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Secrecy Gain of Gaussian Wiretap Codes from 2- and 3-Modular Lattices

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Abstract—Lattice coding over a Gaussian wiretap channel is considered with respect to a lattice invariant called the secrecy gain, which was introduced in [1] to characterize the confusion that a chosen lattice can cause at the eavesdropper: the higher the secrecy gain of the lattice, the more confusion. In this paper, secrecy gains of several 2- and 3-modular lattices are computed. Most are shown to have a secrecy gain larger than the best unimodular lattices can achieve.

I. INTRODUCTION

The wiretap channel was introduced by Wyner [2] as a discrete memoryless broadcast channel where the sender Alice transmits confidential messages to a legitimate receiver Bob, in the presence of an eavesdropper Eve. Both reliable and confidential communication between Alice and Bob should be ensured at the same time, by exploiting the physical difference between the channel to Bob and that to Eve via coding. A survey on this topic is available in [3]. Many results of information theoretical nature are available for various classes of channels capturing the trade-off between reliability and secrecy and aiming at determining the highest information rate that can be achieved with perfect secrecy, the so-called secrecy capacity. Coding results on constructing concrete codes that can be implemented in a specific channel are much fewer (see e.g. [4], [5] for examples of wiretap codes dealing with channels with erasures).

In this paper, we discuss Gaussian wiretap channels, whose secrecy capacity was established in [6]. Examples of existing Gaussian wiretap codes were designed for binary inputs, as in [7]. A different approach was adopted in [1], where lattice codes were proposed, using as design criterion a new lattice invariant called secrecy gain, which was shown to characterize the confusion at the eavesdropper. This suggests the study of the secrecy gain of lattices as a way to understand how to design a good Gaussian wiretap code. So far, the special class of unimodular lattices has been studied with particular emphasis on even unimodular lattices [8], [9]: the secrecy gain of extremal even unimodular lattices was computed, and the asymptotic behavior of the average secrecy gain as a function of the dimension \( n \) was investigated. The results show that maximizing the secrecy gain is meaningful in small dimensions and infinite secrecy gain (implying complete secrecy) is possible as \( n \) grows to infinity. Unimodular lattices in small dimensions, both odd and even, were considered in [10], [11] resulting in a complete classification in dimension \( n \), \( n \leq 23 \), featuring some odd lattices outperforming their even counter parts. It is natural to wonder whether other classes of lattices would have a better secrecy gain. The contribution of this paper is to initiate the study of the secrecy gain of other modular lattices. Since the secrecy gain depends on the theta series of the lattice considered, it makes sense to start with even 2- and 3-modular lattices, whose theta series have a decomposition formula similar to that of even unimodular lattices. Our results show that 2- and 3-modular lattices indeed achieve a secrecy gain larger than that of unimodular lattices in all the dimensions treated so far, but for \( n = 22 \).

II. PRELIMINARIES AND PREVIOUS RESULTS

A Gaussian wiretap channel is a broadcast Gaussian channel modeled by

\[
\begin{align*}
\mathbf{y} &= \mathbf{x} + \mathbf{v}_b \\
\mathbf{z} &= \mathbf{x} + \mathbf{v}_e,
\end{align*}
\]

where \( \mathbf{x} \) is the codeword sent by the transmitter (Alice), \( \mathbf{y} \) and \( \mathbf{z} \) are the received signals at the legitimate receiver (Bob), respectively at the eavesdropper (Eve), with corresponding noise vectors \( \mathbf{v}_b \) and \( \mathbf{v}_e \), whose components are i.i.d. Gaussian distributed with zero mean and respective variance \( \sigma_b^2 \) and \( \sigma_e^2 \).

It is assumed that \( \sigma_b^2 < \sigma_e^2 \) in order to have a positive secrecy capacity [6]. We suppose that \( \mathbf{x} \in \mathbb{R}^n \) is a lattice codeword, where by a lattice \( \Lambda \) we mean a discrete set of points in \( \mathbb{R}^n \), which can be conveniently described by

\[ \Lambda = \{ \mathbf{x} = \mathbf{u}M | \mathbf{u} \in \mathbb{Z}^n \}, \]

where the generator matrix \( M \) stores as row vectors a basis for the lattice.

Lattice encoding for the wiretap channel (1) is done via a generic coset coding strategy [1]: let \( \Lambda_c \subset \Lambda_b \) be two nested lattices, specially chosen such that the quotient group \( \Lambda_b/\Lambda_c \) is of size \( 2^k \). A \( k \)-bit message is then mapped to a coset in \( \Lambda_b/\Lambda_c \), after which a vector is randomly chosen from the coset as the encoded word. The lattice \( \Lambda_c \) will be interpreted as introducing confusion for Eve, while \( \Lambda_b \) as ensuring reliability for Bob. A bound on Eve’s probability \( P_{c,e} \) of correct decoding shows [1] that to minimize \( P_{c,e} \) is to minimize

\[ \sum_{\mathbf{u} \in \Lambda_c} e^{-||\mathbf{u}||^2/2\sigma_e^2} \]

for Bob. A bound on Eve’s probability \( P_{c,e} \) of correct decoding shows [1] that to minimize \( P_{c,e} \) is to minimize

\[ \sum_{\mathbf{u} \in \Lambda_c} e^{-||\mathbf{u}||^2/2\sigma_e^2} \]
where $|t|^2$ is the norm (square root length) of a lattice point. Let $\mathcal{H} = \{a + ib \in \mathbb{C} | b > 0\}$ denote the upper half plane and set

$$q = e^{\pi i \tau}, \quad \tau \in \mathcal{H}.$$ 

Definition 2.1: The theta series of a lattice $\Lambda$ is defined by

$$\Theta_{\Lambda}(\tau) = \sum_{\lambda \in \Lambda} q^{||\lambda||^2}.$$ 

It can be easily recognized that (2) is in fact the theta series of $\Lambda_\varepsilon$ at $\tau = \frac{1}{2\pi i}$. Motivated by the above argument, the confusion brought by the lattice $\Lambda_\varepsilon$ with respect to no coding (namely, use the lattice $\mathbb{Z}^n$ scaled to the same volume, where the volume $\text{vol}(\Lambda)$ of a lattice $\Lambda$ is by definition the determinant of its generator matrix) is measured as follows [11]:

Definition 2.2: Let $\Lambda$ be an $n$-dimensional lattice of volume $V$. The secrecy function of $\Lambda$ is given by

$$\Xi_{\Lambda}(\tau) = \frac{\Theta_{\sqrt{n} \mathbb{Z}^n}(\tau)}{\Theta_{\Lambda}(\tau)}, \quad \tau \in \mathcal{H}.$$ 

The secrecy gain is the maximal value of the secrecy function with respect to $\tau$ and is denoted by $\chi_{\Lambda}$. The theta series of a lattice is in general difficult to analyze except for special classes of lattices. We will take a closer look at one of them, the family of $\ell$-modular lattices. The dual of a lattice $\Lambda$ of dimension $n$ is defined to be

$$\Lambda^* = \{x \in \mathbb{R}^n : x \cdot \lambda \in \mathbb{Z}, \lambda \in \Lambda\}.$$ 

$\Lambda$ is said to be an integral lattice if $\Lambda \subset \Lambda^*$. The norm of any lattice point in an integral lattice $\Lambda$ is always an integer. If the norm is even for any lattice point, then $\Lambda$ is called an even lattice. Otherwise, it is called an odd lattice. The theta series of an integral lattice has a neat representation. Since the norms are all integers, we can combine the terms with the same norm and write

$$\Theta_{\Lambda}(\tau) = \sum_{n=0}^{\infty} A_n q^n.$$ 

A lattice is said to be equivalent, or geometrically similar to its dual, if it differs from its dual only by possibly a rotation, reflection and change of scale. An integral lattice that is equivalent to its dual is called a modular lattice. Or as it was first defined by H.-G. Quebbemann [12], an $n$-dimensional integral lattice $\Lambda$ is modular if there exists a similarity $\sigma \in \mathbb{R}^n$ such that $\sigma(\Lambda^*) = \Lambda$. If $\sigma$ multiplies norms by $\ell$, $\Lambda$ is said to be $\ell$-modular. In the particular case when the similarity factor $\ell = 1$, such lattices are unimodular lattices, corresponding to the more familiar definition $\Lambda = \Lambda^*$. It can be shown that if $\Lambda$ is an $\ell$-modular lattice of dimension $n$, then

$$\text{vol}(\Lambda) = \ell^{\frac{n}{2}}.$$ 

We will need the following concepts and formulae from analytic number theory for our discussion, for which we refer the readers to [13], [14] for more details.

Definition 2.3: The Jacobi theta functions are defined as follows:

$$\begin{align*}
\vartheta_2(\tau) &= \sum_{n\in\mathbb{Z}} q^{(n+\frac{1}{2})^2}, \\
\vartheta_3(\tau) &= \sum_{n\in\mathbb{Z}} q^{n^2}, \\
\vartheta_4(\tau) &= \sum_{n\in\mathbb{Z}} q^{-(\frac{n}{2})^2}.
\end{align*}$$

Definition 2.4: The Dedekind eta function is defined by

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^{2n}).$$

The Jacobi theta functions and the Dedekind eta function are connected as follows [13]:

$$\begin{align*}
\vartheta_2(\tau) &= \frac{2\eta(2\tau)^2}{\eta(\tau)^2}, \\
\vartheta_3(\tau) &= \frac{\eta(2\tau)^2}{\eta(\tau)^2}, \\
\vartheta_4(\tau) &= \frac{\eta(\tau)^2}{\eta(\tau)^2}.
\end{align*}$$

The following lemma [12] will play a crucial role.

Lemma 2.5: The theta series of an even $\ell$-modular lattice of dimension $n = 2k$ when $\ell = 1$, $2$, $3$ belongs to a space of modular forms of weight $k$ generated by the functions $\Theta_{2k}\tau(\tau)\Delta_{2k}\tau(\tau)$ with integers $\lambda$, $\mu \geq 0$ satisfying $k_0\lambda + k_1\mu = k$, where for $\ell = 1, 2, 3, k_0 = 0, 2, 1$ respectively, $k_1 = \frac{24}{\ell^2}, \Theta_{2k_0}\tau(\tau)$ denote the theta series of the modular lattices $E_8, D_4$ and $A_2$, respectively, and $\Delta_{2k_1}(\tau) = (\eta(\tau)\eta(\ell\tau))^{k_1}$.

Example 2.6: If $\ell = 1$, corresponding to the unimodular case, we know from Lemma 2.5 that $k_0 = 4, k_1 = \frac{24}{2^2} = 12, \Theta_{2k_0}\tau(\tau) = \Theta_{E_8}\tau(\tau), \Delta_{2k_1}(\tau) = \eta^{24}(\tau)$, from which we deduce that if $\Lambda$ is an even unimodular lattice of dimension $n = 2k$ then

$$\Theta_{\Lambda}(\tau) = \sum_{4\lambda + 12\mu = k} a_{\lambda,\mu} \Theta_{E_8}\tau(\tau)\Delta_{24}\tau(\tau).$$

The formula (6) was adopted in [8], [9] to compute the secrecy gains of several even unimodular lattices.

III. Secrecy Gain of 2- and 3-Modular Lattices

Lattices that are equivalent to their duals were shown to have a symmetry point, called weak secrecy gain, at $\tau = \frac{1}{\text{vol}(\Lambda)^{\frac{1}{2}}}$ in their secrecy function [9]. It was conjectured in the same paper that for these lattices, the weak secrecy gain is actually the secrecy gain. Here we still use $\chi_{\Lambda}$ to denote the weak secrecy gain. This conjecture was recently proven by A.-M. Ernvall-Hytönen [15] for a special class of lattices called extremal even unimodular lattices and by F. Lin and F. Oggier [11] for unimodular lattices in dimension $n, 8 < n \leq 23$. In this paper, we will compute the weak secrecy gain of known even 2- and 3-modular lattices, and claim that their secrecy gains are larger than that of unimodular lattices in the same dimension, since the secrecy gain is by definition the maximum of the secrecy function.

In order to write the secrecy function, we need to have the theta series of $\mathbb{Z}^n$ scaled to the right volume. Recall the
definition of the theta series of a lattice and the Jacobi theta function \( \vartheta_3(\tau) \). We have that

\[
\begin{align*}
\Theta_{2k}(\tau) &= \vartheta_3(\tau), \\
\Theta_{2k}(\tau) &= \vartheta_3^{k}(\tau), \\
\Theta_{k2}(\tau) &= \vartheta_3(k^2\tau).
\end{align*}
\]  

(7)

Now it follows from (4) and (7) that the theta series is

\[
\Theta_{\mathbb{Z}^n}(\tau) = \vartheta_3^{n}(\sqrt{\ell}\tau).
\]  

(8)

A. 2-modular lattices

According to Lemma 2.5, the theta series of an even 2-modular lattice \( \Lambda \) of dimension \( n = 2k \) can be written as

\[
\Theta_{\Lambda}(\tau) = \sum_{2\lambda+b_\mu=k} a_{\mu}\Theta_{D_4}(\tau)\Delta_{16}^{\mu}(\tau),
\]  

(9)

where

\[
\Theta_{D_4}(\tau) = \frac{1}{2} (\vartheta_3^4(\tau) + \vartheta_4^4(\tau)) = 1 + 24q^2 + 24q^4 + 96q^6 + \cdots
\]  

and

\[
\Delta_{16}(\tau) = (\eta(\tau)\eta(2\tau))^8.
\]  

(10)

By (5), we can write \( \Delta_{16}(\tau) \) in terms of Jacobi theta functions and compute the first few terms:

\[
\Delta_{16}(\tau) = \frac{1}{256} \vartheta_3^5(\tau)\vartheta_4^4(\tau)\vartheta_4^2(\tau) = q^2 - 8q^4 + 12q^6 + \cdots
\]  

(11)

The secrecy function of an even 2-modular lattice \( \Lambda \) of dimension \( n \) is then written as

\[
\Xi_{\Lambda}(\tau) = \sum_{2\lambda+b_\mu=k} a_{\mu}\Theta_{D_4}(\tau)\Delta_{16}^{\mu}(\tau),
\]  

or more conveniently,

\[
\frac{1}{\Xi_{\Lambda}(\tau)} = \sum_{2\lambda+b_\mu=k} a_{\mu}\Theta_{D_4}(\tau)\Delta_{16}^{\mu}(\tau) = \sum_{2\lambda+b_\mu=k} a_{\mu} \left( \frac{\Theta_{D_4}(\tau)}{\vartheta_3^2(\sqrt{2}\tau)} \right)^k.
\]  

Now we only need to know the coefficients \( a_{\mu} \) in order to compute the weak secrecy gain of a 2-modular lattice, once the following two quotients are computed:

\[
\frac{\Theta_{D_4}(\tau)}{\vartheta_3^2(\sqrt{2}\tau)} \bigg|_{\tau = \frac{i}{\sqrt{2}}} \text{ and } \frac{\Delta_{16}(\tau)}{\vartheta_3^2(\sqrt{2}\tau)} \bigg|_{\tau = \frac{i}{\sqrt{2}}}.
\]

Approximating \( \alpha_2 = \frac{\Theta_{D_4}(\tau)}{\vartheta_3^2(\sqrt{2}\tau)} \bigg|_{\tau = \frac{i}{\sqrt{2}}} \)

We recognize that \( \alpha_2 \) is basically the weak secrecy gain of the 2-modular lattice \( D_4 \) (or rather its inverse). Fig. 1 gives a plot of the secrecy function of \( D_4 \), where (i) we set \( y = -i\tau \) and restrict to real positive values of \( y \), since by (2) we are only interested in the values of \( \Theta_{D_4}(\tau) \) with \( \tau = yi, y > 0 \) and (ii) \( y \) is plotted in decibels to transform the multiplicative symmetry point into an additive symmetry point. We can see clearly that the secrecy function has a symmetry point at \( y \approx -1.5 \text{ dB} \) corresponding to \( \tau = \frac{i}{\sqrt{2}} \). The value of the secrecy function \( \Theta_{D_4}(\tau) = \vartheta_3^2(\sqrt{2}\tau) \) at \( \tau = \frac{i}{\sqrt{2}} \) is \( \frac{1}{\sqrt{2}} \), namely, the weak secrecy gain can be approximated by

\[
\lambda_{D_4} \approx 1.08356
\]  

(12)

\[
\beta_2 \approx 0.00283366.
\]  

(14)

We start by explaining how the coefficients \( a_{\mu} \)'s in (9) are computed. By substituting (10) and (11) into (9), we have a formal sum with coefficients represented by the \( a_{\mu} \)'s. Then by comparing this formal sum with (3), we obtain a number of linear equations in the \( a_{\mu} \)'s. When we have enough equations, the \( a_{\mu} \)'s can be recovered by solving a linear system. Now, let us compute the weak secrecy gain of the 16-dimensional 2-modular Barnes-Wall lattice \( BW_{16} \) as an example.

Example 3.1: \( BW_{16} \) is an even lattice with minimum norm 4. The theta series of \( BW_{16} \) looks like

\[
\Theta_{BW_{16}}(\tau) = 1 + 0q^2 + A_4q^4 + \cdots, \quad A_4 \neq 0.
\]
On the other hand, by (9), (10) and (11),
\[
\Theta_{BW_{16}}(\tau) = a_0 \Theta_{A_2}^1(\tau) + a_1 \Delta_{12}(\tau)
\]
\[
= a_0(1 + 24q^2 + \cdots)^4 + a_1(q^2 + \cdots)
\]
\[
= a_0 + (96a_0 + a_1)q^2 + \cdots.
\]
We now have two linear equations in two unknowns \( a_0 \) and \( a_1 \)
\[
\begin{cases}
    a_0 &= 1 \\
    96a_0 + a_1 &= 0
\end{cases}
\]
which gives \( a_0 = 1 \) and \( a_1 = -96 \), yielding the weak secrecy gain
\[
\chi_{BW_{16}} = \frac{1}{\alpha_2^2 - 96\beta_2} \approx 2.20564. \tag{15}
\]
See Figure 3 for a plot of the secrecy function of \( BW_{16} \) for verification.

![Figure 3. Secrecy function of \( BW_{16} \)](image)

In this way, we have computed the weak secrecy gains of the even 2-modular lattices we found in [16] in dimensions smaller than 24 as shown in Table I.

B. 3-modular lattices

According to Lemma 2.5, the theta series of an even 3-modular lattice \( \Lambda \) of dimension \( n = 2k \) can be written as
\[
\Theta_{\Lambda}(\tau) = \sum_{\lambda + \mu = k} a_\mu \Theta_{A_2}^\lambda(\tau) \Delta_{12}^{\mu}(\tau), \tag{16}
\]
where
\[
\Theta_{A_2}(\tau) = \vartheta_2(2\tau)\vartheta_2(6\tau) + \vartheta_3(2\tau)\vartheta_3(6\tau)
\]
\[
= 1 + 6q^2 + 9q^4 + 6q^6 + \cdots \tag{17}
\]
and
\[
\Delta_{12}(\tau) = (\eta(\tau)\eta(3\tau))^6.
\]
We can also compute the first few terms of \( \Delta_{12}(\tau) \):
\[
\Delta_{12}(\tau) = q^2 - 6q^4 + 9q^6 + \cdots. \tag{18}
\]
The secrecy function of an even 3-modular lattice \( \Theta \) of dimension \( n \) is
\[
1 / \Xi_{\Lambda}(\tau) = \sum_{\lambda + \mu = k} a_\mu \Theta_{A_2}^\lambda(\tau) \Delta_{12}^{\mu}(\tau)
\]
\[
= \sum_{\lambda + \mu = k} a_\mu \left( \frac{\Theta_{A_2}(\tau)}{\sigma_2(\sqrt{3}\tau)} \right)^\lambda \left( \frac{\Delta_{12}(\tau)}{\sigma_1(\sqrt{3}\tau)} \right)^\mu.
\]
We can similarly approximate the two quotients
\[
\begin{cases}
    \alpha_3 = \frac{\Theta_{A_2}(\tau)}{\sigma_2(\sqrt{3}\tau)} |_{\tau = \tau_0} \approx 0.982424 \\
    \beta_3 = \frac{\Delta_{12}(\tau)}{\sigma_1(\sqrt{3}\tau)} |_{\tau = \tau_0} \approx 0.00832474
\end{cases}
\]
where we recognize in the first term the inverse of the weak secrecy gain of the 3-modular lattice \( A_2 \). Fig. 4 gives a plot of the secrecy function of \( A_2 \) and the weak secrecy gain is approximated by
\[
\chi_{A_2} \approx 1.01789. \tag{19}
\]
The second term \( \beta_3 \) is approximated numerically as we did for \( \beta_2 \) and is omitted.

![Figure 4. Secrecy function of \( A_2 \)](image)

Now let us compute the weak secrecy gain of the Coxeter-Todd Lattice \( K_{12} \) as an example.

Example 3.2: The Coxeter-Todd Lattice \( K_{12} \) is an even lattice with minimum norm 4. The theta series of \( K_{12} \) looks like
\[
\Theta_{K_{12}}(\tau) = 1 + 0q^2 + 4q^4 + \cdots, \text{ } A_4 \neq 0.
\]
On the other hand, by (16), (17) and (18),
\[
\Theta_{K_{12}}(\tau) = a_0 \Theta_{A_2}^0(\tau) + a_1 \Delta_{12}(\tau)
\]
\[
= a_0(1 + 6q^2 + \cdots)^6 + a_1(q^2 + \cdots)
\]
\[
= a_0(1 + 36q^2 + \cdots) + a_1(q^2 + \cdots)
\]
\[
= a_0 + (36a_0 + a_1)q^2 + \cdots.
\]
We now have two linear equations in two unknowns \( a_0 \) and \( a_1 \)
\[
\begin{cases}
    a_0 = 1 \\
    36a_0 + a_1 = 0
\end{cases}
\]
which gives \( a_0 = 1 \) and \( a_1 = -36 \), yielding the weak secrecy gain
\[
\chi_{K_{12}} = \frac{1}{\alpha_3^0 - 36\beta_3} \approx 1.66839. \tag{20}
\]
and 3-modular lattices. Besides, it is likely that one can prove the conjecture for these particular cases using the same proof technique as in [11], however it would be nice to have a more general proof. Also, the encoding/labeling of the proposed wiretap codes, taking into account both channel parameters and power constraint are being naturally elaborated.

**ACKNOWLEDGMENT**

The research of F. Lin and F. Oggier is supported by the Singapore National Research Foundation under the Research Grant NRF-RF2009-07.

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