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The Universal Covers of Certain Semibiplanes

BARBARA BAUMEISTER AND DMITRII V. PASECHNIK

The universal covers of c.c*-geometries constructed from the length two orbits of certain involutory automorphism of $PG(2, q)$ are determined. It particular, we answer a question from [5].

1. Introduction

Flag-transitive c.c*-geometries (also called semibiplanes) arise naturally from consideration of certain combinatorial objects related to small sporadic and ‘almost sporadic’ groups, such as $L_3(2)$, $L_2(11)$, $3A_6$, $M_{12}$ and $M_{22}$ (see, e.g., Baumeister [2] and Baumeister and Pasechnik [4]), as well as from classical combinatorial objects such as hypercubes and projective planes (see [2]).

They can also be considered as a testing ground for the amalgam technique (see [2], Grams and Meixner [7] and Pasini [9]). The reasons for this are that there exists a reasonably good bound on the number of points (see, e.g., [4]) and also that the amalgams in question do not seem to be too difficult to handle in a certain sense.

The question of classifying c.c*-geometries can be reduced to classifying groups generated by certain rank 3 amalgams. The latter process consists of two steps. The first is to determine the amalgams and the second is to determine their universal closures. The second step is essentially the determination of the universal cover of a geometry defined by a given amalgam. This was addressed in [2–4] for the ‘sporadic’ semibiplanes mentioned above. Here we determine the universal covers of some of the semibiplanes from one of the few known infinite families; namely, of those one related to orbits of an involutory automorphism of $PG(2, q)$. We call this class projective semibiplanes.

The technique used is well-known circuit chasing in the incidence graph (see, e.g., [10]). We demonstrate that it is applicable to infinite families of objects, such that the triangulations (more precisely, splittings into 4-gons in the particular case of semibiplanes) of corresponding circles are not at all straightforward. Namely, here in order to find such triangulations one has to consider sequences of words in the fundamental group of length depending upon the parameter $q$, even though the diameter of the incidence graph is constant.

Let $\Gamma = \Pi^\phi$ be the semibiplane related to the involutory automorphism of $\Pi = PG(2, q)$, as constructed by Hughes [8]. We call $\Gamma$ a projective simibiplane, or, depending on the type of $\phi$, elation, homology, or Baer involution semibiplane. See Section 2 for details. It was proved by Baumeister [1] that the Baer involution semibiplanes are not simply connected for $q > 4$.

We prove the following.

THEOREM 1.1. Let $\Gamma$ be an elation or a homology semibiplane. Then $\Gamma$ is simply connected.
2. Projective Semiplanes

Let $\Pi \simeq PG(2, q)$ and let $\phi$ be an involutory automorphism of $\Pi$. The semiplane $\Gamma = \Pi^\phi$ is an incidence system of length 2 point orbits and length 2 line orbits of $\phi$, with the incidence induced by this of $\Pi$. That is, a point orbit $(p, p^\phi)$ is adjacent to a line orbit $(L, L^\phi)$ iff $p \in L \cup L^\phi$. The point orbits of length 2 will be called Points, and the line orbits of length 2 will be called Circles. (We use capital letters in order to distinguish between elements of $\Pi$ and elements of $\Gamma$.) Let us check that $\Gamma$ is a semiplane (cf. [8]); that is, that there are 0 or 2 Circles on a pair of Points. Let $(p, p^\phi)$ and $(q, q^\phi)$ be two Points. If the lines $pp^\phi$ and $qq^\phi$ are equal, then there are no Circles of the pair of Points. Otherwise, there are exactly 2 Circles $(pq, p^\phi q^\phi)$ and $(pq^\phi, p^\phi q)$ on them. We are done.

According to a classical result of Baer (see, e.g., Dembowski [6]), there are three types of involutory automorphisms of a finite projective plane: namely, elations (fixing a line pointwise), homologies (fixing a line pointwise and a point outside) and Baer involutions (fixing a Baer subplane pointwise). In general, this gives us three possibilities for $\Gamma$. As we already pointed out, Baer involution semiplanes are not simply connected [1]. We will not consider them here.

In order to be able to manipulate with the semiplanes in question, we use the inhomogeneous co-ordinates of the plane $\Pi$ (see, e.g., [6]). The line fixed by $\phi$ pointwise is always the line at infinity.

For an elation semiplane, $q$ is even, and for a homology semiplane, $q$ is odd. Since the involutory automorphisms of these types are all conjugated in $Aut(\Pi)$, the semiplane $\Gamma$ is unique for a given value of $q$. The incidence Point–Circle graph of $\Gamma$ has the distribution diagram given in Figures 1 and 2.

The diagrams deserve a quick explanation. Let the leftmost node of the diagram correspond to a Point $P = (p, p^\phi)$. The Points at distance 2 (respectively, 4) from $P$ are the pairs of points not lying (respectively, lying) on the line $pp^\phi$. In the case $q$ odd, the distinguished $(q - 1)/2$ vertices at distance 3 from $P$ correspond to the pairs of lines intersecting the line at infinity at the same point as the line $pp^\phi$ does.

Finally, note that the middle-type elements of the geometry correspond to the 4-gons of the graph.

3. Triangularizations

In order to show that $\Gamma$ is simply connected, it is sufficient to show that every circuit in the incidence graph can be split into 4-gons. This is a consequence of a general observation by Ronan [10]. Note that, in general, for c.c*-geometries this need not to be the case, since the incidence graph might have non-geometric 4-gons, but in the
case of semibiplanes this criterion works. Since \( \Gamma \) can be reconstructed from its incidence graph, we will abuse the notation and use \( \Gamma \) to denote the incidence graph as well. We say that the circuit is contractible if it can be split into 4-gons.

First, it is sufficient to consider the simple circuits only. In our case it means that we need to consider 6-gons and 8-gons only.

Let \( C \) be one such circuit, and let \( u \) and \( v \) be a pair of opposite vertices. We can assume that \( u \) is the unique vertex in the leftmost node of the distribution diagram, and see which nodes of the diagram the other vertices of \( C \) lie in.

First, consider the case in which \( v \) is at distance 4 from \( u \); that is, \( C \) is an 8-gon. Then the unique 4-gon through \( v \) and the other two vertices \( x, y \) of \( C \) adjacent to \( v \) has its fourth vertex \( z \) at distance 2 from \( u \). This follows from the fact seen on the diagram that there is only one vertex adjacent to \( x \) at distance 4 from \( u \). Hence \( C \) can be split into 6-gons and 4-gons.

Next, consider the case \( q \) odd and \( v \) one of the \((q - 1)/2\) distinguished vertices at distance 3 from \( u \). (Formally speaking, these \((q - 1)/2\) vertices can be characterized by the property of having \( q \) neighbours at distance 2 from \( u \).) Again, it can be seen from the diagram that \( z \) is either at distance 2 from \( u \) or it is one of the remaining \( q(q - 3)/2 \) vertices (from the node other than this of \( v \)) at distance 3 from \( u \). In the former case, \( C \) can be split into 4-gons, and in the latter case into 4-gons and the other type of 6-gons; namely, such that the vertex opposite to \( u \) is from the node of size \( q(q - 3)/2 \).

Let us denote by \( \mathcal{F} \) the set of the circuits remaining. That is, \( \mathcal{F} \) is the set of simple 6-gons, with the restriction for \( q \) odd that the vertex opposite to \( u \) lies in the node of size \( q(q - 3)/2 \). As we just saw, the following holds.

\[ \text{Lemma 3.1. If the elements of } \mathcal{F} \text{ are contractible, then } \Gamma \text{ is simply connected.} \]

In what follows we deal with the contractibility of a 6-gon \( C \in \mathcal{F} \). For a pair of opposite vertices \( u \) and \( v \), define \( O_u = \Gamma(u) \cap \Gamma_3(v) \), \( O_v = \Gamma(v) \cap \Gamma_3(u) \) and \( O = O(u, v) = O_u \cup O_v \).

The following will be proved in the later sections.

\[ \text{Lemma 3.2. The subgraph of } \Gamma \text{ induced on } O \text{ is a disjoint union of circuits.} \]

Let \( O^x \) be a connected component of \( O \) containing \( x \). (Note that we defined \( x \) and \( y \) to be the neighbours of \( v \) in \( C \).) There is unique 6-gon \( C' \) on \( u, v, y \) and \( x' \in O^x \).

\[ \text{Lemma 3.3. The contractibility of } C \text{ follows from the contractibility of } C'. \text{ If } O \text{ is connected, then } C \text{ is contractible.} \]

\[ \text{Proof. The first claim can be checked by induction on the distance between } x \text{ and } x' \text{ in } O. \text{ The second then follows from the first.} \]

The following lemma will be proved in the later sections.

\[ \text{Lemma 3.4. Suppose that } O \text{ is not connected. Then there exists } C'' \in \mathcal{F} \text{ containing } u, \ y, \ x' \in O^x, \ v'' \in \Gamma_3(u) \text{ such that the number of connected components of } O(u, v'') \text{ is strictly less than that of } O(u, v). \]

Lemma 3.4 completes the proof of Theorem 1.1. Indeed, in view of Lemma 3.3, \( C \)
can be replaced with \( C' \). The contractibility of \( C' \) clearly follows from the contractibility of \( C'' \). The latter now follows by induction on the number of connected components of \( O \) and the second part of Lemma 3.3.

4. Final Part of the Proof for Elation Semiplanes

Here we prove Lemmas 3.2 and 3.4 for the case \( \Gamma \) an elation semiplane. Here \( q \) is even. We choose the elation \( \phi \) to act as follows: \( \phi((x, y)) = (x, y + 1) \), the Point \( u \) to be \(((0, 0), (0, 1))\), and the Circle \( v \) to be \(((Y = aX + b), [Y = aX + b + 1])\), where \( b \neq 0, 1 \). Then

\[
O_u = \{(Y = dX), [Y = dX + 1] \mid d \neq a\},
\]

\[
O_v = \{(x, ax + b), (x, ax + b + 1) \mid x \neq 0\}.
\]

As the elements of \( O_v \) are determined by the parameter \( x \), we will write \( x \in O_v \) instead of \(((x, ax + b), (x, ax + b + 1)) \in O_v \). Similarly, we will write \( d \in O_u \) instead of \(((Y = dX), [Y = dX + 1]) \in O_u \).

Let \( x \in O_v \). Then \( x \) is incident to \( d \in O_u \) iff \( ax + b = dx \) or \( ax + b = dx + 1 \). Hence \( d = a + b/x \) and \( d = a + (b + 1)/x \) are the two neighbours of \( x \) in \( O_v \). Similarly, given \( d \in O_u \), one finds the two neighbours \( x = b/(a + d) \) and \( x = (b + 1)/(a + d) \) of \( d \) in \( O_v \). Thus the components of \( O \) are (ordinary) polygons. This completes the proof of Lemma 3.2 for the case \( q \) even.

Now we have to analyse the connected components of \( O \) more explicitly. Let \( O^x \) be the component containing \( x \in O_v \). The vertex \( d = a + b/x \in O_u \) is adjacent to \( x \) and \( x(b + 1)/b \in O_v \). Continuing from the latter vertex, we find that \( O^x \cap O_v \) is the coset \( xH_b \) of the subgroup \( H_b = \langle (b + 1)/b \rangle \) of the multiplicative group of \( GF(q) \). In particular, the number of connected components of \( O \) equals \( [GF(q)^\times : \langle b + 1 \rangle] \).

By the assumption of Lemma 3.4, we have \( y \not\in xH_b \). Therefore there are at least \( |H_b| \) vertices \( y \in \Gamma_3(u) \) of the form \(((Y = a^\prime X + b^\prime), [Y = a^\prime X + b^\prime + 1])\) adjacent to \( y \) and to some \( x^\prime \in xH_b \); namely, one can choose \( a^\prime = a + 1/(y + x^\prime), b^\prime = b + (1 + x^\prime/y)^{-1} \). In order to verify this, it suffices to check that the points \((y, ay + b) \) and \((x', ax' + b + 1) \) lie on the line \([Y = a^\prime X + b^\prime]\).

As above, we see that the connected components of \( O(u, v') \) are in one-to-one correspondence with the cosets of \( H_b = \langle b + 1 \rangle \). Note that \( b^\prime \not= 0 \) by the choice of \( y \). We show that \( x^\prime \in xH_b \) can always be chosen in such a way that \( |H_b | > |H_b| \). It suffices to prove that \( t = 1 + 1/b^\prime \) can be written as \( \lambda(1 + b/y) \) for some \( \lambda \not\in H_b \). (Note that \( H_b = \langle \rangle \) by definition.)

Indeed, \( ord t > |H_b| \); otherwise, \( ord \lambda \mid |H_b| \), which would imply that for a certain prime \( r \) the Sylow \( r \)-subgroup of \( GF(q)^\times \) is not cyclic, a nonsense.

Denote \( z = x^\prime/y \). Write

\[
1 + 1/b^\prime = 1 + \frac{1 + z}{b + bz + 1} = \lambda(1 + b)/b.
\]

Hence \((b + bz + z)b = \lambda(b(b + bz + z) + 1) \). Thus

\[
\mu(z) = 1/\lambda = 1 + (b(b + bz + z))^{-1}.
\]

Observe that \( \mu(z) = \mu(z') \) iff \( z = z' \). As \( z \) can take \( |H_b| \) distinct values, so can \( \lambda = \mu(z)^{-1} \). On the other hand, \( \lambda \neq 1 \) for any \( z \). Hence there is no bijection between the elements of \( H_b \) and the range of values of \( \lambda \). Therefore, for some \( z \), we have \( \lambda \not\in H_b \), as claimed.
Thus there always exists \( b^* = b'' \) such that \( H_{b^*} > H_b \). We construct \( v'' \) required in Lemma 3.4 by choosing \( b'' = b^* \) and the corresponding \( a'' \) (the latter can be computed explicitly from \( b'' \), for \( b'' \) determines \( x' \) uniquely). OED.

5. Final Part of the Proof for Homology Semiplanes

Here we prove Lemmas 3.2 and 3.4 for the case \( \Gamma \) a homology semiplane. Here \( q \) is odd. We choose the homology \( \phi \) to act as follows: \( \phi((x, y)) = (-x, -y) \), the Point \( u \) to be \((0, 1), (0, -1)\), and the Circle \( v \) to be \([Y = aX + b], [Y = aX - b]\), where \( b \neq \pm 1 \). Then

\[
O_u = \{(Y = dX + 1), [Y = dX - 1] \mid d \neq a\},
\]

\[
O_v = \{(x, ax + b), (-x, -ax - b) \mid x \neq 0\}.
\]

As the elements of \( O_v \) are determined by the parameter \( x \), we will write \( x \in O_v \) instead of \((x, ax - b), (-x, -ax - b) \in O_v \). Similarly, we will write \( d \in O_u \) instead of \((Y = dX + 1), [Y = dX - 1] \in O_u \).

Let \( x \in O_v \). Then \( x \) is incident to \( d \in O_u \) if \( ax + b = dx + 1 \) or \( ax + b = dx - 1 \). Hence \( d = a + (b - 1)/x \) and \( d = a + (b + 1)/x \) are the two neighbors of \( x \) in \( O_u \). Similarly, given \( d \in O_u \), one finds the two neighbors \( x = (b - 1)/(d - a) \) and \( x = (b + 1)/(d - a) \) of \( d \) in \( O_v \). Thus the components of \( O \) are (ordinary) polygons. This completes the proof of Lemma 3.2 for the case \( q \) odd.

As above, we analyse the connected components of \( O \). In the same vein, we find that \( O \cap O_v \) is the coset \( xH_b \) of the subgroup \( H_b = (b + 1)/(b - 1) \) of the multiplicative group of \( GF(q) \). In particular, the number of connected components of \( O \) equals \([GF(q)]^*: H_b\].

By the assumption of Lemma 3.4, we have \( y \notin xH_b \). Therefore there are at least \( |H_b| \) vertices \( v'' \) in \( \Gamma_3(u) \) of the form \([Y = a''X + b''], [Y = a''X - b'']\) adjacent to \( y \) and to some \( x' \in xH_b \); namely, one can choose \( a'' = a + 2b/(y + x') \) and \( b'' = -b + 2by/(y + x') \). In order to verify this, it suffices to check that the points \((-y, -ay - b)\) and \((x', ax' + b)\) lie on the line \([Y = a''X + b'']\).

As above, we see that the connected components of \( O(u, v'') \) are in one-to-one correspondence with the cosets of \( H_{b''} = (b'' + 1)/(b'' - 1) \). We show that \( x' \in xH_b \) can always be chosen in such a way that \( |H_{b''}| > |H_b| \). It suffices to prove that \( t = 1 + 2/(b'' - 1) \) (note that \( H_{b''} = (t) \) can be written as \( \lambda(1 + 2/(b - 1)) \) for some \( \lambda \neq H_b \). As this is quite similar to the computation done in the previous section, we leave it to the reader.

Thus we construct \( v'' \) required in Lemma 3.4 by choosing \( b'' = b^* \) and the corresponding \( a'' \) (the latter can be computed explicitly from \( b'' \), for \( b'' \) determines \( x' \) uniquely). OED.

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References


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