)$ induces flag-transitive action isomorphic to $L_3(4) \rtimes 2, 3 \cdot L_3(4) \rtimes 2, 2^8 \cdot L_3(2)$ and $(L_3(2) \times L_3(2)) \rtimes 2$, for $i = 1, 2, 3$ and 4.

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1. Introduction

The article contributes to the classification of geometries with the diagram

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  c
1 -- 2 -- 2 -- t -- t
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(where $t = 1, 2$ or 4) for which the following conditions hold (the types of elements on diagram increase rightward from 1 to 5):

(a) two elements of type 1 are incident to at most one common element of type 2;
(b) three elements of type 1 are pairwise incident to common elements of type 2 if and only if they are incident to a common element of type 5.

In order to simplify the notation we call such geometries $c.F_4(t)$ geometries.

It is shown in [7] that there exists a unique $c.F_4(4)$-geometry which is flag-transitive with the automorphism group isomorphic to the Baby Monster sporadic simple group. Also in [7] a number of useful general results on $c.F_4(t)$-geometries are established (some of these results are cited below). In [8] the geometries $F(Fi_{22})$ and $F(3.Fi_{22})$ related to the Fischer 3-transposition group $Fi_{22}$ are characterised in the class of $c.F_4(1)$-geometries by a certain combinatorial condition which holds in every flag-transitive geometry.

In Section 4 of [7] four flag-transitive examples $F(2^6.2E_6(2))$, $F(3.2^6.2E_6(2))$, $F(2^{26}.F_4(2))$ and $F(E_6(2))$ of $c.F_4(2)$-geometries are mentioned. The main purpose of the present article is to provide some information on these geometries and their fixed-point subgeometries to be used in the classification of such geometries (possibly under the flag-transitivity assumption) started in [12]. We calculate the suborbit diagrams of the collinearity graphs and describe the subgeometries induced by the elements fixed by certain subgroups of order 3 and 7. We conjecture that the four geometries exhaust the class of flag-transitive $c.F_4(2)$-geometries. We have used a computer package [5] and have also produced some computer-free calculations. The arguments for the computer-free calculations are mostly ad hoc and we do not always present them in full.

2. $F_4(2)$-building

Let $\mathcal{F}$ be the building of the group $F = F_4(2)$ (cf. [11]). Then $\mathcal{F}$ belongs to the diagram

```
  2 -- 2 -- 2 -- 2
```

and $F$ is the only (and hence the full) flag-transitive automorphism group of $\mathcal{F}$ (the outer automorphism of $F$ performs a diagram automorphism of $\mathcal{F}$). In terms of $F$ the geometry $\mathcal{F}$ can be described as follows. Let $\Psi$ be a conjugacy class of central (root) involutions in $F$ (there are two such classes, say $\Psi$ and $\Psi'$ fused in Aut $F$). Then $\Psi$ is the set of points (elements of type 1) of $\mathcal{F}$. Two points $p, q \in \Psi$ are collinear (incident to a common element of type 2) if and only if $q \in O_2(C_F(p))$ (equivalently if $p \in O_2(C_F(q))$). The collinearity graph (which we denote by the same letter $\Psi$) has
We follow the standard notation for suborbit diagrams (cf. [6]). The above diagram shows that the stabiliser $F(p)$ of $p \in \Psi$ in the group $F$ has exactly five orbits: $\{p\}$, $\Psi_1(p)$, $\Psi_2^1(p)$, $\Psi_2^4(p)$ and $\Psi_3(p)$ on $\Psi$ (we have $F(p) = C_F(p) \cong 2^{1+6+8}$. $\text{Sp}_6(2)$). The orbit $\Psi_2^4(p)$, for instance, has length 34560 and if $q \in \Psi_2^4(p)$ then $q$ is adjacent in $\Psi$ to 1 vertex in $\Psi_1(p)$, 14 vertices in $\Psi_2^1(p)$, 127 vertices in $\Psi_2^4(p)$ and 128 vertices in $\Psi_3(p)$. Furthermore, $F(p) \cap F(q)$ has three orbits on $\Psi_1(q) \cap \Psi_2^4(p)$ with lengths 1, 14 and 112.

Every element $r \in F$ will be identified with the subgraph in $\Psi$ induced by the vertices incident to $r$ in $F$, so that the incidence relation is via inclusion. If the type of $r$ is 1, 2, 3 or 4 then $r$ is a vertex, a triangle, a complete subgraph on 7 vertices or the collinearity graph of the polar space of $\text{Sp}_6(2)$, respectively. If $p$ and $q$ are distinct vertices contained in an element of type 4, then $q \in \Psi_1(p) \cup \Psi_2^2(p)$ and if $q \in \Psi_2^2(p)$ then there is exactly one such element of type 4.

The following two lemmas are standard [4].

**Lemma 2.1.** (i) If $q \in \Psi_1(p) \cup \Psi_2^2(p)$ then $p$ and $q$ commute (as involutions in $F$) and $q \in \Psi_1(p)$ if and only if $pq \in \Psi$.

(ii) If $q \in \Psi_2^4(p)$ then $\langle p, q \rangle \cong D_8$ and the central involution of $\langle p, q \rangle$ is in $\Psi_1(p) \cap \Psi_1(q)$.

(iii) If $q \in \Psi_3(p)$ then $\langle p, q \rangle \cong D_6$.

**Lemma 2.2.** The orbits of $O_2(F(p))$ on $\Psi_1(p)$, $\Psi_2^2(p)$, $\Psi_2^4(p)$ and $\Psi_3(p)$ are of length, respectively, 2, $2^3$, $2^8$ and $2^{15}$; the orbits on $\Psi_1(p)$ and $\Psi_2^4(p)$ naturally correspond to the elements of type 2 incident to $p$ while the orbits on $\Psi_2^2(p)$ correspond to the elements of type 4 incident to $p$.

We are interested in the graph $\Lambda$ on $\Psi$, in which two vertices $p$ and $q$ are adjacent if and only if $q \in \Psi_1(p) \cup \Psi_2^2(p)$. In view of the above we have the following.

**Lemma 2.3.** Two distinct points $p, q \in \Psi$ are adjacent in $\Lambda$ if and only if one of the following equivalent conditions holds:

(i) $p$ and $q$ commute (as involutions in $F$);

(ii) $p$ and $q$ are incident in $F$ to a common element of type 4.
Notice that an element of type 4 in $\mathcal{F}$ induces in $\Lambda$ a complete graph on 63 vertices. By Lemma 2.3 $\Lambda$ is the commuting graph of $\Psi$ which means that two involutions from $\Psi$ are adjacent in $\Lambda$ if and only if they commute.

It is known (cf. [3,10]) that $F$ contains two conjugacy classes of subgroups isomorphic to $\text{Sp}_8(2)$. The representatives $P$ and $P'$ of these classes can be chosen in such a way that the central involutions of $P$ and $P'$ are contained in $\Psi$ and $\Psi'$, respectively. Then the class $\mathcal{K}$ of 255 central involutions in $P$ (which is a class of 3-transpositions) is a subset of $\Lambda$ and the subgraph induced by $\mathcal{K}$ is clearly the commuting graph. For $\varepsilon = +$ or $-$ let $Q^\varepsilon$ be (the unique up to conjugation) subgroup in $P$ isomorphic to $\Omega^\varepsilon_8(2):2$.

From the subgroup structure of $F$ (cf. [3,10]) we deduce the following.

**Lemma 2.4.** Let $Q^\varepsilon \triangleleft H \triangleleft F$ for $\varepsilon = -$ or $+$. Then either $H \leq N_F(O^2(Q^\varepsilon))$ or $H = P$. Furthermore, $N_F(O^2(Q^-)) = Q^{-}$ while $N_F(O^2(Q^+)) \cong \Omega^+_8(2):S_3$ (a maximal subgroup of $F$).

The orbit $\mathcal{K}$ of $P$ on $\Lambda$ can be identified as the set of points of the polar space $\mathcal{P}$ of $P$. The remaining vertices in $\Lambda$ can be classified depending on the set of the nearest vertices in $\mathcal{K}$. This enables to describe the vertices of $\Lambda$ in terms of $\mathcal{P}$ and to determine the orbits of $P$ on $\Lambda$ as given in the next lemma.

**Lemma 2.5.** The subgroup $P \cong \text{Sp}_8(2)$ of $F$ acting on $\Lambda$ has four orbits with lengths 255, 2295, 32640 and 34425.

Using the description of $\Lambda$ in terms of the polar space $\mathcal{P}$ of $P$ we can deal with the restrictions of the action of $P$ on $\Lambda$ to $Q^-$ and $Q^+$ and obtain the next two lemmas.

**Lemma 2.6.** The subgroup $Q^- \cong \Omega^-_8(2):2$ of $F$ acting on $\Lambda$ has seven orbits with lengths 119, 136, 2295, 16065, 16320 (twice) and 18360.

**Lemma 2.7.** The subgroup $Q^+ \cong \Omega^+_8(2):2$ of $F$ acting on $\Lambda$ has nine orbits with lengths 120, 135, 240, 270, 2025, 4050, 14175, 16200 and 32400.

Similarly $F$ contains two conjugacy classes of subgroups isomorphic to $^3D_4(2)$ and the representatives $D$ and $D'$ of these classes can be chosen so that the central involution of $D$ and $D'$ are contained in $\Psi$ and $\Psi'$, respectively.

**Lemma 2.8.** The subgroup $D' \cong ^3D_4(2)$ of $F$ acting on $\Lambda$ has four orbits with lengths 17199 and 17472 (three times).

When considering fixed-point subgeometries we keep the following principle in mind (cf. [13, Theorem 3.5]).

**Lemma 2.9.** Let $G$ be a group acting transitively on a set $\Omega$. Let $\alpha \in \Omega$, $G(\alpha)$ be the stabiliser of $\alpha$ in $G$ and $V$ be a subgroup in $G(\alpha)$. Suppose that
(i) whenever a conjugate $V'$ of $V$ in $G$ is contained in $G(x)$, it is conjugate to $V$ in $G(x)$;
(ii) $V$ is fully normalized in $G(x)$, which means that $N_{G(x)}(V)/C_{G(x)}(V) \cong \text{Aut } V$.

Then $C_G(V)$ acts transitively on the set of elements in $\Omega$ fixed by $V$.

**Lemma 2.10.** Let $S$ be a Sylow 7-subgroup in $F(p)$, $\Sigma$ the subgraph in $\Delta$ induced by the vertices fixed by $S$ and $\Sigma^h_a(p) = \Sigma \cap \Psi^h_a(p)$. Then

(i) $|\Sigma_1(p)| = 4$, $|\Sigma_2^a(p)| = |\Sigma_3(p)| = 8$ and $\Sigma_2^b(p) = \emptyset$, so that $|\Sigma| = 21$;
(ii) $\Sigma$ is the graph on the flags of the projective plane of order 2 in which two distinct flags are adjacent if and only if they have a non-empty intersection;
(iii) $C_F(S) \cong 7 \times L_3(2)$ induces on $\Sigma$ an action of $L_3(2)$;
(iv) $N_F(S) \cong (F_3^7 \times L_3(2)) : 2 \cong (F_3^7 \times \text{PGL}_2(7))^+$ induces on $\Sigma$ the full automorphism group of $\Sigma$, isomorphic to $\text{PGL}_2(7)$ and the kernel is the Frobenius group $F_3^7$ of order 21.

**Proof.** The claim (i) is easy to deduce from Lemma 2.3 and the permutation characters of $F(p)/O_2(F(p)) \cong \text{Sp}_6(2)$ on the elements of type 2 and 4 incident to $p$ [3]. The remainder can be deduced from the character table and the list of maximal subgroups in $F$ given in [3]. There are two classes $\mathscr{S}$ and $\mathscr{S}'$ of elements of order 7 in $F$ and every such element is conjugate to all its powers of order 7. It is easy to use the power map to conclude that $\mathscr{S}$ and $\mathscr{S}'$ are fused in $\text{Aut } F$. Next, the centraliser of an element of order 7 has order $1176 = 7 \times |L_3(2)|$. Since there is a (maximal) subgroup in $F$ of the form $(L_3(2)^2 \times L_3(2)) : 2$ we have (iii). To get (iv) all we have to do is to exclude the possibility $N_F(S) \cong F_3^7 \times L_3(2)$ by noticing that $F$ has no elements of order 42. Finally (ii) follows from (i), (iii) and (iv). $\square$

The action of $\text{PGL}_2(7)$ on $\Sigma$ is distance-transitive with the following suborbit diagram:

$$
\begin{array}{c|cccc|cc}
1 & 4 & 1 & 4 & 1 & 8 & 2 & 2 & 8 \\
|p| & \Sigma_1(p) & \Sigma_2^a(p) & \Sigma_3(p) & \\
\end{array}
$$

The incidence system in which the elements of type 1 are the flags and the elements of type 2 are the points together with the lines of the projective plane of order 2 is a generalized hexagon $\mathcal{H}$ with parameters $(2, 1)$. In terms of $\Sigma$ the elements of $\mathcal{H}$ are the vertices and triangles with the natural incidence relation.

**Lemma 2.11.** Let $T$ be a Sylow 3-subgroup of the kernel (isomorphic to $F_3^3$) of the action of $N_F(S)$ on $\Sigma$, $\Theta$ be the subgraph in $\Delta$ induced by the vertices fixed by $T$ (clearly $\Sigma \subseteq \Theta$) and $\Theta^h_a(p) = \Theta \cap \Psi^h_a(p)$. Then

(i) $|\Theta_1(p)| = 18$, $|\Theta_2^a(p)| = 24$, $|\Theta_2^b(p)| = 144$, $|\Theta_3(p)| = 128$, so that $|\Theta| = 315$;
(ii) $C_F(T)/T \cong \text{Sp}_6(2)$ acts faithfully on $\Theta$;
(iii) $\Theta$ is isomorphic to the graph on the set of elements of type 2 in the polar space $\mathcal{P}$ of $C_F(T)/T \cong \text{Sp}_6(2)$ with the diagram

$\begin{array}{c}
\circ \quad \circ \quad \circ \\
2 \quad 2 \quad 2
\end{array}$

in which two elements are adjacent if and only if they are incident to a common element of type 1.

**Proof.** First we identify the image $\tilde{T}$ of $T$ in $H = F(p)/O_2(F(p)) \cong \text{Sp}_6(2)$. The stabiliser (isomorphic to $2^6:L_3(2)$) of a point in the dual polar space of $H$ contains $F^3_7$ (hence it contains a conjugate of $\tilde{T}$) and the permutation character on the points is known. Direct calculation shows that $\tilde{T}$ is generated by a $3C$-element. Now (i) can be easily deduced from Lemma 2.3. To identify the class of $T$ in $F$ we notice that up to conjugation in $\text{Aut} F$ there are two classes of elements of order 21. This and the power map shows that $|C_F(T)| = 3 \times |\text{Sp}_6(2)|$, there are no subgroups of order 3 in $F$ with strictly larger centraliser and $T$ is uniquely determined up to conjugation in $\text{Aut} F$. Since $P \cong \text{Sp}_6(2)$ contains a subgroup isomorphic to $S_3 \times \text{Sp}_6(2)$ we obtain (ii). By Lemma 2.9 $C_F(T)$ acts transitively on $\Theta$. By Lie theory every subgroup of $\text{Sp}_6(2)$ containing a Sylow 2-subgroup is a parabolic subgroup from which it is easy to deduce that every subgroup of index 315 is the stabiliser of an element of type 2 in the polar space $\mathcal{P}$. \qed

Below we present the suborbit diagram with respect to the action of $C_F(T)$ of the subgraph in $\mathcal{P}$ induced by $\Theta$.

$\begin{array}{c}
1 \\
\{p\}
\end{array}$  \\
$\begin{array}{c}
18 \\
\Theta_1(p)
\end{array}$  \\
$\begin{array}{c}
18 \\
\Theta_2(p)
\end{array}$  \\
$\begin{array}{c}
144 \\
\Theta_3(p)
\end{array}$  \\
$\begin{array}{c}
24 \\
\Theta_2(p)
\end{array}$

In terms of $\Theta$ the polar space $\mathcal{P}$ of $C_F(T)/T$ can be described as follows. The elements of type 2 the vertices of $\Theta$, the planes are the elements of type 3 in $\mathcal{F}$ contained in $\Theta$ and the points are the intersection of size 15 of $\Theta$ with the elements of type 4. A point and a plane are incident if their intersection is an element of type 2 in $\mathcal{F}$.

3. $c.F_4(2)^*$-geometries

Let $\mathcal{E}$ be a $c.F_4(2)^*$-geometry and $\Gamma(\mathcal{E})$ be the collinearity graph of $\mathcal{E}$. The diagram of $\mathcal{E}$ is

$\begin{array}{c}
\circ \\
C \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ
\end{array}$
(the elements of type 1 are points and the elements of type 2 are lines). It follows directly from properties (a) and (b) that $\Gamma(\mathcal{E})$ is locally $\Delta$ (recall that a graph $\Gamma$ is locally $\Delta$ if for every vertex $x \in \Gamma$ there is an isomorphism

$$i_x : \Gamma(x) \to \Delta$$

of graphs, where $\Gamma(x)$ denotes the subgraph in $\Gamma$ induced by the vertices adjacent to $x$).

The following result is Lemma 3.3 in [7].

**Lemma 3.1.** Let $\Gamma$ be a graph which is locally $\Delta$. Then $\Gamma = \Gamma(\mathcal{E})$ for a $c.F_4(2)^*$-geometry $\mathcal{E}$.

The geometry $\mathcal{E}$ in the above lemma can be defined as follows. The elements of type 5 are complete 64-vertex subgraphs of the form $\{x\} \cup X_{BEl}$, where $i_x(X_{BEl})$ is an element of type 4 in $\mathcal{E}$. The elements of types 1, 2, 3 and 4 are the intersection of size 1, 2, 4 and 8 of two or more elements of type 5 in $\mathcal{E}$; the incidence relation is via inclusion.

In the remainder of this section $\Gamma$ is a graph which is locally $\Delta$ and $\mathcal{E}$ is the corresponding $c.F_4(2)^*$-geometry. Let $\pi = (z,x,y)$ be a 2-path in $\Gamma$ such that $z$ and $y$ are not adjacent (which means they are at distance 2). Then by Lemma 2.1 $\langle i_x(z), i_x(y) \rangle \cong D_8$ or $D_6$ and we say that $\pi$ is of type $D_8$ or $D_6$, respectively. For $j = 3$ or 4 and a vertex $x$ of $\Gamma$ let $I^j_2(x)$ denote the set of vertices $y$ at distance 2 from $x$ in $\Gamma$ such that there is a 2-path of $D_{2j}$-type, joining $x$ and $y$. Then $I_2(x)$ (the set of vertices at distance 2 from $x$ in $\Gamma$) is the union of $I_2^4(x)$ and $I_2^3(x)$ (in general we do not exclude the possibility that these two subsets intersect, but compare Lemma 3.5).

The following result is established in Section 6 of [7].

**Lemma 3.2.** For $y \in I_2^4(x)$ put

$$M = \{z \in \Gamma(x) \cap \Gamma(y) \mid (x,z,y) \text{ is of } D_8\text{-type}\}.$$  

Then

(i) $|M| = 144$ and the subgraph in $\Gamma(x)$ induced by $M$ is connected;
(ii) the stabiliser $X$ of $i_x(M)$ in $F$ is of the form $2^{1+6+8} \cdot U_3(3) \cdot 2 \cong 2^{1+6+8} : G_2(2)$;
(iii) $X$ has exactly 7 orbits $A_1, \ldots, A_7$ on the vertex-set of $\Delta$ with lengths 1, 126, 144, 2016, 16128, 18432 and 32768, respectively (so that $i_x(M) = A_3$);
(iv) if $u \in \Gamma(x)$ then either $u$ is adjacent to a vertex from $M$ or $i_x(u) \in A_7$.

In particular

$$|I_2^4(x)| = |\Gamma(x)| \cdot |\Psi^4_2(p)|/144 = 16 \, 707 \, 600 = 2^4(2^{12} - 1)(2^8 - 1).$$

The next lemma is a summary of Section 7 in [7] for the case $t = 2$.

**Lemma 3.3.** For $y \in I_2^3(x)$ put

$$N = \{z \in \Gamma(x) \cap \Gamma(y) \mid (x,z,y) \text{ is of } D_6\text{-type}\}.$$
Let $N_c$ be a connected component of the subgraph in $\Gamma(x)$ induced by $N$ and let $Y$ be the stabiliser of $i_* (N_c)$ in $F$. Then one of the following holds:

(i) $|N_c| = 136$ and $Y$ is a conjugate of the subgroup $Q^- \cong \Omega^-_8(2):2$ as in Lemma 2.6;
(ii) $|N_c| = 120$ and $Y$ is a conjugate of the subgroup $Q^+ \cong \Omega^+_8(2):2$ as in Lemma 2.7;
(iii) $|N_c| = 128$ and $N_c$ is a double cover of the complete graph with the automorphism group of the form $2^{1+6} : \text{Sp}_8(2)$.

From now on we assume that $E$ possesses a flag-transitive automorphism group $E$. Then $E$ acts vertex- and edge-transitively on $\Gamma$ and for a vertex $x \in \Gamma$ the stabiliser $E(x)$ induces on $\Gamma(x)$ its full automorphism group $F \cong F_4(2)$. We will assume that the kernel $K(x)$ of this action is of order at most 2 (this is proved in [12] and holds in all our examples anyway). Since $E$ acts transitively on the set of 2-paths of both $D_8$- and $D_6$-types, we have the following.

**Lemma 3.4.** In the above terms $E(x)$ acts transitively both on $I^4_2(x)$ and $I^3_2(x)$. Furthermore for $y \in D_2(x)$ the subgroup $E(x) \cap E(y)$ acts transitively both on the set $M$ as in Lemma 3.2 and the set $N$ as in Lemma 3.3.

**Lemma 3.5.** In the flag-transitive case the sets $I^4_2(x)$ and $I^3_2(x)$ are disjoint.

**Proof.** By Lemma 3.4 if the claim fails then $I^4_2(x) = I^3_2(x)$. The size of $I^4_2(x)$ is given in Lemma 3.2. By Lemmas 3.3 and 3.4

$$|I^3_2(x)| = |\Gamma(x)| \cdot |\Psi_3(p)|/(n \cdot \mu),$$

where $\mu \in \{136, 120, 128\}$ and $n$ is the number of connected components of $N$. Direct calculation shows that $I^4_2(x)$ and $I^3_2(x)$ are always of different size. \(\square\)

The elementwise stabiliser in $F$ of $\Delta(p)$ is of order 2, generated by $p$ and it acts faithfully both on $\Psi^2_5(p)$ and $\Psi_5(p)$. This shows that $K(x)$ (assumed to be of order 2) acts faithfully both on $I^4_2(x)$ and $I^3_2(x)$.

**Lemma 3.6.** For $y \in I^4_2(x)$ there is exactly one other vertex $y'$ in $I^4_2(x)$ such that $\Gamma(x) \cap \Gamma(y') = \Gamma(x) \cap \Gamma(y)$. The action of $E(x) \cap E(y)$ on $\Gamma(x)$ is a conjugate of the subgroup $X$ as in Lemma 3.2 if $|K(x)| = 2$ and of a subgroup of index 2 in $X$ if $K(x) = 1$.

**Proof.** In view of the paragraph before the lemma the result follows from the fact that the size of $I^4_2(x)$ is twice the index of $X$ in $F$. \(\square\)

The next two lemmas are refinements of Lemma 3.3 in the flag-transitive case.

**Lemma 3.7.** Let $y \in I^3_2(x)$. Then in terms of Lemma 3.3 all the connected components of the subgraph in $\Gamma(x)$ induced by $N$ are isomorphic. Let $C$ denotes the stabiliser
in \( E(x) \cap E(y) \) of a component \( N_c \). Then one of the following holds:

(i) \( |N_c| = 136 \), \( C \) is a conjugate of \( Q^- \) if \(|K(x)| = 2\) and of \( O^2(Q^-) \) if \( K(x) = 1\);
(ii) \( |N_c| = 120 \), \( C \) is a conjugate of \( Q^+ \) if \(|K(x)| = 2\) and of \( O^2(Q^+) \) if \( K(x) = 1\);
(iii) \( |N_c| = 128 \), \( K(x) = 1 \) and \( C \cong \text{Aut} N_c \cong 2^{1+6}:\text{Sp}_6(2)\).

**Proof.** The result follows from Lemma 3.3 by comparison of \( |T^3_2(x)| \) as in the proof of Lemma 3.5 and the index of the stabiliser of \( N_c \) in \( F \). \( \square \)

**Lemma 3.8.** In terms of the previous lemma one of the following holds:

(i) the subgraph induced by \( N \) is connected on 136 vertices;
(ii) the subgraph induced by \( N \) has 1, 2 or 3 connected components of size 120 each;
(iii) if \( C < H < F \) then \( H \) is either the centraliser of a central involution in \( F \) or \( H \cong \text{Sp}_8(2) \).

**Proof.** We know that \( E(x) \cap E(y) \) contains \( C \) with index \( n \) equal to the number of connected components in the subgraph induced by \( N \) and \( E(x) \cap E(y) \) has an orbit on \( \Gamma(x) \) of length \( |N_c| \cdot n \). In (i) and (ii) we apply Lemmas 2.4–2.7. In case (iii), since the centre of \( C \) is of order 2, \( H \) must contain an involution with centraliser containing \( C \). Then the result follows from the list of maximal subgroups in \( F \). \( \square \)

The following result provides us with a sufficient condition for existence of flag-transitive \( c.F_4(2)^* \)-geometries.

**Lemma 3.9.** Let \( E \) be a group containing a subgroup \( \tilde{F} \) such that \( \tilde{F}/K \cong F \cong F_4(2) \) for a normal subgroup \( K \) of order at most 2 in \( \tilde{F} \). Suppose that the valencies of the non-diagonal orbitals of the permutation action of \( E \) on the cosets of \( \tilde{F} \) are \( n_1 = 69,615, n_2, \ldots, n_m \). Suppose that the orbital \( \Gamma \) corresponding to the subdegree \( n_1 \) is self-paired and for every \( 2 \leq i \leq m \) neither \( n_1 \cdot 270 \), nor \( n_1 \cdot 2016 \) is divisible by \( n_i \). Then for every \( x \in \Gamma \) there is a bijection \( i_x \) of the vertex-set of \( \Gamma(x) \) onto the vertex-set of \( \Lambda \), which commutes with the action of \( \tilde{F} \) (the action on \( \Lambda \) is assumed to be natural with kernel \( K \)) such that the preimage of an edge in \( \Lambda \) is an edge in \( \Gamma(x) \).

**Proof.** We assume that \( x \) is the coset containing the identity. Since \( n_1 \) is odd, \( \tilde{F} \) induces on \( \Gamma(x) \) an action of \( F \) (so that \( K \) is the kernel). By the list of maximal subgroups in \( F \) (or by Lie theory) we can assume without loss of generality that the action of \( F \) on \( \Gamma(x) \) is similar to that on \( \Lambda \) and we take \( i_x \) to be the bijection which establishes the similarity. Let \( p \in \Lambda \) and \( q \in \Psi_1(p) \) or \( \Psi_2(p) \). Then the orbital containing \( (i_x^{-1}(p),i_x^{-1}(q)) \) is non-diagonal and of valency dividing \( |\Gamma(x)| \cdot |\Psi_1(p)| \) or \( |\Gamma(x)| \cdot |\Psi_2(p)| \), respectively. By the assumption this orbital must be \( \Gamma \) and the result follows. \( \square \)

By the above lemma \( \Gamma \) is either locally \( \Lambda \) or contains “extra” triangles, in which case there is at most one orbit of \( E(x) \) on \( \Gamma_2(x) \). In order to exclude the latter possibility we consider fixed points of a subgroup of order 7. Let \( S \) be a Sylow 7-subgroup in \( E(x) \cap E(y) \) where \( x \) and \( y \) are adjacent in \( \Gamma \) (compare Lemma 2.10). By Lemma 2.9
$C_E(S)$ acts transitively on the set $A$ of vertices in $\Gamma$ fixed by $S$. By Lemmas 2.10 and 3.9 $|A \cap \Gamma(x)| = 21$. We leave the proof of the following lemma as an exercise.

**Lemma 3.10.** Under the hypothesis of Lemma 3.9 and in terms of the above, let $A$ denote also the orbital corresponding to $A \cap \Gamma(x)$. Suppose that $C_E(S)$ has two orbits of different lengths on the set of vertices at distance 2 from $x$ in $A$. Then $\Gamma$ is locally $\Delta$.

In terms of the above lemma the graph $A$ is locally $\Sigma$ and we can associate with $A$ a geometry $\mathcal{G}$ of rank three whose elements are the vertices, the edges and the complete 4-vertex subgraphs (equivalently these are the elements of type 1, 2 and 3 contained in $A$); the incidence relation is via inclusion. It is easy to see that $\mathcal{G}$ has the following diagram:

- **Diagram:**
  
  ![Diagram](attachment:image.png)

In the above terms let $T$ be a subgroup of order 3 in $E(x) \cap E(y)$ which normalises $S$ (cf. Lemma 2.11) then $C_E(T)$ acts transitively on the set $\Xi$ of vertices in $\Gamma$ fixed by $T$. If $\Xi$ also denotes the subgraph in $\Gamma$ induced by this set then $\Xi$ is locally $\Theta$. Define $\mathcal{F}$ to be a geometry whose elements of type 1 are the vertices in $\Xi$ and for $2 \leq j \leq 4$ the elements of type $j$ are the complete subgraphs of the form $z \cup \Pi$ where $z \in \Xi$ and $i_z(\Pi)$ is an element of type $j - 1$ in the polar space $\mathcal{P}$ of $C_E(T)/T$. It is easy to see that the diagram of $\mathcal{F}$ is:

- **Diagram:**
  
  ![Diagram](attachment:image.png)

4. **Examples**

In four subsection of this section we construct four examples of flag-transitive c.F4(2)*-geometries. We start with a group $E$ (isomorphic, respectively to $^2E_6(2):2$, $^3D_4(2):2$, $^2E_6(2):2$, $^2O_{10}:F_4(2)$ and $E_6(2):2$) and its subgroup $\widetilde{F}$ (isomorphic, respectively, to $F_4(2) \times 2$, $F_4(2) \times 2$, $F_4(2)$ and $F_4(2) \times 2$). Put $K = O_2(\widetilde{F})$, so that $\widetilde{F}/K \cong F$. We consider the permutation action of $E$ on the set $\Gamma$ of cosets of $\widetilde{F}$. The subdegrees of this action can be deduced from [9] or [2]. It turns out that in every case the subdegrees are pairwise different and hence the orbitals are self-paired. There is always a subdegree 69615 and we denote by $\Gamma$ also the orbital corresponding to this subdegree. Let $x$ be the vertex of $\Gamma$ fixed by $\widetilde{F}$ (the coset containing the identity). Then $\widetilde{F}$ acts on $\Gamma(x)$ with kernel $K$ and $F$ acts on $\Gamma(x)$ as on a class of its central involutions. Hence, we can assume that there is a bijection $i_x$ (of the set $\Gamma(x)$) onto the vertex set of $A$ which commutes with the action of $\widetilde{F}$. Our first goal is to show that $i_x$ is an isomorphism of graphs (i.e., that $\Gamma$ is locally $\Delta$). In all cases Lemma 3.9 applies. Next, we apply
Table 1
The action of $3^{1+6+8}$ on the cosets of $F_4(2)$ [9]

<table>
<thead>
<tr>
<th>Subdegree</th>
<th>Multiplicity</th>
<th>Stabiliser</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$3$</td>
<td>$F_4(2)$</td>
</tr>
<tr>
<td>$(2^4 + 1)(2^{12} − 1) = 69 615$</td>
<td>$3$</td>
<td>$2^{1+6+8} : \text{Sp}_6(2)$</td>
</tr>
<tr>
<td>$2^4(2^{12} − 1)(2^8 − 1) = 16 707 600$</td>
<td>$3$</td>
<td>$2^{1+6+7} : U_5(3).2$</td>
</tr>
<tr>
<td>$2^{12}(2^4 + 1)(2^8 + 2^4 + 1) = 19 009 536$</td>
<td>$1$</td>
<td>$\Omega_4^+(2)$</td>
</tr>
</tbody>
</table>

Table 2
The action of $2^{26} : F_4(2)$ on the cosets of $F_4(2)$ [2]

<table>
<thead>
<tr>
<th>Subdegree</th>
<th>Multiplicity</th>
<th>Stabiliser</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$F_4(2)$</td>
</tr>
<tr>
<td>$(2^4 + 1)(2^{12} − 1) = 69 615$</td>
<td>$1$</td>
<td>$2^{1+6+8} : \text{Sp}_6(2)$</td>
</tr>
<tr>
<td>$2^4(2^{12} − 1)(2^8 − 1) = 16 707 600$</td>
<td>$1$</td>
<td>$2^{1+6+7} : U_5(3).2$</td>
</tr>
<tr>
<td>$2^8(2^{12} − 1)(2^4 + 1) = 17 821 440$</td>
<td>$1$</td>
<td>$2^{1+6} : \text{Sp}_6(2)$</td>
</tr>
<tr>
<td>$2^8(2^8 + 2^4 + 1) = 69 888$</td>
<td>$1$</td>
<td>$\text{Sp}_6(2)$</td>
</tr>
<tr>
<td>$2^{12}(2^{12} − 1) = 16 773 120$</td>
<td>$1$</td>
<td>$\Omega_4^+(2)$</td>
</tr>
<tr>
<td>$2^{12}(2^8 − 1)(2^4 − 1) = 15 667 200$</td>
<td>$1$</td>
<td>$\Omega_4^+(2)$</td>
</tr>
</tbody>
</table>

Lemma 3.10 after calculating the suborbit diagrams of the subgraphs $\Lambda$ and $\Xi$ induced by the vertices fixed by $S$ and $T$. Here $S$ and $T$ are subgroups in $E(x) \cap E(y)$ for a vertex $y$ adjacent to $x$, where $S$ is of order 7 and $T$ is of order 3 normalising $S$. We identify the isomorphism type of $C_E(S)/S$ and construct on a computer its action on the cosets of $C_E(S)/S$. Similar calculations are performed for $T$.

In three case when $K$ is non-trivial, the fact that $\Gamma$ is locally $\Lambda$ can be established easier. Indeed in these cases the set $\Gamma$ can be identified with the conjugacy class $\mathcal{X}$ in $E$ of the unique involution in $K$. Since $\tilde{F} = C_E(k)$, it is easy to deduce from the subdegrees that two vertices in $\Gamma$ are adjacent if and only if the corresponding involutions commute. Thus $\Gamma$ is the commuting graph of $\mathcal{X}$ and hence $\Gamma(x)$ is the commuting graph of $\Psi$, which is $\Lambda$.

Finally, we calculate the suborbit diagram of $\Gamma$ with respect to the action of $E$. For the case $E \cong 2^{26} : F_4(2)$ the 2-point stabilisers are given in [2]. For the remaining three cases we have learned the 2-point stabilisers from [9] and a private communication from R. Lawther. We are pleased to thank Ross for the helpful cooperation.

The 2-point stabilisers given in Tables 1–3 should be understood (up to conjugation) as follows:

(a) $2^{1+6+8} : \text{Sp}_6(2)$ is $F(r)$;
(b) $2^{1+6+7} : U_5(3).2$ is a subgroup of index 2 in $X$ as in Lemma 3.2(ii);
(c) $\Omega_4^+(2)$ is $O^2(Q^4)$ as in Lemma 2.5 or 2.6 depending on $e$;
(d) $2^{1+6} : \text{Sp}_6(2)$ is the centraliser of a central involution in $P$;
(e) $\text{Sp}_8(2)$ is the subgroup $P$ as in Lemma 2.5;
(f) $3D_4(2)$ is the subgroup $D'$ as in Lemma 2.7.
Table 3
The action of $E_6(2)$ on the cosets of $F_4(2)$ [9]

<table>
<thead>
<tr>
<th>Subdegree</th>
<th>Multiplicity</th>
<th>Stabiliser</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$</td>
<td>$1$</td>
<td>$F_4(2)$</td>
</tr>
<tr>
<td>$(2^4 + 1)(2^{12} - 1) = 69615$</td>
<td>$1$</td>
<td>$2^{1+6+8} : \text{Sp}_6(2)$</td>
</tr>
<tr>
<td>$2^4(2^{12} - 1)(2^8 - 1) = 16707600$</td>
<td>$1$</td>
<td>$2^{1+6+7} : U_6(3).2$</td>
</tr>
<tr>
<td>$2^{12}(2^{12} - 1) = 16773120$</td>
<td>$1$</td>
<td>$\Omega^-_6(2)$</td>
</tr>
<tr>
<td>$2^{12}(2^8 - 1)(2^4 - 1) = 15667200$</td>
<td>$2$</td>
<td>$3D_4(2)$</td>
</tr>
</tbody>
</table>

Thus, by the mentioned lemmas in every case we know the orbits of the corresponding 2-point stabiliser on $A$ (up to a possibility that some orbits might split into two orbits of equal length). Now for an orbit $I_\gamma(x)$ of $E(x)$ on $\Gamma$ and for $y \in I_\gamma(x)$ we basically know the orbit lengths of $E(x) \cap E(y)$ on the set of vertices adjacent to $y$ in $\Gamma$ and we have to allocate these orbits. This we achieve by using the divisibility conditions and the suborbit diagram of $A$ which gives the intersection numbers of $\Gamma$ modulo 7.

In order to check that the allocation is correct, we calculate the first eigenmatrix of the corresponding association scheme and then the intersection of the other orbital graphs. These numbers must be integers which is a very strong condition and enabled us to eliminate all the wrong allocations. Below on the intersection diagrams of $\Gamma$ (but not of $A$ and $\Xi$) we present the intersection numbers as sums of the orbit lengths of the corresponding 2-point stabilisers. Since the first eigenmatrices are interesting by themselves we also include them. The rightmost column of the matrix contains the multiplicities of the association scheme arising from the permutation action of $E$ under consideration (cf. [1, Section 2.4]). In our cases these multiplicities are the degrees of the irreducible components of the permutational character. The remaining columns form the first eigenmatrix in the sense of [1], from which one can calculate all the intersection numbers of the association scheme.

4.1. $^2E_6(2)$-geometry

In this subsection $E = ^2E_6(2) : 2$ and $\tilde{F}_1 \cong F_4(2) \times 2$. The subdegrees of the action of $E$ on the set $\Gamma$ of cosets of $\tilde{F}$ are easy to deduce from Table 1 which gives the subdegrees of $3^2E_6(2)$ on the cosets of $F_4(2)$. It is easy to see that when the action is folded to $E' = \cong ^2E_6(2)$ the three triples of equal subdegrees are fused and the longest subdegree becomes three times shorter.

Consider the fixed vertices of the subgroups $S$ and $T$.

**Lemma 4.1.** (i) $C_\tilde{F}(S)/S \cong L_3(4) : 2$ acts on the set $A$ of size 120 as on the cosets of $C_\tilde{F}(S)/S \cong L_3(2) \times 2$;
(ii) $C_\tilde{F}(T)/T \cong U_6(2) : 2$ acts on the set $\Xi$ as on the cosets of $C_\tilde{F}(T)/T \cong \text{Sp}_6(2) \times 2$.

**Proof.** First we identify the classes of $S$ and $T$ in $E' = ^2E_6(2)$. By Conway et al. [3] $E'$ contains two classes of subgroups of order 7 with representatives $S_1$ and $S_2$ such that an
involution commuting with $S_1$ is central and with $S_2$ is not central. We claim that $S$ is a conjugate of $S_2$. The subgroup $H = E'(x) \cap E'(y) \cong 2^{1+6+8} : \text{Sp}_6(2)$ contains $S$ and it is the centraliser of a central involution, say $\tau$ in $E'(x) \cong F$. We need to show that $\tau$ is not central in $E'$. Indeed, otherwise $C_{E'}(\tau) \cong 2^{1+20} : U_6(2)$ and $H$ is self-normalised in $C_{E'}(\tau)$. On the other hand, an element in $E'$ which swaps $x$ and $y$ clearly normalises $H$ and centralises $\tau$. Now $|C_{E'}(S_1)| = 7 \cdot |L_3(2)|$ and $|C_{E'}(S_2)| = 7 \cdot |L_3(4)|$ and since there is a maximal subgroup $B \cong (L_3(2) \times L_3(4)).2$ in $E'$, (i) follows. Since $T$ is in the kernel of the action of $C_{E'}(S)$ on $A$, it is easy to see that $T$ is contained in the $L_3(2)$-direct factor of $O^2(B)$. Hence $C_E(S)/S$ contains $L_3(4)$ and $\text{Sp}_6(2)$. This and the list of maximal subgroups show that $T$ is generated by a $3A$-element and $C_{E'}(S) \cong 3 \times U_6(2)$, (ii) follows.

The suborbit diagram of the orbital of valency 21 of $L_3(4):2$ acting on the cosets of $L_3(2) \times 2$ is

```
12  6  8  9
  56
  3
  8  8
  21
  4
  1
```

The suborbit diagram of the orbital graph of valency 315 of $U_6(2):2$ acting on the cosets of $\text{Sp}_6(2) \times 2$ is

```
135  2240  162  96  207  3760
  18  12
  128  144
  315  42
  1
```

The above information enables us to apply the strategy described at the beginning of the section and to calculate the suborbit diagram of $2E_6(2):2$ acting on the cosets of
$F_4(2) \times 2$:

4.2. $3^2E_6(2) : 2$

Let $E = 3^2E_6(2) : 2$, non-split extension of $^2E_6(2) : 2$ by a normal subgroup of order 3 inverted by outer automorphism. The crucial observation for calculations in this section is that $C_E(S)/S$ and $C_E(T)/T$ do not split over $O_E(E)$. This specifies the centralisers up to isomorphism and enables us to calculate the suborbit diagram of $3 \cdot L_3(4)$: acting on the cosets of $L_3(2) \times 2$. 
Similarly we obtain the diagram of $3 \cdot U_6(2):2$ acting on the cosets of $\text{Sp}_6(2) \times 2$.

and finally of $3 \cdot 2^2 E_6(2):2$ on the cosets of $F_4(2) \times 2$:

- $2^{1+4+8} : \text{Sp}_6(2)$
- $\Omega^+_8(2) : 2$
- $F_4(2)$
- $F_4(2) \times 2$
The first eigenmatrix is the following:

\[
\begin{pmatrix}
1 & 69615 & 19009536 & 16707600 & 33415200 & 139230 & 2 & 1 \\
1 & 3951 & 98304 & -36720 & -73440 & 7902 & 2 & 48320 \\
1 & 4335 & 61200 & -61200 & -4335 & -1 & 93366 \\
1 & 495 & -12288 & 3600 & 7200 & 990 & 2 & 2909907 \\
1 & -273 & 0 & 4368 & -4368 & 273 & -1 & 14389650 \\
1 & -81 & 1536 & -432 & -864 & -162 & 2 & 20155200 \\
1 & 111 & 0 & -2160 & -2160 & -111 & -1 & 31744440
\end{pmatrix}
\]

4.3. \(2^{26} : Fr(2)\)-geometry

Let \(E\) be the semidirect product of \(F \cong Fr(2)\) and a 26-dimensional GF(2)-module \(V\) of \(F\). It is well known \([10]\) that there are two such modules permuted by the outer automorphism of \(F\). The stabilisers in \(F\) of vectors in \(V\) were calculated in \([2]\) which are the same as the 2-point stabilisers of the permutation action of \(E\) on the cosets of \(\tilde{F} \cong F\).

Let us turn to the subgroups \(S\) and \(T\) and their fixed vertices (equivalently fixed vectors in \(V\)).

**Lemma 4.2.**

(i) \(CE(S)/S \cong 2^8 : L_3(2)\), a semidirect product of \(C_F(S)/S \cong L_3(2)\) and its Steinberg module;

(ii) \(CE(T)/T \cong 2^{14} : Sp_6(2)\), a semidirect product of \(C_F(T)/T \cong Sp_6(2)\) and its 14-dimensional GF(2)-module.

**Proof.** One can determine the submodule structure of \(V\) with respect to \(N_F(S) \cong (\mathbb{F}_2^5 \times \text{PGL}_2(7))^+\) as in Lemma 2.10 and to \(N_F(T) \cong S_3 \times \text{Sp}_6(2)\) from the very basic principles.

The suborbit diagram of \(2^8 : L_3(2)\) on the cosets of \(L_3(2)\) is the following:
the one of $2^{14} : \text{Sp}_6(2)$ on the cosets of $\text{Sp}_6(2)$ is

Finally we get the diagram of $2^{26} : F_4(2)$ on the cosets of $F_4(2)$.
Eigenmatrix of the (affine) association scheme is the following.

\[
\begin{bmatrix}
1 & 69,615 & 17,821,440 & 16,707,600 & 69,888 & 16,773,120 & 15,667,200 & 1 \\
1 & 4079 & 61,184 & -4080 & 4352 & -4096 & -61440 & 69,615 \\
1 & 239 & -4096 & -240 & 0 & 4096 & 0 & 17,821,440 \\
1 & -17 & -256 & 16 & 256 & -4096 & 4096 & 16,707,600 \\
1 & 4335 & 0 & 61,200 & -4096 & -61,440 & 0 & 69,888 \\
1 & -17 & 4352 & -4080 & -256 & 0 & 0 & 16,773,120 \\
1 & -273 & 0 & 4368 & 0 & 0 & -4096 & 15,667,200 \\
\end{bmatrix}
\]

4.4. $E_6(2)$-geometry

Finally let $E = E_6(2):2$ and $\tilde{F} \cong F_4(2) \times 2$. The 2-point stabilisers are given in Table 3.

**Lemma 4.3.**

(i) $C_E(S)/S \cong (L_3(2) \times L_3(2)).2$ acts on the set $\Lambda$ of size 168 as on the cosets of $L_3(2) \times 2$;

(ii) $C_E(T)/T \cong L_6(2):2$ acts on the cosets of $\text{Sp}_6(2) \times 2$.

**Proof.** If the character table of $E_6(2)$ would be available for us we could probably proceed as in the proof of Lemma 4.1. But we were not able to obtain this table. Although luckily we already collected enough information. Namely, every 2-point stabiliser in Table 3 already appeared either in Table 1 or 2 and thus we know exactly how many vertices in the corresponding orbital are fixed by $S$ and $T$. Thus we know the orders of $C_E(S)$ and $C_E(T)$. Finally, looking at the maximal subgroups of $E'$ in [3] we identify the isomorphism type of the centralisers. \(\square\)

The suborbit diagram of the graph of valency 21 of $L_3(2) \times L_3(2)$ on the cosets of $L_3(2)$ is given below; two orbitals of the same valency are fused under the automorphism which permutes the direct factors.

The suborbit diagram of the graph of valency 315 of $L_6(2)$ on the cosets of $\text{Sp}_6(2)$ is given below. Similarly, the orbitals of equal valencies are fused under the contragredient
automorphism.

Finally, we get the suborbit diagram of $E_6(2)$ on the cosets of $F_4(2)$ and the outer automorphism fuses the orbitals of equal valencies.
The first eigenmatrix is the following:

\[
\begin{bmatrix}
1 & 69,615 & 16,773 & 120 & 16,707 & 600 & 15,667 & 200 & 15,667 & 200 & 1 \\
1 & 4207 & 28,672 & 28,560 & −30,720 & −30,720 & 137,020 \\
1 & −273 & 0 & 4368 & −2048 & −2048 & \sqrt{7} & −2048 & +2048 & \sqrt{7} & 8,655 & 975 \\
1 & −273 & 0 & 4368 & −2048 & +2048 & \sqrt{7} & −2048 & −2048 & \sqrt{7} & 8,655 & 975 \\
1 & −17 & 4096 & −4080 & 0 & 0 & 21,622 & 965 \\
1 & 175 & −3584 & 336 & 1536 & 1536 & 25,812 & 800
\end{bmatrix}
\]

References