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<td>Xing, Chaoping; Ling, San</td>
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A Class of Linear Codes with Good Parameters from Algebraic Curves
Chaoping Xing and San Ling

Abstract—A class of linear codes with good parameters is constructed in this correspondence. It turns out that linear codes of this class are subcodes of the subfield subcodes of Goppa’s geometric codes. In particular, we find 61 improvements on Brouwer’s table [1] based on our codes.

Index Terms—Algebraic curves, algebraic-geometry codes, subfield subcodes.

I. INTRODUCTION

Algebraic-geometry codes constructed by Goppa [2] make use of algebraic curves with many rational points. These codes have excellent asymptotic parameters. In particular, the $q$-ary Gilbert–Varshamov bound was broken by Goppa’s geometric codes for some sufficiently large $q$ [8], [3].

However, for small $q$, it seems difficult to find many good codes by Goppa’s construction. The reason is that the number of rational points of an algebraic curve over $\mathbb{F}_q$ is not satisfactorily to construct good Goppa’s geometric codes for small $q$. In order to increase the length of geometric codes, researchers have been looking for possibilities to use points over some extensions of $\mathbb{F}_q$ to construct good codes [5], [10], [11], [4], [12]. In this correspondence, we make use of curves defined over some extension of $\mathbb{F}_q$ to construct codes over $\mathbb{F}_q$ with larger length. It turns out that codes constructed in this way are subcodes of the subfield subcode of Goppa’s geometric codes. It is well known that one cannot expect any good results if we just use a general lower bound (see (1) of Section II) to estimate dimensions of subfield subcodes of algebraic-geometry codes. However, by using our explicit construction, we can determine dimensions of our codes that exceed the lower bound (1) of Section II on dimensions of subfield subcodes. Consequently, we obtain some linear codes with good parameters. In particular, 61 improvements on Brouwer’s table are found for $q = 8$ and 9 by using elliptic curves.

The correspondence is organized as follows. In Section II, we introduce Goppa’s construction of algebraic-geometry codes and a general lower bound on dimensions of subfield subcodes. Section III is devoted to construction of a class of linear codes that are subcodes of subfield subcodes of algebraic-geometry codes. Parameters of these codes are estimated in this section. Finally, in Section IV, we list some improvements on Brouwer’s table based on the codes of Section III.

II. BACKGROUND

In this section, we briefly introduce Goppa’s geometric codes and subfield subcodes.

Let $r$ be a prime power and $\mathcal{X}$ a projective, absolutely irreducible, complete algebraic curve over $\mathbb{F}_r$ (we simply express this fact by saying that $\mathcal{X}/\mathbb{F}_r$ is an algebraic curve). Let $\mathcal{F}_r(\mathcal{X})$ be the function field of $\mathcal{X}/\mathbb{F}_r$. For a divisor $G$ (we always mean a rational divisor whenever a divisor is mentioned in this correspondence) of $\mathcal{X}/\mathbb{F}_r$, the linear space

$$\mathcal{L}(G) = \{ f \in \mathcal{F}_r(\mathcal{X}) \mid \text{div}(f) + G \geq 0 \} \cup \{ 0 \}$$

is a finite-dimensional linear space over $\mathbb{F}_r$. Moreover, by the Riemann–Roch theorem [6], [7] have

$$l(G) := \dim \mathcal{L}(G) \geq \deg(G) + 1 - g$$

where $g$ is the genus of $\mathcal{X}$. The equality holds if $\deg(G) \geq 2g - 1$.

Let $P_1, P_2, \ldots, P_n$ be $n$ distinct rational points of $\mathcal{X}$ satisfying

$$\text{Supp}(G) \cap \{ P_1, P_2, \ldots, P_n \} = \emptyset.$$ 

We denote by $C(P_1, P_2, \ldots, P_n; G)$ the set

$$C(P_1, P_2, \ldots, P_n; G) = \{ (f(P_1), f(P_2), \ldots, f(P_n)) \mid f \in \mathcal{L}(G) \}.$$

It is clear that $C(P_1, P_2, \ldots, P_n; G)$ is a linear code over $\mathbb{F}_r$. Furthermore, we have

**Proposition 2.1 (Goppa):** Let $\mathcal{X}/\mathbb{F}_r$ be an algebraic curve with $n$ distinct points $\{ P_1, P_2, \ldots, P_n \}$. Suppose $G$ is a divisor of $\mathcal{X}$ with $\text{Supp}(G) \cap \{ P_1, P_2, \ldots, P_n \} = \emptyset$ and $\deg(G) < n$. Then $C(P_1, P_2, \ldots, P_n; G)$ is an $[n, k, d]$ linear code over $\mathbb{F}_r$ with

$$k = l(G) \geq \deg(G) + 1 - g \quad d \geq n - \deg(G).$$

Next we introduce a general lower bound on dimensions of subfield subcodes of linear codes.

Let $r = q^s$ for some prime power $q$ and a positive integer $s$. Suppose that $C$ is a linear code over $\mathbb{F}_r$ with parameters $[N, K, D]$. Consider the intersection

$$C|\mathbb{F}_q := C \cap \mathbb{F}_q^N.$$ 

It is clear that $C|\mathbb{F}_q$ is a linear code over $\mathbb{F}_q$. It is called the subfield subcode of $C$. $C|\mathbb{F}_q$ has length $N$ and distance of at least $D$.

**Lemma 2.2 (see [6], [7]):** If $C$ is a linear code over $\mathbb{F}_r$, $\mathbb{F}_{r^k}$ with parameters $[N, K, D]$, then $C|\mathbb{F}_q$ is a linear code over $\mathbb{F}_q$ with parameters $[N, K, D]$, satisfying $d \geq D$ and

$$d \geq sK + (s - 1)N. \tag{1}$$

III. CONSTRUCTIONS

In this section, we construct a class of linear codes over $\mathbb{F}_q$ based on curves over $\mathbb{F}_{q^2}$. In fact, codes constructed in this section are subcodes of subfield subcodes of algebraic-geometry codes over $\mathbb{F}_{q^2}$.

First we fix some notations for this section.

$q$ a prime power;

$$\mathcal{X}/\mathbb{F}_{q^2}$$ an algebraic curve;

$\mathcal{G} := \mathcal{G}(\mathcal{X})$ the genus of $\mathcal{X}/\mathbb{F}_{q^2}$;

$\mathcal{F}_{q^2}(\mathcal{X})$ the function field of $\mathcal{X}/\mathbb{F}_{q^2}$;

$$\mathcal{L}(\mathcal{F}_{q^2})$$ the set of $\mathcal{F}_{q^2}$-rational points of $\mathcal{X}$.

We label the elements of $\mathcal{L}(\mathcal{F}_{q^2})$ as $P_0, P_1, \ldots, P_n$, where $n = |\mathcal{X}(\mathbb{F}_{q^2})| - 1$.

For $m \geq g$, we have $l(mP_0) \geq m + 1 - g \geq 1$ by the Riemann–Roch theorem. Therefore, we can find $m + 1 - g$ monomials

$$0 = v_0 < v_2 < \cdots < v_{m+1-g} \leq mP_0,$$

i.e., there exist $m + 1 - g$ functions $f_1, f_2, \ldots, f_{m+1-g}$ of $\mathcal{L}(mP_0)$ such that

$$v_{m+1-g}(f_i) = -v_i. \tag{2}$$
for all $1 \leq i \leq m + 1 - g$, where $\nu_{P_0}$ denotes the normalized discrete valuation of $P_0$. Since $v_i$ are strictly increasing, we obtain

$$v_i \leq v_{m+1-g} - ((m+1-g) - i) \leq m - (m+1-g - i) = g + i - 1.$$  \hfill (3)

Put

$$e_{ij} = f^j_i f_j + f_i f^j_j$$

for all $1 \leq i < j \leq m + 1 - g$, and

$$e_{ii} = f^{i+1}_i$$

for all $1 \leq i \leq m + 1 - g$.

Choose an element $\gamma$ of $\mathbb{F}_q^2 - \mathbb{F}_q$ and put

$$e'_{ij} = \gamma f^j_i f_j + \gamma f_i f^j_j$$

for all $1 \leq i < j \leq m + 1 - g$.

**Lemma 3.1:** For any $\mathbb{F}_q$-rational point $P \neq P_0$ of $\mathcal{X}/\mathbb{F}_q$, both $e_{ij}(P)$ and $e'_{ij}(P)$ are elements of $\mathbb{F}_q$, for all $i, j$.

**Proof:** It is clear that $P_0$ is the unique possible pole of $e_{ij}$ and $e'_{ij}$. Therefore, $e_{ij}(P)$ and $e'_{ij}(P)$ are well defined.

Considering

$$e_{ij}(P)^3 = (f^j_i f_j + f_i f^j_j)^3 = f^{i+1}_i (P)^3 + f^j_j (P)^3 = e_{ij}(P)$$

and

$$e'_{ij}(P)^3 = (\gamma f^j_i f_j + \gamma f_i f^j_j)^3 = (\gamma^3 f^j_i f_j + \gamma f^j_j (P))^3 = \gamma f_i (P)^3 f^j_j (P) + \gamma^3 f^j_j f_i (P) = e'_{ij}(P)$$

for all $1 \leq i < j \leq m + 1 - g$, and

$$e_{ii}(P)^3 = (f^{i+1}_i (P))^3 = f^{i+1}_i (P) = e_{ii}(P)$$

for all $1 \leq i \leq m + 1 - g$.

Let $U_m$ be the $\mathbb{F}_q$-linear span of the set

$$\{e_{ij} | 1 \leq i \leq j \leq m + 1 - g\} \cup \{e'_{ij} | 1 \leq i < j \leq m + 1 - g\}.$$  

**Lemma 3.2:** The dimension of the $\mathbb{F}_q$-linear space $U_m$ is equal to $(m + 1 - g)^2$ if $m < q$.

**Proof:** The cardinality of the set

$$\{e_{ij} | 1 \leq i \leq j \leq m + 1 - g\} \cup \{e'_{ij} | 1 \leq i < j \leq m + 1 - g\}$$

is equal to $(m + 1 - g)^2$, we have to prove that these $(m + 1 - g)^2$ elements are $\mathbb{F}_q$-linearly independent. First, for $1 \leq i < j \leq m + 1 - g$, we have

$$\nu_{P_0}(f^j_i f_j) = -q v_i - v_j > -q v_i - v_j = \nu_{P_0}(f_i f^j_j).$$

Therefore, both $e_{ij}$ and $e'_ij$ are nonzero functions for all $i, j$. Suppose

$$\sum_{1 \leq i < j \leq m + 1 - g} a_{ij} e_{ij} + \sum_{1 \leq i < j \leq m + 1 - g} a'_{ij} e'_{ij} = 0$$  \hfill (4)

for some $a_{ij}, a'_j \in \mathbb{F}_q$.

Rewrite (4) into the form

$$\sum_{1 \leq i < j \leq m + 1 - g} (a_{ij} e_{ij} + a'_j e'_{ij}) + \sum_{i=1}^{m+1-g} a_{ii} e_{ii} = 0$$

i.e.,

$$\sum_{1 \leq i < j \leq m + 1 - g} \left(\langle a_{ij} + a'_j \gamma \rangle f^j_i f_j + \langle a_{ij} + a'_j \gamma \rangle f_i f^j_j\right) + \sum_{i=1}^{m+1-g} a_{ii} f^{i+1}_i = 0.$$

Noting the following three facts:

i) for all $1 \leq i < j \leq m + 1 - g, a_{ij} = a'_j = 0$ (this is equivalent to

$$\langle a_{ij} + a'_j \gamma \rangle f^j_i f_j + \langle a_{ij} + a'_j \gamma \rangle f_i f^j_j = 0$$

since

$$\nu_{P_0}(f^j_i f_j) = -(q v_i + v_j) > -(q v_i + v_j) = \nu_{P_0}(f_i f^j_j)$$

and 1. $\gamma$ are $\mathbb{F}_q$-linearly independent, or

$$\nu_{P_0}(\langle a_{ij} + a'_j \gamma \rangle f^j_i f_j + \langle a_{ij} + a'_j \gamma \rangle f_i f^j_j) = -(q v_i + v_j);$$

ii) for $1 \leq i \leq m + 1 - g, a_{ii} = 0$ or $\nu_{P_0}(a_{ii} f^{i+1}_i) = -(q + 1) v_i$;

and

iii) if $1 \leq i < j \leq m + 1 - g$ and $1 \leq u \leq w \leq m + 1 - g$ satisfy

$$(q v_j + v_i = -(q v_w + v_u)$$

then $(i, j) = (u, w)$ (Suppose $-(q v_j + v_i = -(q v_w + v_u)$.

Then $q(v_j - v_i) = v_w - v_i$. Hence $v_u - v_i \equiv 0 \pmod{q}$. By the condition $m < q$, we have $0 \leq v_i, v_u \leq m < q$. It follows that $v_i = v_u$, i.e., $i = u$. This yields $v_j = v_w$ by (5), i.e., $j = w$);

we find that all $a_{ij}, a'_j$ are equal to zero. This completes the proof.

We construct a code as follows:

$$C_m = \{(f(P_1), f(P_2), \ldots, f(P_m)) | f \in U_m\}.$$  

This code is defined over $\mathbb{F}_q$ by Lemma 3.1 and it is clear that $C_m$ is a linear code of length $n$.

Looking at

$$\min\{\nu_{P_0}(e_{ij})| 1 \leq i \leq j \leq m + 1 - g\} = \min\{-(q v_j + v_i) | 1 \leq i \leq j \leq m + 1 - g\} = -(q v_{m+1-g} + v_{m+1-g}) \geq -m(q + 1)$$

and

$$\min\{\nu_{P_0}(e'_{ij})| 1 \leq i < j \leq m + 1 - g\} = \min\{-(q v_j + v_i) | 1 \leq i < j \leq m + 1 - g\} = -(q v_{m+1-g} + v_{m+1-g}) \geq -m(q + 1)$$

we find that all $e_{ij}, e'_ij$ are elements of $\mathcal{L}(m(q + 1) P_0)$, i.e., $U_m$ is a subset of $\mathcal{L}(m(q + 1) P_0)$. Hence, $C_m$ is a subcode of the subfield subcode

$$\mathcal{C}(P_1, P_2, \ldots, P_m; m(q+1) P_0) \mid_{\mathbb{F}_q}.$$
Theorem 3.3: Let $X/F_{q^2}$ be an algebraic curve of genus $g$ with $n+1$ rational points $P_0, P_1, \ldots, P_n$. Suppose $g \leq m < \min\{q, n/(q+1)\}$. Then $C_m$ is an $[n, (m+1-g)^2, d]$ linear code over $F_q$ with 
$$d \geq n - m(g+1).$$

Proof: We can directly show the result on the minimum distance. However, since $C_m$ is a subcode of the subfield subcode $C(P_1, P_2, \ldots, P_n; m(q+1)P_0)\mid_F q$ and $n > \deg(m(q+1)P_0)$, it follows from Proposition 2.1 that 
$$d = d(C_m) \geq d(C(P_1, P_2, \ldots, P_n; m(q+1)P_0)\mid_F q) \geq n - m(g+1).$$

In order to show that the dimension of $C_m$ is equal to $(m+1-g)^2$, it is sufficient to show that the map 
$$U_m \rightarrow F_q^m, \quad f \mapsto (f(P_1), f(P_2), \ldots, f(P_n))$$
is injective since $\dim F_q(U_m) = (m+1-g)^2$ by Lemma 3.2. Suppose $f \in U_m$ satisfies $(f(P_1), f(P_2), \ldots, f(P_n)) = 0$, i.e., 
$$f \in \mathcal{L}(m(q+1)P_0 - \sum_{i=1}^n P_i).$$

Then, $f = 0$ as 
$$\deg(m(q+1)P_0 - \sum_{i=1}^n P_i) < 0.$$

The proof is complete.

Remark: Since $C_m$ is a subcode of $C(P_1, P_2, \ldots, P_n; m(q+1)P_0)\mid_F q$, we have 
$$\dim F_q(C(P_1, P_2, \ldots, P_n; m(q+1)P_0)\mid_F q) \geq \dim F_q(U_m) = (m+1-g)^2. \quad (6)$$

On the other hand, applying the lower bound (1) in Section II, we only get 
$$\dim F_q(C(P_1, P_2, \ldots, P_n; m(q+1)P_0)\mid_F q) \geq 2(m(q+1)P_0 - n) \geq 2((m+1)P_0 + 1 - g) - n. \quad (7)$$

For small genus $g$, bound (6) is obviously better than bound (7). For instance, taking $g = 0$ gives 
$$(m+1-g)^2 - (2(m+1) - (m+1) - 1 - g) = 0,$$
and taking an elliptic curve over $F_{q^2}$ with $q^2 + 2q + 1$ rational points (such an elliptic curve exists [9]) gives 
$$(m+1-g)^2 - (2(m+1) - q^2 - 2) = (q-m)^2 - 1.$$

2) Consider an elliptic curve $X$ over $F_{q^2}$ with $q^2 + 2q + 1$ rational points. Then 
$$n = |\mathcal{L}(F_{q^2})| = q^2 - 2q.$$

By Theorem 3.3, there exists a $[q^2, (m+1)^2, d \geq q^2 - m(q+1)]$ linear code over $F_q$.

Corollary 3.4: If $1 \leq m < q$, then there exist
1) a $[q^2, (m+1)^2, d \geq q^2 - m(q+1)]$ linear code over $F_q$, and
2) a $[q^2 + 2q, m^2, d \geq q^2 + 2q - m(q+1)]$ linear code over $F_q$.

Proof: 1) Consider the projective line $X$ over $F_{q^2}$. Then 
$$n = |\mathcal{L}(F_{q^2})| - 1 = q^2.$$ 

By Theorem 3.3, there exists a $[q^2, (m+1)^2, d \geq q^2 - m(q+1)]$ linear code over $F_q$. 

By Theorem 3.3, there exists a $[q^2, (m+1)^2, d \geq q^2 - m(q+1)]$ linear code over $F_q$. 

For the code $C_m$, its dimension is uniquely determined by $m$. In order to obtain more codes, we add another parameter $l$ to control the dimension. For $1 \leq l \leq q - m$, let $U_{ml}$ be the $F_q$-linear span of the set 
$$\{v_{ij} | 1 \leq i \leq j \leq m - g\} \cup \{\epsilon_{1,m+1-g,\epsilon_{2,m+1-g},\ldots,\epsilon_{l,m+1-g}}\} \cup \{\epsilon_{1,m+1-g,\epsilon_{2,m+1-g},\ldots,\epsilon_{l,m+1-g}}\}.$$ 

It is clear by the proof of Lemma 3.2 that the dimension of $U_{ml}$ is equal to $(m-g)^2 + 2l$.

Define the code 
$$C_{ml} = \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in U_{ml}\}.$$ 

Then $C_{ml}$ is a subcode of the subfield subcode 
$$C(P_1, P_2, \ldots, P_n; (qm + g + l - 1)P_0)\mid_F q$$ 

since 
$$\min\{\nu_{F_q}(f) | f \in U_{ml}\} = \nu_{F_q}(v_{ml}) = -q^{m+1-g} - v_i \geq -qm - (g+l-1) = -qm - g - l - 1.$$ 

This is because $C_{ml}$ is a subcode of $C(P_1, P_2, \ldots, P_n; qm P_0)$ since 
$$\min\{\nu_{F_q}(f) | f \in U_{ml}\} = \nu_{F_q}(v_{ml}) = -q^{m+1-g} - v_i \geq -qm.$$ 

Therefore, by Proposition 2.1 the minimum distance of $C_{ml}$ is at least $n - qm - g - l + 1$. In the case where $l = 1$, we have a better result on minimum distance, namely, 
$$d(C_{ml}) \geq n - qm.$$

This is because $C_{ml}$ is a subcode of $C(P_1, P_2, \ldots, P_n; qm P_0)$ since 
$$\min\{\nu_{F_q}(f) | f \in U_{ml}\} = \nu_{F_q}(v_{ml}) = -q^{m+1-g} - v_i \geq -qm.$$ 

Theorem 3.5: If $g \leq m < \min\{q, n/(q+1)\}$ and $1 \leq l \leq m - g$, then $C_{ml}$ is an $[n, (m-g)^2 + 2l]$ linear code over $F_q$ with 
$$d \geq \left\{ \begin{array}{ll} n - (qm + v_i) & \geq n - (qm + l + g - 1), \\ \min(qm, & i = 1. \end{array} \right.$$ 

Proof: From the above analysis, we know that the result on the minimum distance is true. As in the proof of Theorem 3.3, it is easy to show that the dimension of $C_{ml}$ is equal to the dimension of $U_{ml}$, i.e., 
$$\dim F_q(C_{ml}) = (m-g)^2 + 2l.$$

Corollary 3.6: 
1) If $1 \leq m \leq q$ and $1 \leq l \leq m$, then there exists a $[q^2, m^2 + 2l, d]$ linear code with $d \geq q^2 - qm - l + 1$.
2) If $1 \leq m \leq q$ and $1 \leq l \leq m - 1$, then there exists a $[q^2 + 2q, (m-1)^2 + 2l, d]$ linear code with 
$$d \geq \left\{ \begin{array}{ll} q^2 + 2q - qm - l, & \text{if } 2 \leq l \leq m - 1, \\ q^2 + 2q - qm, & \text{if } l = 1. \end{array} \right.$$ 

Proof: Considering the projective line over $F_{q^2}$ and an elliptic curve over $F_{q^2}$ with $q^2 + 2q + 1$ rational points, respectively, yields the desired result by Theorem 3.5.
For the code $C_m$, the length is equal to $|X(F_q^2)| - 1$. In fact, we can increase the length by one if a certain divisor exists.

We only discuss this idea for elliptic curves over $F_q$ with $q^2 + 2q + 1$ rational points though it can be extended to arbitrary curves.

**Lemma 3.7:** Let $X(F_q^2)$ with $q \geq 4$ be an elliptic curve of $q^2 + 2q + 1$ rational points $P_0$ and $P_0$ be a fixed rational point. Then for any rational point $P' \in X(F_q^2)$, there exists a closed point $Q$ of degree 2 such that $Q - P_0 - P$ is a principal divisor.

**Proof:** By the Riemann-Roch theorem, one has $\ell(P + P_0) = 2$. Thus there are $((q^2)^2 + 1)/(q^2 - 1) = q^2 + 1$ positive divisors equivalent to $P_0 + P$. Since $\lambda(1) = 1$, $X(F_q^2)$ forms an Abelian group isomorphic to the divisor class group of degree 0 of $Y$. Therefore, the set

$S := \{R \in X(F_q^2) \mid 2R \sim P_0 + P\}$

has at most four elements. For any rational point $P' \in X(F_q^2)$, there is a unique rational point $P''$ such that $P' + P'' \sim P_0 + P$. Hence the number of positive divisors $D$ with $D \sim P + P_0$ and

$\text{Supp}(D) \cap X(F_q^2) \neq \emptyset$

is equal to

$$\left|S\right| + \frac{|X(F_q^2)| - |S|}{2} = \frac{(q^2 + 2q + 1) + |S|}{2} \leq \frac{q^2 + 2q + 5}{2},$$

that is less than $q^2 + 1$ as $q \geq 4$. This implies that there is a closed point $Q$ of degree 2 equivalent to $P + P_0$, i.e., $Q - P - P_0$ is principal.

**Lemma 3.8:** Let $X(F_q^2)$ be an elliptic curve with $q^2 + 2q + 1$ rational points. If $4 \leq m < q$, then there exists a function $h_m \in F_q^2(X)$ such that the zero divisor of $h_m$ is

$$(h_m)_0 := \sum_{\nu_P(h_m) > 0} \nu_P(h_m)P = mP_0$$

and the pole divisor

$$(h_m)_\infty := -\sum_{\nu_P(h_m) < 0} \nu_P(h_m)P$$

satisfies

$\text{Supp}((h_m)_\infty) \cap X(F_q^2) = \emptyset$.

**Proof:** Choose a place $T$ of degree $m - 2 \geq 2$. Then there exists a rational point $P$ of $X$ such that $P - T$ is equivalent to $T - (m - 2)P_0$ since $X(F_q^2)$ is isomorphic to the class group of $X$. Let $Q \sim P$ be a closed point of degree 2 of $X$ such that $Q - P_0 + P$ (this is possible by Lemma 3.7), then $2P_0 - Q - P_0 = P$. Hence

$$2P_0 - Q \sim T - (m - 2)P_0,$$

i.e.,

$$T + Q \sim mP_0.$$ 

Let $h_m \in F_q^2(X)$ satisfy $\text{div}(h_m) = mP_0 - (T + Q)\nu$, then $h_m$ is the desired function.

**Theorem 3.9:** If $4 \leq m < q$, then there exists a

$$[q^2 + 2q + 1, m^2, q^2 + 2q + 1 - m(q + 1)]$$

linear code over $F_q$.

**Proof:** Let $X(F_q^2)$ be an elliptic curve with $q^2 + 2q + 1$ rational points. By Lemma 3.8, there exists a function $h_m \in F_q^2(X)$ such that

$$(h_m)_0 = mP_0; \quad \text{Supp}((h_m)_\infty) \cap X(F_q^2) = \emptyset.$$

Put

$$\overline{U_m} = h_m^{+1}U_m = \{h_m^{+1}f \mid f \in U_m\}.$$

Then $\overline{U_m}$ is an $F_q$-linear space and

$$\text{dim} F_q \overline{U_m} = \text{dim} F_q U_m = m^2.$$

Moreover, we have the following:

i) $\text{div}(h_m^{+1}f) = \text{div}(h_m^{+1}) + \text{div}(f) \geq m(q + 1)P_0 - (q + 1)(h_m)\infty + (f)P_0 + \nu_P(\epsilon_{m, m})P_0 \geq -q + 1(h_m)\infty$

i.e., $\overline{U_m} \subseteq \mathcal{L}((q + 1)(h_m)\infty)$, and

ii) $(h_m^{+1}f)(P) \in F_q$ for all $f \in U_m$ and $P \in X(F_q^2)$.

Put

$$\overline{C_m} = \{(h_m^{+1}f)(P_0), (h_m^{+1}f)(P_1), \ldots, (h_m^{+1}f)(P_n) \mid f \in U_m\}$$

where $n = q^2 + 2q$. Then it is easy to see that $\overline{C_m}$ is a subcode of the subfield subcode $C(P_0, P_1, \ldots, P_n; (q + 1)(h_m)\infty)$.

The dimension of this code is equal to the dimension of $U_m$, that is, $m^2$ and the minimum distance satisfies

$$d(\overline{C_m}) \geq d(C(P_0, P_1, \ldots, P_n; (q + 1)(h_m)\infty)) \geq q^2 + 2q + 1 - m(q + 1).$$

**IV. TABLES**

In this section, we first list all improvements on Brouwer’s table [1] for $q = 8$ and 9 by directly using the results in Section III. Then we apply some well-known propagation rules to our codes to obtain more improvements on Brouwer’s table [1] for $q = 8$ and 9.

We need to explain the symbols in Tables I and II. Corollaries 3.4, 3.6 and Theorem 3.9 are used to construct in Tables I and II.

| $l, m$ | parameters in Corollaries 3.4, 3.6 and Theorem 3.9; |
| $n, k$ | length and dimension of codes, respectively, obtained from Corollaries 3.4, 3.6, or Theorem 3.9 for given parameters $l, m$; |
| $d$ | lower bound on minimum distance obtained from Corollaries 3.4, 3.6, or Theorem 3.9; |
| $d_B$ | the lower bound on minimum distance of codes with given length $n$ and dimension $k$ quoted from Brouwer’s table. |

**TABLE I**

<table>
<thead>
<tr>
<th>$q = 8$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary or Theorem</td>
<td>$m$</td>
</tr>
<tr>
<td>3.6(2)</td>
<td>5</td>
</tr>
<tr>
<td>3.4(2)</td>
<td>6</td>
</tr>
<tr>
<td>3.6(2)</td>
<td>7</td>
</tr>
<tr>
<td>3.6(2)</td>
<td>7</td>
</tr>
<tr>
<td>3.4(2)</td>
<td>7</td>
</tr>
<tr>
<td>3.6(2)</td>
<td>8</td>
</tr>
<tr>
<td>3.9</td>
<td>5</td>
</tr>
<tr>
<td>3.9</td>
<td>6</td>
</tr>
<tr>
<td>3.9</td>
<td>7</td>
</tr>
</tbody>
</table>
Next we use the propagation rules to get more improvements on Brouwer’s table.

**Proposition 4.1 (see [7, Exercise 1.2.24]):**

1. If there is an \([n, k, d]\) linear code over \(F_q\), then there exists an \([n-s, k-s, d]\) linear code over \(F_q\) for any \(s \leq n-d\).
2. If there is an \([n, k, d]\) linear code over \(F_q\), then there exists an \([n-s, k-d-s]\) linear code over \(F_q\) for any \(s \leq n-k\).

By considering subcodes of a linear code, we obtain the following result.

**Proposition 4.2:** If there is an \([n, k, d]\) linear code over \(F_q\), then there exists an \([n, k-s, d]\) linear code for any \(0 \leq s \leq k-1\).

We need to explain the symbols in Tables III and IV. We use the codes in Tables I, II and Proposition 4.1 to construct the codes in Tables III and IV.

<table>
<thead>
<tr>
<th>Proposition</th>
<th>original code</th>
<th>(s)</th>
<th>(n)</th>
<th>(k)</th>
<th>(d)</th>
<th>(d_B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1(1)</td>
<td>([81,49,\geq 18])</td>
<td>5</td>
<td>75</td>
<td>44</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,48,\geq 18])</td>
<td>4</td>
<td>64</td>
<td>44</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([81,49,\geq 18])</td>
<td>4</td>
<td>64</td>
<td>45</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,48,\geq 18])</td>
<td>3</td>
<td>77</td>
<td>45</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,51,\geq 16])</td>
<td>3</td>
<td>77</td>
<td>48</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([81,49,\geq 18])</td>
<td>3</td>
<td>77</td>
<td>48</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,36,\geq 26])</td>
<td>2</td>
<td>78</td>
<td>34</td>
<td>26</td>
<td>25</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,48,\geq 18])</td>
<td>2</td>
<td>78</td>
<td>46</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,49,\geq 17])</td>
<td>2</td>
<td>78</td>
<td>47</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,51,\geq 16])</td>
<td>2</td>
<td>78</td>
<td>49</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,46,\geq 19])</td>
<td>1</td>
<td>79</td>
<td>45</td>
<td>19</td>
<td>18</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([81,36,\geq 27])</td>
<td>2</td>
<td>79</td>
<td>34</td>
<td>27</td>
<td>26</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,36,\geq 26])</td>
<td>1</td>
<td>79</td>
<td>35</td>
<td>26</td>
<td>25</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([81,49,\geq 18])</td>
<td>2</td>
<td>79</td>
<td>47</td>
<td>18</td>
<td>17</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,51,\geq 16])</td>
<td>2</td>
<td>79</td>
<td>50</td>
<td>16</td>
<td>15</td>
</tr>
<tr>
<td>4.1(1)</td>
<td>([80,49,\geq 17])</td>
<td>1</td>
<td>79</td>
<td>48</td>
<td>17</td>
<td>16</td>
</tr>
</tbody>
</table>

\(d\) is the lower bound on minimum distance obtained from Proposition 4.1 for given \(s\) and original code; 

\(d_B\) is the lower bound on minimum distance of codes with given length \(n\) and dimension \(k\) quoted from Brouwer’s table.

**REFERENCES**

