On the number of inductively minimal geometries

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Abstract

We count the number of inductively minimal geometries for any given rank by exhibiting a correspondence between the inductively minimal geometries of rank \(n\) and the trees with \(n + 1\) vertices. The proof of this correspondence uses the van Rooij–Wilf characterization of line graphs (see [11]).

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1. Introduction

1.1. Motivation

The availability of performant computer algebra systems such as MAGMA [6] allows us to easily generate a lot of examples of finite incidence geometries whose group of automorphisms is isomorphic to (a supergroup of) some prescribed finite group. The properties of these examples can also be studied by computer. Algorithms used for this are described in [8]. For small almost simple groups, fairly long lists of geometries together with some of their properties have been compiled in atlases (see e.g. [5] or [3]).

These atlases have been a useful source of information for several research projects. One example is the concept of inductively minimal geometry. The consideration of geometries of maximal rank for the symmetric groups in the atlases led to a new infinite family of geometries which is described in [4].

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Inductively minimal geometries are defined as incidence structures, together with a group of automorphisms, satisfying further conditions. They were completely classified in [4] and some of their properties were studied in [2, 7].

In this paper we solve the problem of determining the number of non-isomorphic inductively minimal geometries of a given rank (see below for definitions).

1.2. Definitions and known results

Since our terminology is not always standard, we give some definitions concerning graphs. A **graph** \( \mathcal{G} \) is a pair \((V, \sim)\) where \( V \) is a finite, non-empty set whose elements are called **vertices** and \( \sim \) is an antireflexive, symmetric relation on \( V \). If for \( u, v \in V \) we have \( u \sim v \), we say that \( u \) and \( v \) are **adjacent** and that the set \( \{u, v\} \) is an **edge** of \( \mathcal{G} \). By abuse of notation, we often write \( v \in \mathcal{G} \) when we actually mean \( v \in V \). The adjacency relation is always denoted by \( \sim \).

A \( k \)-tuple \((v_0, v_1, \ldots, v_k)\) of vertices of \( \mathcal{G} \) is called a (simple) **path** of **length** \( k \) if we have \( v_i \sim v_{i+1} \) for \( 0 \leq i < k \) and all vertices are different. A graph is said to be **connected** provided that for every choice of two different vertices \( u \) and \( v \), one can find a path with \( u \) as the first and \( v \) as the last vertex. A \( k \)-tuple \((v_0, v_1, \ldots, v_k)\) of vertices is called a (simple) **cycle** of **length** \( k \) provided \( k \) is at least 3, the vertices \( v_0 \) and \( v_k \) are equal and \((v_0, v_1, \ldots, v_{k-1})\) is a path with \( v_{k-1} \sim v_k \). The cycle \((v_0, v_1, \ldots, v_k)\) is called **chordless** if \( v_i \not\sim v_j \) whenever \( |j - i| \neq 1 \) mod \( k \) (in [4], the words **minimal circuit** are used instead). A **tree** is a connected graph which has no cycles.

A **subgraph** of \( \mathcal{G} \) is a graph \((W, \sim_W)\) whose set of vertices \( W \) is a subset of \( V \) and whose adjacency relation \( \sim_W \) is the restriction of \( \sim \) to \( W \). We remark that this definition implies that a subgraph is completely determined by its vertex set and allows us to denote a subgraph by its vertex set. A subgraph of \( \mathcal{G} \) in which every vertex is adjacent to every other vertex is called a **clique** of \( \mathcal{G} \). The set of cliques in a graph is ordered by inclusion. A **coclique** of a graph \( \mathcal{G} \) is a subgraph in which no two vertices are adjacent.

An (incidence) **geometry** of **rank** \( n \in \mathbb{N}_0 \) is a graph \((V, \sim)\) together with a surjection \( t : V \rightarrow \{1, 2, \ldots, n\} \) such that \( t^{-1}(i) \) is a coclique for each \( i \in \{1, 2, \ldots, n\} \) and that every clique of \((V, \sim)\) is contained in at least two cliques \( C \) with \( t(C) = \{1, 2, \ldots, n\} \). The map \( t \) is then called **type function** and for \( v \in V \) (resp. \( A \subseteq V \)), \( t(v) \) (resp. \( t(A) \)) is the **type** of the vertex \( v \) (resp. set \( A \)). A complete bipartite graph defines a geometry of rank 2 which is called a **digon**, i.e. a geometry in which all elements of one type are incident to all elements of the other type. A standard references for incidence geometries are \([10, 1]\); in Chapter 3 of the latter one can find a clear introduction to the topic and the concepts in use.

Given two graphs, an **isomorphism** is a bijection between their vertex sets which preserves adjacency. An **isomorphism of geometries** of the same rank with respective type functions \( t_1 \) and \( t_2 \) is an isomorphism \( \tau \) of their underlying graphs such that \( t_2 \tau = t_1 \). The isomorphisms of a geometry to itself (i.e. the **automorphisms**) form a group under composition. If this group is transitive on the set of cliques of type \( \{1, 2, \ldots, n\} \) it is said to be **flag-transitive**.
Consider a clique \( A \) in a geometry \((V, \sim)\) of rank \( n \) with type function \( t \); the **residue** of \( A \) is a new geometry of rank \( n - |t(A)| \) whose underlying graph is the subgraph determined by the set of all \( v \in V \setminus A \) such that \( A \cup \{v\} \) is a clique. For a geometry of rank \( n \), we define the **basic diagram** to be the graph with vertex set \( \{1, 2, \ldots, n\} \) in which two vertices \( i \) and \( j \) are adjacent if and only if there exists a clique of type \( \{1, 2, \ldots, n\} \setminus \{i, j\} \) whose residue is not a digon.

**Definition 1.** An **IMG diagram** is a connected graph \((I, \sim)\) satisfying the following three properties:
1. \((I, \sim)\) has no chordless cycle of length \( l \geq 3 \);
2. Every edge of \((I, \sim)\) is in a unique maximal clique;
3. Each vertex of \((I, \sim)\) is either in one or in two maximal cliques of \((I, \sim)\).

In [4], **inductively minimal geometries** were defined as pairs consisting of a geometry together with a group such that certain axioms are fulfilled. Afterwards it turned out (see Theorem 1 of [4]) that we could as well define them in the following way.

An **inductively minimal geometry** of rank \( n \) is a geometry of rank \( n \) whose basic diagram is an IMG diagram and such that its group of automorphisms, which is assumed to be flag-transitive, is the smallest possible for that rank. By Theorem 1 of [4], such a geometry is uniquely defined by the diagram.

This definition shows that in order to find the number of inductively minimal geometries for a given rank \( n \), we can equivalently count the number of connected graphs with \( n \) vertices satisfying conditions (1)–(3) of Definition 1. We shall prove the following.

**Theorem 1.** For each integer \( n \geq 1 \), there exists a one-to-one correspondence between the set of non-isomorphic inductively minimal geometries of rank \( n \) and the set of non-isomorphic trees with \( n + 1 \) vertices.

We can conclude that the number of inductively minimal geometries of given rank \( n \) is equal to the number of trees with \( n + 1 \) vertices.

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**2. Proof of the theorem**

The symbol \( K_{1,3} \) denotes the graph with vertex set \( \{1, 2, 3, 4\} \) and edge set \( \{\{1, 2\}, \{1, 3\}, \{1, 4\}\} \). We recall that the **line graph** \( L\mathcal{G} \) of a given graph \( \mathcal{G} \) is defined as follows:
- the vertices of \( L\mathcal{G} \) are the edges of \( \mathcal{G} \);
- two vertices of \( L\mathcal{G} \) are adjacent if they have a common vertex in \( \mathcal{G} \).

The next result is known as the van Rooij–Wilf characterization of line graphs [11] and can be found as Theorem 8.4(3) in [9].
Result 1. A graph $L$ is a line graph of some graph if and only if

- $LG_1$ $L$ has no subgraph isomorphic to $K_{1,3}$;
- $LG_2$ if two odd triangles have a common edge then the subgraph defined by their 4 vertices is a clique.

An odd triangle is a particular type of cycle of length 3 in $L$. We do not need to define it here, since our graph $L$ will be an IMG diagram and in such a diagram any two cycles of length 3 with a common edge generate a clique because of (2) in Definition 1. Hence $LG_2$ is satisfied in every IMG diagram. From (3) in Definition 1 it is also clear that $LG_1$ is satisfied in every IMG diagram. As we now know that every IMG diagram is the line graph of some graph $G$, we can investigate the nature of $G$. We use

Result 2 (Harary [9, Theorem 8.3]). Let $G$ and $G'$ be connected graphs with isomorphic line graphs. Then $G$ and $G'$ are isomorphic unless one is a cycle of length 3 and the other is $K_{1,3}$.

Lemma 1. If the line graph $L = LG$ of a graph $G$ is an IMG diagram, then $G$ is a tree uniquely determined by $L$. The number of vertices of $G$ is one more than the number of vertices of $L$.

Proof. By result 2, $G$ is unique up to isomorphism as long as $L \neq K_3$, and in case $L = K_3$, we can assume $G = K_{1,3}$, which is a tree. This settles the case for $n \leq 3$.

Next, observe that if $G$ contains a 4-vertex subgraph containing a cycle, of length 4 then either condition (1) or (2) of Definition 1 is violated. This gives us a starting point for induction: assume that all the connected $k$-vertex subgraphs of $G$ are trees, for $k \geq 4$. Hence any $(k + 1)$-vertex subgraph of $G$ containing a cycle is a cycle of length $(k + 1)$ in $G$, again contradicting (1) in Definition 1. Therefore $G$ is a tree with $n$ edges and hence $n + 1$ vertices.

Now we also prove the converse.

Lemma 2. The line graph $LT$ of any tree $T$ is an IMG diagram.

Proof. Consider a tree $T$. The line graph $L = LT$ evidently satisfies (1) of Definition 1, while (3) of that same definition is just $LG_1$. Let $\{\{1,2\}, \{1,3\}\}$ be an edge of $L$. The only two possible maximal cliques on $\{\{1,2\}, \{1,3\}\}$ are the clique of edges on the vertex 1 in $T$ and the clique $\{\{1,2\}, \{1,3\}, \{2,3\}\}$. The latter clique is impossible since $T$ is a tree, implying (2).

Putting the lemmas together, we see that the line graph construction $L$ yields a bijection between the set of non-isomorphic IMG diagrams with $n$ vertices and the set of non-isomorphic trees with $n + 1$ vertices.
3. Remarks

1. Theorem 1 settles the enumeration of inductively minimal geometries which is now equivalent to enumeration of trees, which is discussed in detail in Chapter 15 of [9]. In the same book, one can find a table (Table A3) with the number $N_{n+1}$ of trees with $n + 1$ vertices.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{n+1}$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>11</td>
<td>23</td>
<td>47</td>
<td>106</td>
</tr>
</tbody>
</table>

In [5], the first 6 values are found when counting the inductively minimal geometries of the given ranks.

2. The tree $\mathcal{T}$ associated with the IMG diagram $L\mathcal{T}$ can easily be constructed as follows. The nodes of $\mathcal{T}$ are the maximal cliques of $L\mathcal{T}$ together with the set $V_1$ of vertices of $L\mathcal{T}$ that lie in just one maximal clique. Two cliques are adjacent if they have a nontrivial intersection and the adjacency between vertices and maximal cliques is inclusion (thus, $V_1$ corresponds to the set of leaves of $\mathcal{T}$).

3. An open question is to find how the tree associated to the basic diagram of an inductively minimal geometry can be found directly from the geometry.

References