<table>
<thead>
<tr>
<th>Title</th>
<th>A new family of extended generalized quadrangles.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Fra, Alberto Del.; Pasechnik, Dmitrii V.; Pasini, Antonio.</td>
</tr>
<tr>
<td>Date</td>
<td>1997</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10220/9281">http://hdl.handle.net/10220/9281</a></td>
</tr>
<tr>
<td>Rights</td>
<td>© 1997 Academic Press Limited. This is the author created version of a work that has been peer reviewed and accepted for publication by European Journal of Combinatorics, Academic Press Limited. It incorporates referee’s comments but changes resulting from the publishing process, such as copyediting, structural formatting, may not be reflected in this document. The published version is available at: DOI [<a href="http://dx.doi.org/10.1006/eujc.1995.0091">http://dx.doi.org/10.1006/eujc.1995.0091</a>].</td>
</tr>
</tbody>
</table>
A New Family of Extended Generalized Quadrangles†

ALBERTO DEL FRA, DMITRII V. PASECHNIK AND ANTONIO PASINI

For every hyperoval $O$ of $PG(2, q)$ ($q$ even), we construct an extended generalized quadrangle with point-residues isomorphic to the generalized quadrangle $T_2^2(O)$ of order $(q - 1, q + 1)$. These extended generalized quadrangles are flag-transitive only when $q = 2$ or 4. When $q = 2$ we obtain a thin-lined polar space with four planes on every line. When $q = 4$ we obtain one of the geometries discovered by Yoshiara [28]. That geometry is produced in [28] as a quotient of another one, which is simply connected, constructed in [28] by amalgamation of parabolics. In this paper we also give a ‘topological’ construction of that simply connected geometry.

1. Introduction

An extended generalized quadrangle (also called $C_2$ geometry) is a (connected) geometry belonging to the following Beukenhout diagram:

(c.$C_2$)

The orders $s$ and $t$ are assumed to be finite. Given a finite generalized quadrangle $Q$ and an extended generalized quadrangle $\Gamma$, if all point-residues of $\Gamma$ are isomorphic to $Q$ then we say that $\Gamma$ is an extension of $Q$.

In the context of extended generalized quadrangles, the following property is equivalent to the Intersection Property ([21, Lemma 7.25]):

(LL) no two distinct lines are incident with the same points.

We warn the reader that some authors use the name ‘extended generalized quadrangle’ only for $c.C_2$ geometries satisfying (LL). This convection is adopted in [6], for instance.

We follow the notation of [24, chapter 3] for finite generalized quadrangles, but we take the liberty of writing $T_2^2(O)_{q}$ instead of $T_2^2(O)$, in order to remind ourselves of the order $q$ of the projective plane to which the hyperoval $O$ belongs. (We recall that, given a hyperoval $O$ of the plane at infinity of $AG(3, q)$, $q$ even, the generalized quadrangle $T_2^2(O)_{q}$ consists of the points of $AG(3, q)$ and of the lines of $AG(3, q)$ with the point at infinity belonging to $O$. The orders of $T_2^2(O)_{q}$ are $q - 1$ and $q + 1$. More properties of $T_2^2(O)_{q}$ will be recalled in Section 1.2.1.)

In Section 1.1 we briefly survey the known examples of extended generalized quadrangles.

In Section 2, for every hyperoval $O$ of $PG(2, q)$ ($q = 2^n$, $n$ any positive integer) we construct an extension $\Gamma_{q}$ of $T_2^2(O)_{q}$ satisfying (LL). Our construction is entirely geometric. As we will explain in Section 2.5, it imitates an idea of [17] and [19], which goes back to Beukenhout and Hubaut [2].

As there are different types of hyperovals in $PG(2, q)$ when $q > 4$, the isomorphism type of $\Gamma_{q}$ depends on the type of $O$. But it also depends on the choice of a line $l_\infty$ of

† In memory of Giuseppe Tallini.
The automorphism group of $\Gamma_q$ is also investigated in Section 2. It turns out that $\Gamma_q$ is flag-transitive iff $q = 2$ or 4. In fact, $\Gamma_2$ is the thin-lined polar space with four planes on every line, whereas $\Gamma_4$ is a quotient of an extended generalized quadrangle discovered by Yoshiara [28]. A ‘topological’ construction of the latter, inspired by Cameron [3], is given in Section 3 (Corollary 19). In fact, we do more than this: assuming to have chosen a classical hyperoval and a conic for $O$ and $O^\circ \{u^\circ\}$ respectively, we prove that $\Gamma_q$ admits a $q/2$-fold cover.

Some problems are suggested in Section 4.

1.1. A survey of the known examples. The examples of extended generalized quadrangles presently known are of the following types.

1.1.1. Extensions of Dual Grids. Extensions of dual grids are precisely thin-lined $C_3$ geometries; in particular, thin-lined polar spaces of rank 3 (when $(LL)$ holds in them). A description of these geometries can be found in [25] (see also [20]).

1.1.2. Extended Grids. There are many families of extensions of grids (extended grids, for short). The reader is referred to [15] for a census of flag-transitive extended grids known at 1992. However, that list is not complete: one new family has recently been discovered by Shult [26]. Perhaps, more flag-transitive examples can also be obtained by the gluing construction of [16].

Many families of non-flag-transitive extended grids also exist. (Some of them are described in [1], [4], [6], [3] and [17].)

1.1.3. Extensions of Classical Thick Generalized Quadrangles. Not so many extensions exist of classical thick generalized quadrangles, but perhaps they are the most interesting. There are precisely 13 flag-transitive examples of this kind, with point-residues and automorphism groups as follows [10]:

<table>
<thead>
<tr>
<th>Example</th>
<th>Point-residue</th>
<th>Automorphism group</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$W(2)$</td>
<td>$2^5; S_6$</td>
</tr>
<tr>
<td>(2) (quotient of (1))</td>
<td>$W(2)$</td>
<td>$2^4; S_6$</td>
</tr>
<tr>
<td>(3)</td>
<td>$W(2)$</td>
<td>$U_4(2): 2$</td>
</tr>
<tr>
<td>(4)</td>
<td>$W(2)$</td>
<td>$S_8$</td>
</tr>
<tr>
<td>(5)</td>
<td>$Q_7(2)$</td>
<td>$O_6(2)$</td>
</tr>
<tr>
<td>(6)</td>
<td>$Q_7(2)$</td>
<td>$2.S_6(2)$</td>
</tr>
<tr>
<td>(7) (quotient of (6))</td>
<td>$Q_7(2)$</td>
<td>$S_6(2)$</td>
</tr>
<tr>
<td>(8)</td>
<td>$W(3)$</td>
<td>$U_5(2)$</td>
</tr>
<tr>
<td>(9)</td>
<td>$Q_7(3)$</td>
<td>$McL$</td>
</tr>
<tr>
<td>(10)</td>
<td>$H_3(2^2)$</td>
<td>$3.O_6(3)$</td>
</tr>
<tr>
<td>(11) (quotient of (10))</td>
<td>$H_3(2^2)$</td>
<td>$O_6(3)$</td>
</tr>
<tr>
<td>(12)</td>
<td>$H_3(3^2)$</td>
<td>$Suz$</td>
</tr>
<tr>
<td>(13)</td>
<td>$H_3(3^2)$</td>
<td>$Aut(HS)$</td>
</tr>
</tbody>
</table>
The property \((LL)\) holds in all of the above examples. A non-flag-transitive extension of the generalized quadrangle \(H_3(2^5)\) has also been discovered by Pasechnik [17]. The \((LL)\) property holds in it.

1.1.4. **Some Extensions of \(AS(q)\).** J. Thas [27] has constructed an extension of the Arens–Székely generalized quadrangle \(AS(q)\) for every odd prime power \(q\). Let us call that extension \(\mathcal{F}_q\).

The generalized quadrangle \(AS(q)\) is flag-transitive only when \(q = 3\) (see [13]). Thus, its extension \(\mathcal{F}_q\) is flag-transitive only if \(q = 3\). When \(q = 3\), \(\mathcal{F}_q\) is Example (7) of Section 1.1.3, which is flag-transitive.

Cameron [3] has shown that \(\mathcal{F}_q\) admits an \(m\)-fold cover for any divisor \(m\) of \((q + 1)/2\). Quite recently, Kasikova and Shult [13] have constructed an extension of \(AS(q)\) larger than \(\mathcal{F}_q\). The extension that they have found seems to be a \((q + 1)/2\)-fold cover of \(\mathcal{F}_q\). (Perhaps it coincides with the \((q + 1)/2\)-fold cover of \(\mathcal{F}_q\) described in [3].)

1.1.5. **A Family of Flat Extensions of \(T^3_2(O)_q\).** Extensions of \(T^3_2(O)_q\) are constructed in [22] (see also [5]), with \(O\) a classical hyperoval of \(PG(2, q)\), \(q\) any power of 2. That extension is flat; namely, all of its points are incident with all its planes (whence \((LL)\) fails to hold in it).

It is well known that \(T^3_2(O)_q\) is flag-transitive only when \(q = 2, 4\) or 16, with \(O\) the Lunelli–Sce hyperoval when \(q = 16\) (see also Section 1.2.1). Thus, the geometries of [22] are not flag-transitive, except possibly when \(q = 2\) or 4. When \(q = 2\) the construction of [22] gives us a flat thin-lined \(C_3\) geometry which is in fact a flag-transitive quotient of a thin-lined polar space of rank 3. It is not at all clear how things go when \(q = 4\); the automorphisms recognizable from the construction of [22] are definitely not sufficient to give flag-transitivity even if \(q = 4\), but it might be that more automorphisms exist which cannot be seen from that construction.

We do not know if the examples of [22] can be obtained as quotients of the extensions that we will construct in this paper. (Compare with Section 2.4.)

1.1.6. **Three Simply Connected Examples by Yoshihara.** A simply connected flag-transitive extension of \(T^6_2(O)_q\) has been discovered by Yoshihara [28] as a coset geometry in a group constructed by amalgamation of a certain triple of subgroups. We denote this geometry by \(Y_1\). We have \(Aut(Y_1) = 2^{5+8} \cdot (A_5 \times A_3)_2\), the center \(Z\) of \(Aut(Y_1)\) has order 2 and \(Y_1\) can be factorized by \(A\). We will denote the quotient \(Y_1/Z\) by \(Y_1/2\). Clearly, \(Y_1/2\) is flag-transitive and \(Aut(Y_1/2) = 2^8 \cdot (A_5 \times A_3)_2\).

Two flag-transitive simply connected extensions of the dual of \(T^6_2(O)_q\) are also constructed in [28]. Let us denote them by \(Y_2\) and \(Y_3\). Their automorphism groups are \(2^{12} \cdot 3S_7\) and \(2^{8+8} \cdot L_3(2)\) respectively. Yoshihara has obtained both \(Y_2\) and \(Y_3\) by amalgamation of feasible triples of parabolics, but a geometric construction is also offered in [28] for one of them, namely for \(Y_2\).

The following is also proved in [28]. Let \(\Gamma\) be a simply connected flag-transitive extension of \(T^6_2(O)_q\) (or of its dual), satisfying \((LL)\) and such that the stabilizer in \(Aut(\Gamma)\) of a point contains a normal subgroup acting regularly on the set of lines incident with that point. Then \(\Gamma \simeq Y_1\) (respectively, \(\Gamma \simeq Y_2\) or \(Y_3\)).

1.2. **Appendix.** In the following two paragraphs we give some information on \(Aut(T^3_2(O)_q)\) and on hyperovals of \(PG(2, 4)\), to be used in Section 2.
1.2.1. The Automorphism Group of $T_2^2(O)_q$. It follows from (11) that, when $q > 2$, all lines and planes of the affine geometry $AG(3, q)$ in which $T_2^2(O)_q$ is embedded and the hyperoval $O$ can be recovered from $T_2^2(O)_q$. Consequently, when $q > 2$ we have $\text{Aut}(T_2^2(O)_q) = V \cdot Z_{q-1} \cdot G_O$, where $V$ is the group of translations of $AG(3, q)$, $Z_{q-1}$ is the cyclic group of order $q - 1$ and $G_O$ is the stabilizer of $O$ in $\text{PGL}(3, q)$. (Note that this is false when $q = 2$. Indeed, $T_2^2(O)_2$ is the dual of a $(4 \times 4)$ grid. Its automorphism group is $(S_2 \times S_2\rangle 2$, whereas $V \cdot Z_{q-1} \cdot G_O = 5^3 \cdot S_4$ in this case.)

The above makes it clear that $T_2^2(O)_q$ is flag-transitive iff $G_O$ is transitive on $O$. This happens only when $q = 2, 4$ or 16, with $O$ the Lunelli–Sce hyperoval in the latter case (see [14, Theorem 4.5]).

The hyperovals of $PG(2, 2)$ are the complements of the lines of $PG(2, 2)$. Thus, $G_O = S_4$ when $q = 2$.

Let $q = 4$. All hyperovals of $PG(2, 4)$ are classical (that is, they consist of a conic plus its nucleus). Let $O$ be one of them. Then $G_O = S_6$, acting faithfully on the six points of $O$. Therefore $\text{Aut}(T_2^2(O)_4) = 2^6 \cdot 3 \cdot S_6$.

Let $q = 16$ and let $O$ be the Lunelli–Sce hyperoval. We have $G_O = 2 \times 3^2 \cdot 8$ and the Sylow 3-subgroup of $G_O$ has two orbits of size 9 on $O$ (see [14]; also [23]). The stabilizer $G_{O,L}$ of $O$ and a line $L$ external to $O$ is not transitive on $O$. Indeed, assume the contrary. Then 18 divides $|G_{O,L}|$; hence at least one of the Sylow 3-subgroups of the stabilizer $G_L$ of $L$ stabilizes $O$. Therefore, given any Sylow 3-subgroup $X$ of $G_L$, there is a hyperoval $O_X$ stabilized by $X$ and isomorphic to the Lunelli–Sce hyperoval. $X$ has two orbits of size 9 on $O_X$. However, a computer-aided calculation shows that no hyperoval can be obtained by joining two of the orbits of $X$ of size 9. Therefore $G_{O,L}$ is not transitive on $O$.

1.2.2. Hyperovals of $PG(2, 4)$. There are 168 hyperovals in $PG(2, 4)$ and $PGL(3, 4)$ transitively permutes them. Let $O$ be one of them. There are six lines of $PG(2, 4)$ external to $O$, transitively permuted by the stabilizer $G_O$ of $O$ in $PGL(3, 4)$. Let $l$ be one of those six lines. The stabilizer $G_{O,L}$ of $l$ in $G_O$ is $S_5$ ($= PGL(2, 4) = PGL(2, 5)$). It acts as $PGL(2, 4)$ on $l$ and as $PGL(2, 5)$ on the six points of $O$ (as if $O$ were a model of $PG(1, 5)$). There are 48 hyperovals of $PG(2, 4)$ external to $l$ and the stabilizer of $l$ in $PGL(3, 4)$ transitively permutes them. Given a point $u$ of $PG(2, 4)$, there are 48 hyperovals of $PG(2, 4)$ that contain $u$. They are transitively permuted by the stabilizer of $u$ in $PGL(3, 4)$.

2. The New Examples

2.1. Notation. Given a plane $u^\omega$ of $PG(4, q)$ ($q$ even) and a hyperoval $O$ of $u^\omega$, let $l^\omega$ be a line of $u^\omega$ external to $O$. Without loss, we can assume that $u^\omega$ and $l^\omega$ are represented by the following systems of equations:

$$x_4 = x_5 = 0 \quad \text{(for } u^\omega\text{)},$$

$$x_3 = x_4 = x_5 = 0 \quad \text{(for } l^\omega\text{)}.$$

We denote by $St(l^\omega)$ the star of $l^\omega$; namely, the projective plane formed by the planes and the 3-spaces on $l^\omega$. Let $u_0, u_1, \ldots, u_q$ be a family of planes on $l^\omega$ such that the family $O^\omega = \{u_i\}_{i=0}^q \cup \{u^\omega\}$ is a hyperoval of $St(l^\omega)$. We set $S_0 = \bigcup_{i=0}^q (u_i \cap O^\omega)$. Clearly, $|S_0| = (q + 1)q^2$.

We denote by $S_1$ the set of lines of $PG(4, q)$ skew with $u^\omega$ and intersecting $S_0$ in some points. Every plane of $PG(4, q)$ not intersecting $l^\omega$ meets $u^\omega$ in precisely one point. We
denote by $S_2$ the set of the planes of $PG(4, q)$ that do not meet $l^\infty$ but intersect $u^\infty$ in a point of $O$.

We denote by $\bar{A}$ the stabilizer of $l^\infty$ in $Aut(u^\infty)$ and by $A$ the subgroup of $\bar{A}$ consisting of the linear elements of $\bar{A}$; namely, $A = PGL(3, q) \cap \bar{A}$. By $\bar{A}_O$ and $A_O$ we denote the stabilizers of $O$ in $\bar{A}$ and in $A$, respectively. Similarly, $\bar{A}^*$ is the stabilizer of $u^\infty$ in $Aut(St(l^\infty))$, $A^* = \bar{A}^* \cap PGL(3, q)$, and $\bar{A}_O^*$ and $A_O^*$ are the stabilizers of $O^*$ in $\bar{A}^*$ and in $A^*$, respectively.

It is easy to see that the stabilizer $PGL(5, q)_{l^\infty,u^\infty}$ of $l^\infty$ and $u^\infty$ in $PGL(5, q)$ has the following structure:

$$PGL(5, q)_{l^\infty,u^\infty} = T \cdot (A \times A^*)$$

where $T$ is the elementwise stabilizer of $u^\infty$ and $St(l^\infty)$ in $PGL(5, q)$. The subgroups $T \cdot A$ and $T \cdot A^*$ are the elementwise stabilizers in $PGL(5, q)$ of $St(l^\infty)$ and $u^\infty$ respectively. The subgroup $T$ consists of the elements of $PSL(5, q)$ represented by matrices of the following form:

$$\begin{pmatrix}
1 & 0 & 0 & t_{1,1} & t_{1,2} \\
0 & 1 & 0 & t_{2,1} & t_{2,2} \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$  

Hence $T$ is elementary abelian of order $q^4$ and acts transitively on the $q^2$ points of $u\backslash l^\infty$, for any plane $u$ on $l^\infty$ different from $u^\infty$. The extensions $T \cdot A$ and $T \cdot A^*$ split. The non-singular matrices of the following form,

$$\begin{pmatrix}
a_{1,1} & a_{1,2} & a_1 & 0 & 0 \\
a_{2,1} & a_{2,2} & a_2 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

represent the elements of $A$. The elements of $A^*$ can be represented by matrices as follows:

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & b_1 & b_2 \\
0 & 0 & b_{1,1} & b_{1,2} & 0 \\
0 & 0 & b_{2,1} & b_{2,2} & 0
\end{pmatrix}.$$  

Note that $A$ and $A^*$ commute modulo $T$, but they do not commute as subgroups of $PGL(5, q)$. Obviously,

$$PGL(5, q)_{l^\infty,u^\infty} = T \cdot (A \times A^*) \cdot Aut(GF(q)).$$  

We have

$$\bar{A}_O = A_O \cdot F_O \quad \text{and} \quad \bar{A}_O^* = A_O^* \cdot F_O^*$$

for suitable subgroups $F_O$ and $F_O^*$ of $Aut(GF(q))$ (these subgroups need not be the stabilizers of $O$ and $O^*$ in $Aut(GF(q))$). Let $L$ (respectively, $\bar{L}$) be the stabilizers of $O$ and $O^*$ in $PGL(5, q)_{l^\infty,u^\infty}$ (in $PGL(5, q)_{l^\infty,u^\infty}$). Then,

$$L = T \cdot (A_O \times A_O^*) \quad \text{and} \quad \bar{L} = T \cdot (A_O \times A_O^*) \cdot F_O$$
for some subgroup $F_O$ of $F_O \cap F_{O^*}$. Note that, if $X$ is a subgroup of $A_{O^*}^\sigma$ transitive (or regular) on $O^* \setminus \{u^\sigma\}$, then $T \cdot X$ is transitive (regular) on $S_0$.

2.2. The construction

**Lemma 1.** Every line of $S_1$ meets $S_0$ in precisely two points.

**Proof.** Let $l \in S_1$ and let $a \in l \cap S_0$. The 3-space $\langle l, l^\sigma \rangle$ of $PG(4, q)$ spanned by $l$ and $l^\sigma$ contains precisely two planes of $O^*$. The plane $\langle l^\sigma, a \rangle$ is one of them and neither of them is $a^\sigma$. The line $l$ meets each of those two planes in precisely one point. □

**Lemma 2.** Given $u \in S_2$, let $a = u \cap u^\sigma$. Then $(u \cap S_0) \cup \{a\}$ is a hyperoval of the plane $u$.

**Proof.** As $u \cap l^\sigma = \emptyset$, the plane $u$ meets every plane on $l^\sigma$ in precisely one point. The statement follows from Lemma 1. □

Let $\mathcal{G}_0$ be the graph defined on $S_0$ by declaring two distinct points to be adjacent when the line joining them belongs to $S_1$.

**Lemma 3.** The graph $\mathcal{G}_0$ is a complete $(q + 1)$-partite graph. Its classes are the sets $u_0 \setminus l^\sigma, u_1 \setminus l^\sigma, \ldots, u_q \setminus l^\sigma$.

**Proof.** Clearly, no two points of $S_0$ belonging to the same plane of $O^*$ are adjacent in $\mathcal{G}_0$. Let $a$ and $b$ be points of $S_0$ belonging to distinct members $u$ and $v$ of $O^*$, and let $l$ be the line joining them. Clearly, $l$ is skew with $l^\sigma$. If $l$ meets $u^\sigma$, then the 3-space $\langle l, l^\sigma \rangle$ contains at least three members of $O^*$; namely, $u$, $v$ and $u^\sigma$. This is impossible. Therefore $l \notin S_1$. □

Let $\Gamma_q$ be the triple $(S_0, S_1, S_2)$, equipped with the incidence relation inherited from $PG(4, q)$. By Lemma 3, this incidence structure is a connected geometry.

**Theorem 4.** The geometry $\Gamma_q$ is an extension of $T_2^8(O_q)$.

**Proof.** It is obvious that, given $a \in S_0$, the lines of $S_1$ on $a$ and the planes of $S_2$ on $a$ form a model of $T_2^8(O_q)$. By lemmas 1 and 2, the points of $S_0$ and the lines of $S_1$ in a given plane $u \in S_2$ form a complete graph with $q + 1$ vertices. □

Clearly, $(LL)$ holds in $\Gamma_q$. Therefore, and since $T_2^8(O_2)$ is a dual grid with four lines on every point, $\Gamma_2$ is the thin-lined polar space with four planes on every line. That is, $\Gamma_2$ is the system of vertices, edges and 3-cliques of a complete 3-partite graph with all classes of size 4. Thus, $\Gamma_2$ is unique: it does not depend on the particular choice of $O$, $l^\sigma$.
and $O^*$. This is also clear if we consider that, when $q = 2$, up to automorphisms of $PG(4, 2)$ there is only one way of choosing $I^*$, $O$ and $O^*$ as above. The same is true when $q = 4$, as follows.

**Proposition 5.** Up to isomorphism, $\Gamma_4$ does not depend on the particular choice of $O$, $I^*$ and $O^*$.

**Proof.** As $PGL(5, 4)$ is transitive on the line–plane flags of $PG(4, 4)$, the particular choice of the flag $(l^*, u^*)$ has no influence on the isomorphism type of $\Gamma_4$. Given $l^*$ and $u^*$, it follows from (1a) and from the information given in Section 1.2.2 on hyperovals of $PG(2, q)$, that $PGL(5, 4)_{l^*,u^*}$ transitive permutes the hyperovals of $u^*$ external to $l^*$ and the hyperovals of $St(l^*)$ containing $u^*$. Hence the isomorphism type of $\Gamma_4$ is also independent on the particular choice of $O$ and $O^*$.

On the other hand, there are different types of hyperovals in $PG(2, q)$ when $q > 4$ (see [12]). Furthermore, when $q > 4$, even if we take a classical hyperoval as $O^*$, there are still two possibilities for $O^*$, depending on whether or not $O^* \setminus \{u^*\}$ is a conic. When the hyperoval $O$ is classical and $O^* \setminus \{u^*\}$ is a conic, we say that $\Gamma_q$ is regular. (Clearly, both $\Gamma_4$ and $\Gamma_2$ are regular.)

2.3. The group $Aut(\Gamma_q)$. Given an element $x$ of $\Gamma_q$, we denote by $G_x$ its stabilizer in $G = Aut(\Gamma_q)$ and by $K_x$ the kernel of the action of $G_x$ on the residue of $x$. We set $L_x = G_x \cap L$ and $\bar{L}_x = G_x \cap \bar{L}$.

Given a point $a$ of $\Gamma_q$, we denote by $(l^*)_a$, $(u^*)_a$ and $(O)_a$ the plane of $PG(4, q)$ spanned by $l^* \cup \{a\}$, the 3-space of $PG(4, q)$ spanned by $u^* \cup \{a\}$ and the set of lines joining $a$ to points of $O$, respectively. Let $St(a)$ be the residue of $a$ in $PG(4, q)$. We denote by $A_{u^*, l^*, O}$ the stabilizer of $(u^*)_a$, $(l^*)_a$ and $(O)_a$ in $Aut(St(a))$. Clearly,

$$A_{u^*, l^*, O} = V \cdot A_O \cdot F_O$$

(4)

with $A_O$ and $F_O$ as in (2), and $V$ is the group of translations of the 3-dimensional affine geometry obtained by removing $(u^*)_a$ from $St(a)$. (We warn the reader that $V$ is larger than the stabilizer of $a$ in $T$.)

**Lemma 6.** Let $q > 2$. It is clear from Section 1.2.1 that $G_a/K_a$ is a subgroup of $A_{u^*, l^*, O}$.

**Proof.** Let $q > 2$. It is clear from Section 1.2.1 that $G_a/K_a$ is a subgroup of the stabilizer of $(u^*)_a$ and $(O)_a$ in $Aut(St(a))$. We need to prove that $G_a$ also stabilizes $(l^*)_a$.

Let $g \in G_a$. As $G_a/K_a \leq Aut(St(a)) = PTL(4, q)$, $g$ acts on $St(a)$ as an element of $PTL(4, q)$. Therefore it sends the plane $(l^*)_a$ of $PG(4, q)$ to some plane $w$ of $PG(4, q)$ containing $a$. The plane $(l^*)_a$ contains $q^2$ points of $\Gamma_q$.

Let $u$ be a plane of $PG(4, q)$ on $a$ different from $(l^*)_a$ (whence $u$ does not contain $l^*$). If $u$ is skew with $l^*$, then $u$ meets every plane of $O^*$ in just one point. In this case $u$ contains $q + 1$ points of $\Gamma_q$. Assume that $u$ meets $l^*$ in a point, say $b$. Then $u \cup l^*$ span a 3-space of $PG(4, q)$, which contains just two members $v_1$ and $v_2$ of $O^*$. The plane $u$ intersects each of $v_1$ and $v_2$ in a line through $b$, whereas $u \cap v = \{b\}$ for any other plane $v \in O^*$. In this case $u$ contains either $q$ or $2q$ points of $\Gamma_q$, according to whether or not $u^* \in \{v_1, v_2\}$. In any case, and since we have assumed $2 < q$, $u$ contains less than $q^2$
points of $\Gamma_q$. Therefore, $(l^\circ)_a$ is the only plane of $PG(4, q)$ on $a$ containing $q^2$ points of $\Gamma_q$. This forces $g$ to stabilize $(l^\circ)_a$. \hfill \Box \\

**Theorem 7.** If $q > 4$, then $\Gamma_q$ is not flag-transitive.

**Proof.** If $\Gamma_q$ is flag-transitive, then $A_{u^\circ, l^\circ, O}$ is flag-transitive on $St(a)$. Consequently, the group induced by $A_{u^\circ, l^\circ, O}$ on $u^\circ$, which is the stabilizer of $l^\circ$ and $O$ in $Aut(u^\circ)$, acts transitively on $O$. As we have remarked in Section 1.2.1, this forces $q \leq 4$. \hfill \Box

We shall now give more information on $Aut(\Gamma_q)$.

**Lemma 8.** Let $q > 2$. Then $K_a = 1$ for every point $a$ of $\Gamma_q$.

**Proof.** Let $g \in K_a$. As $g$ fixes all lines on $a$, it is forced to fix all points collinear with $a$. Hence $g \in K_a$ for every plane $u$ on $a$. Let $b$ be a point collinear with $a$. The element $g$ fixes the line $l$ on $b$ and $a$, all planes on $l$ and all lines on $b$ belonging to some of those planes. Then $g \in K_b$, by Lemma 6. Therefore $K_a \leq K_b$. Also $K_b \leq K_a$, by symmetry. Hence $K_a = 1$, by the connectedness of $\Gamma_q$. \hfill \Box

Obviously, $\tilde{L} \leq G$. Thus $\tilde{L} \preceq G_a$, with $a$ a point of $\Gamma_q$. Furthermore, $G_a \leq A_{u^\circ, l^\circ, O}$, by Lemmas 6 and 8.

**Proposition 9.** Let $q > 2$, let $\tilde{L}$ be transitive on the set of points of $\Gamma_q$ and let $\tilde{L}_a = A_{u^\circ, l^\circ, O}$. Then $G = \tilde{L}$.

**Proof.** We have $G = \tilde{L}G_a$ because $\tilde{L}$ is point-transitive. Hence $G = \tilde{L}G_a$ because $G_a \leq A_{u^\circ, l^\circ, O} = \tilde{L}_a$. \hfill \Box

The hypothesis $q > 2$ is essential in the previous proposition. Indeed, $Aut(\Gamma_2) = S_4 \times S_4 \times S_4$, whereas $\tilde{L} = L = 2^4 \cdot (S_4 \times S_4)$ when $q = 2$. Note that the statement of Lemma 8 also fails to hold when $q = 2$.

From now on, we focus on the case of $q = 4$.

**Lemma 10.** Let $q = 4$. Then $L = 2^8(A_5 \times A_5)$ and $\tilde{L} = L \cdot 2$.

**Proof.** The first claim follows from (3) and well-known properties of stabilizers in $PGL(3, 4)$ of hyperovals of $PG(2, 4)$ (see Section 1.2.2). Let us prove that $\tilde{L} = L \cdot 2$.

By Proposition 5, we can assume that $O = C \cup \{(0, 0, 1, 0, 0), \}$, where $C$ is the conic of $u^\circ$, represented by the equation

$$x_3^2 = \varepsilon x_1 x_2 + x_1^2 + x_2^2,$$

with $\varepsilon \in GF(4) \setminus \{0, 1\}$. As $l^\circ$ is represented by the system of equations $x_3 = x_4 = x_5 = 0$, given a plane $u$ on $l^\circ$, the co-ordinates of any two points of $u \setminus l^\circ$ coincide at the last three entries (modulo a factor). Thus, we can take $(x_3, x_4, x_5)$ as the triple of co-ordinates of a plane on $l^\circ$ (that is, of a point of $St(l^\circ)$). By Proposition 5, we can also assume that $O^u \setminus \{u^\circ\}$ is the conic of $St(l^\circ)$ represented by the equation $x_3^2 = x_4 x_5$. (Note that $u^\circ$ is the nucleus of $O^u \setminus \{u^\circ\}$.)

Let $\sigma$ be the element of $PGL(5, 4)$ defined by the following clause:

$$\sigma(x_1, x_2, x_3, x_4, x_5) = (\varepsilon x_1^2 + x_2^2, x_1^2 + \varepsilon x_2^2, x_3^2, x_4^2, x_5^2).$$

It is clear that $\sigma \in \tilde{L}$. Hence $\tilde{L} = L \langle \sigma \rangle = L \cdot 2$. \hfill \Box
Given a point \( a \) of \( \Gamma_4 \), let \( L_a \) and \( \bar{L}_a \) be the stabilizers of \( a \) in \( L \) and \( \bar{L} \), respectively.

**Lemma 11.** Let \( q = 4 \). Then \( L_a = 2^4(A_4 \times A_5) = 2^6(Z_3 \times A_5) \).

**Proof.** The first equality easily follows from (3) and from well known properties of stabilizers in \( PGL(3, 4) \) of hyperovals of \( PG(2, 4) \) (see Section 1.2.2). The second equality is obtained by comparing the first one with the description of \( Aut(T_{\infty}^\delta(O_4)) \) given in Section 1.2.1 and noticing that, since \( K_a = 1 \) by Lemma 8, \( L_a \) acts faithfully on \( \bar{S}t(a) \).

**Lemma 12.** Let \( q = 4 \). Then \( L \) is transitive on the set of points of \( \Gamma_4 \).

**Proof.** By (3) and by well known properties of stabilizers in \( PGL(3, 4) \) of hyperovals of \( PG(2, 4) \), the group \( L \) is transitive on the set \( O^\delta \setminus \{u^\infty\} \). On the other hand, the elementwise stabilizer \( T \) of \( \bar{S}t(l^\infty) \) and \( u^\infty \) in \( PGL(5, 4) \) acts transitively on every plane on \( l^\infty \) other than \( u^\infty \). As \( T \equiv L \) (see (3)), \( L \) is point-transitive on \( \Gamma_4 \).

**Lemma 13.** Let \( q = 4 \). Then \( \bar{L}_a = 2^6(Z_3 \times A_5)2 = Aut(T_{\infty}^\delta(O_4)) \).

**Proof.** By Lemma 11, we only need to prove that \( \bar{L}_a = L_a \cdot 2 \). With the same assumptions as made in the proof of Lemma 10, the point \((0, 0, 0, 0, 1)\) belongs to \( \Gamma_4 \). By Lemma 12, we can assume that \( a \) is that point. Then the semilinear transformation \( \sigma \) considered in the proof of Lemma 10 stabilizes \( a \). Hence \( \bar{L}_a = L_a \cdot 2 \).

**Theorem 14.** The geometry \( \Gamma_4 \) is flag-transitive, with

\[
Aut(\Gamma_4) = \bar{L} = 2^8(A_5 \times A_5)2.
\]

**Proof.** As \( T_{\infty}^\delta(O_4) \) is flag-transitive, \( \Gamma_4 \) is flag-transitive, by Lemmas 12 and 13. The equality \( Aut(\Gamma_4) = \bar{L} \) follows from Proposition 9. The equality \( \bar{L} = 2^8(A_5 \times A_5)2 \) follows from Lemma 10.

By this theorem and by Theorem 1.1 of [28], \( \Gamma_4 \) is a quotient of the geometry \( Y_1 \) mentioned in Section 1.1.6, which is simply connected. On the other hand, \( Aut(Y_1) = 2^{1+8}(A_5 \times A_5)2 \). Therefore, we have the following.

**Corollary 15.** We have \( \Gamma_4 \cong Y_1 / 2 \) and \( Y_1 \) is the universal cover of \( \Gamma_4 \).

### 2.4. Quotients of \( \Gamma_4 \)

The subgroup \( T \) of \( Aut(\Gamma_4) \) admits subgroups acting regularly on \( u \setminus l^\infty \), for every plane \( u \) containing \( l^\infty \) and different from \( u^\infty \). Let us call these subgroups of \( T \) regular subgroups. For instance, the reader can check that, given \( a, b \in GF(q) \) such that the polynomial \( i^2 + at + b \) is irreducible over \( GF(q) \), the elements of \( PGL(5, q) \) represented by the following matrices form a regular subgroup of \( T \).

\[
M(x, y) = \begin{pmatrix}
1 & 0 & 0 & b & y \\
0 & 1 & 0 & ax + y & x \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
Let $T_0$ be a regular subgroup of $T$. It is easy to see that $T_0$ acts semi-regularly on the set of lines and on the set of planes of $\Gamma_q$. Hence it defines a quotient of $\Gamma_q$. As $T_0$ acts regularly on the $q^2$ points of $\kappa^\pi$ for every plane $u \in O^\pi \setminus \kappa^\pi$, the quotient $\Gamma_q/T_0$ has $q + 1$ points. Hence it is flat (all points of $\Gamma_q/T_0$ are incident with all planes).

Clearly, every non-trivial subgroup $X$ of $T_0$ also defines a proper quotient of $\Gamma_q$. As $\Gamma_q$ has diameter 2, the property $(LL)$ fails to hold in $\Gamma_q/X$.

**Remark.** Let $T_0$ be the regular subgroup of $T$ consisting of the elements represented by the above matrices. It is straightforward to prove that the normalizer of $T_0$ in $\text{Aut}(\Gamma_q)$ is flag-transitive on $\Gamma_q$ iff $q = 2$.

2.5. **Revisiting our construction.** The extended generalized quadrangle $\Gamma_q$ is a subgeometry of a certain rank 3 geometry considered in [9], which we shall denote by $\Delta_q$ here. The geometry $\Delta_q$ is defined as follows. Its points are the points of $PG(4, q)$ outside $u^\infty$, its lines are the lines of $PG(4, q)$ skew with $u^\infty$ and its planes are the planes $u$ of $PG(4, q)$ such that $u \cap u^\infty$ is a point of $O$. The following is a diagram for $\Delta_q$:

\[
(A\kappa^\pi, C_2)
\]

\[
\begin{array}{c c c}
q & q - 1 & q + 1 \\
npoints & nlines & nplanes
\end{array}
\]

Note also that $\Delta_q$ is a subgeometry of the affine grassmannian [7] obtained from $PG(4, q)$ by removing $u^\infty$, its points and its lines, the lines of $PG(4, q)$ meeting $u^\infty$ in a point, the planes meeting $u^\infty$ in a line and the 3-spaces containing $u^\infty$. The set $S_0$ is a *hyperoval* of $\Delta_q$ (in the meaning of [17] and [18]). The extended generalized quadrangle $\Gamma_q$ is the ‘intersection’ of $\Delta_q$ with $S_0$. Namely, its elements are the points of $S_0$ and the lines and the planes of $\Delta_q$ incident with some points of $S_0$. Thus, our construction of $\Gamma_q$ is analogous to the construction used in [17] to produce extensions of the generalized quadrangles $Q^m_5(4)$ and $H_4(2^3)$ from hyperovals of the rank 3 polar spaces $Q^m_5(4)$ and $H_6(2^2)$.

**Remark.** The regular subgroups of $T$ also define flat quotients of the $\Delta_q$. Thus, if $T_0$ is a regular subgroup of $T$, the quotient $\Gamma_q/T_0$ is a subgeometry of $\Gamma_q/T_0$. Note that flat $A\kappa^\pi \cdot C_2$ geometries are also defined in [22]. The flat extended generalized quadrangles defined in [22] are subgeometries of those flat $A\kappa^\pi \cdot C_2$ geometries.

3. **A Topological Construction of $Y_1$**

In this section, assuming $\Gamma_q$ regular, we construct a $q/2$-fold cover $\tilde{\Gamma}_q$ of $\Gamma_q$. As a by-product, we obtain a description of the universal cover $Y_1$ of $\Gamma_q$ (compare with Corollary 15). Indeed, $\tilde{\Gamma}_q = Y_1$.

Note that $\tilde{\Gamma}_2 = \Gamma_2$. In fact $\Gamma_2$, being a polar space, is simply connected.

3.1. **Preliminaries.** Algebraic covers of graphs and of certain rank 3 geometries have been introduced by Cameron [3]. We shall recall the basic ideas of that theory here.
Let $A$ be an elementary abelian 2-group. (The theory of [3] is developed for any abelian group, but elementary abelian 2-groups are enough for our purposes.) Given a connected graph $\mathcal{G}$, let $\mathcal{H}(\mathcal{G})$ be the simplicial complex of vertices, edges and cliques of $\mathcal{G}$. A simplex of $\mathcal{H}(\mathcal{G})$ containing $d+1$ vertices is called a $d$-simplex. For $d = 0, 1, 2, \ldots$, a $d$-cochain of $\mathcal{G}$ over $A$ is a mapping from the set of $d$-simplices of $\mathcal{H}$ to $A$. Clearly, the set $\mathcal{C}_d$ of $d$-cochains over $A$ is a vector space over $GF(2)$. For $d > 0$, let $\delta_d$ be the $d$-coboundary operator; namely, the linear mapping from $\mathcal{C}_{d-1}$ to $\mathcal{C}_d$ defined by the clause

$$(\delta_d(\gamma))\{a_0, a_1, \ldots, a_d\} = \sum_{l=0}^{d} \gamma(\{a_0, \ldots, a_{l-1}, a_{l+1}, \ldots, a_d\})$$

(for any $\gamma \in \mathcal{C}_{d-1}$). Note that $\delta_{d+1}\delta_d = \delta_d\delta_{d-1} = 0$. A $d$-cocycle is an element of $\text{Ker}(\delta_{d+1})$, whereas a $d$-coboundary is an element of $\text{Im}(\delta_d)$. The $d$th-cohomology group $H^d(\mathcal{G}, A)$ of $\mathcal{G}$ over $A$ is defined to be $\text{Ker}(\delta_{d+1})/\text{Im}(\delta_d)$. (Note that the elements of $\text{Ker}(\delta_1)$ are the constant 0-cochains.)

Now let $\Gamma$ be a (connected) geometry belonging to a diagram of the following form:

```
   c
  / \
 /   \ /
/     /
/     /
  s     t
   X
```

with $X$ denoting a class of (connected) partial planes (for instance, $X$ might be the class of generalized quadrangles). Assume, furthermore, that $(LL)$ holds in $\Gamma$. This implies that the intersection property also holds in $\Gamma$ [21, Lemma 7.25]. In particular, distinct planes are incident with distinct sets of points. Thus, the planes of $\Gamma$ can be viewed as distinguished sets of points.

Let $\mathcal{G}$ be the collinearity graph of $\Gamma$; namely, the graph with the points and the lines of $\Gamma$ as vertices and edges. Assume that

$$H^1(\mathcal{G}, A) = H^2(\mathcal{G}, A) = 0. \quad (5)$$

Denote by $S_0$ the set of points of $\Gamma$, let $\tilde{S}_0 = S_0 \times A$. Given a 2-cocycle $\gamma_2$ over $A$, let $\gamma_1 \in \delta_2^{-1}(\gamma_2)$. (Note that $\delta_2^{-1}(\gamma_2) \neq \emptyset$ because $H^2(\mathcal{G}) = 0$.)

We define a new graph $\tilde{\mathcal{G}}$ on $\tilde{S}_0$ as follows: given two distinct points $a, b \in S_0$ and $x, y \in A$, we declare $(a, x)$ and $(b, y)$ to be adjacent in $\tilde{\mathcal{G}}$ when $\gamma_1([a, b]) = x + y$ and $\{a, b\}$ is an edge of $\mathcal{G}$.

Note that, up to isomorphism, $\tilde{\mathcal{G}}$ does not depend on the particular choice of $\gamma_1$ in $\delta_2^{-1}(\gamma_2)$. Indeed, let $\gamma'_1$ be another element of $\delta_2^{-1}(\gamma_2)$ and let $\tilde{\mathcal{G}}'$ be defined in the same way as $\tilde{\mathcal{G}}$, but replacing $\gamma_1$ with $\gamma'_1$. We have $\gamma_1 + \gamma'_1 \in \text{Ker}(\delta_2)$, as $\delta_2(\gamma_1) = \delta_2(\gamma'_1) = \gamma_2$. On the other hand, $\text{Ker}(\delta_2) = \text{Im}(\delta_1)$ by $(5)$. Therefore $\gamma'_1 = \gamma_1 + \delta_1(\gamma_0)$ for some 0-cochain $\gamma_0$. The permutation of $\tilde{S}_0$ mapping $(a, x)$ onto $(a, x + \gamma_0(a))$ is an isomorphism from $\tilde{\mathcal{G}}$ to $\tilde{\mathcal{G}}'$.

Assume that $\gamma_2$ satisfies the following two properties:

We have $\gamma_2([a, b, c]) = 0$ for any triple of points $a, b, c$ of $\Gamma$ contained in the same plane.

There is at least one point $a$ of $\Gamma$ such that, for every $x \in A$, we have $\gamma_2(x) = x$ for some 3-clique $X$ of $\mathcal{G}$ containing $a$.

Then $\mathcal{G}$ is connected and the natural projection $\pi: \tilde{S}_0 \rightarrow S_0$ (which maps $(a, x)$ onto $a$) is an $|A|$-fold covering from $\tilde{\mathcal{G}}$ to $\mathcal{G}$ (see [3, §3]). Furthermore, for every plane $X$ of $\Gamma$, the graph induced by $\tilde{\mathcal{G}}$ on $\pi^{-1}(X)$ is a family of mutually disjoint complete graphs [3, §3]. We denote by $\tilde{S}_2$ the family of the complete graphs obtained in this way. That is, $\tilde{S}_2$ is the family of the connected components of the preimages of the planes of $\Gamma$ via $\pi$. Let
FIG. 1. (Color online) Remanent state MFM images showing the magnetic domain patterns after applying different initialization fields along the ⟨100⟩ direction. The scan size is 3 μm × 3 μm. In (a) the slight deviation of the stripe directions from the ⟨010⟩ direction is due to the misalignment between the sample position and field direction. The insets show the FFT images of the corresponding MFM patterns. The brightest spot of the FFT image indicates the most dominant periodic feature and its angular position with respect to the ⟨100⟩ crystal axis indicates the relative angle (α) between the current direction (along the ⟨100⟩ axis) and the normal to the DW plane.

To determine the DW resistivity by exploiting the remanent state electronic transport characteristics, the knowledge of stability of the magnetic domain landscape after removal of the external initialization field (H) and the orientation of the DW plane is a priori. In order to examine such prerequisites in our La0.7Sr0.3MnO3 sample, we first applied a small field of 400 Oe to align the stripe domains along the ⟨010⟩ crystal axis and subsequently took the MFM image [Fig. 1(a)] over a 3-μm × 3-μm square area region. Upon initializing the magnetic state, the external field was increased in small steps along the ⟨100⟩ direction. After every step of increase, the field was swept back to zero and kept there for 5 min, in order to facilitate the remanent state MFM imaging. As shown in Figs. 1(b)–1(d), on increasing the field strength, the remanent state stripe domains gradually rotate toward the ⟨100⟩ direction, suggesting the presence of a rotatable anisotropy, as present in typical 3d transition metal ferromagnets and their alloys. 25–27 Furthermore, no deformation of magnetic domains related to the slow relaxation procedure24 was noticed in a short timescale of few minutes. The insets in Figs. 1(a)–1(d) show the brightness-coded fast Fourier transform (FFT) images of the corresponding MFM landscapes, where the angular position of the brightest spot represents the most dominant periodic feature’s relative angle (α) between the normal (n) to the DW plane and the ⟨100⟩ crystal axis. The field dependence of thus determined α is plotted in Fig. 2(a). It is worthy to note that in spite of having a certain degree of inhomogeneity in the stripe domain pattern, the overall magnetic landscape is stable and the relative angle α increases with the initialization field above 50 to 300 Oe. The intermediate values of the relative angle suggest that the rotation of n is governed by the vector sum of the external field and the rotatable anisotropy field, where the latter can be roughly estimated to be around 50 Oe, above what the stripes start to rotate.

3.2. A q/2-fold cover of Γq, when Γq is regular. From now on, we assume that Γq is regular (Section 2.2): that is, the hyperoval O is classical and O*:\{u\} is a conic. Given an element ε ∈ GF(q) such that the polynomial t^2 + εt + 1 is irreducible over GF(q), we can assume that O consists of the conic of u^2 represented by the equation

εx_1 x_2 + x_1^2 + x_2^2 + x_3^2 = 0

and of its nucleus (0, 0, 1, 0, 0). Let χ be the function from u^2 \cup ε to A = GF(q)/GF(2) defined as follows:

χ(x_1, x_2, 1, 0, 0) = εx_1 x_2 + x_1^2 + x_2^2 (mod GF(2))

(where GF(q)/GF(2) is the quotient of the additive group of GF(q) over the additive group of GF(2)). A point (x_1, x_2, 1, 0, 0) of the affine plane u^2/ε belongs to O iff

χ(x_1, x_2, 1, 0, 0) = 0 (mod GF(2)).

By Lemma 3, the collinearity graph G of Γq is a complete (q + 1)-partite graph, with all classes of size q^2. Therefore (5) of Section 3.1 holds on G for any abelian group A (see [3, Proposition 2.2(ii)]).

By Lemma 1, there are no lines of PG(4, q) containing three distinct points of Γq forming a clique of G. Given a 3-clique {a, b, c} of G, let ⟨a, b, c⟩ be the plane of PG(4, 9) spanned by {a, b, c}. As the line of PG(4, q) spanned by any two of a, b and c is skew with u^2, the planes ⟨a, b, c⟩ and u^2 meet in one point. We set γ_2({a, b, c}) = χ(⟨a, b, c⟩ ∩ u^2).

L E M M A 16. The triple Γ = (S_0, S_1, S_2) is a connected geometry and π induces an |A|-fold covering from Γ to Γ.

We call Γ the A-cover of Γ defined by the 2-cocycle γ_2.

L E M M A 17. The 2-cochain γ_2 is a 2-cocycle.

P R O O F. Let γ_3 = δ_3(γ_2). Given a 4-clique X = {a, b, c, d} of G, we call faces of X the 3-subsets of X and we set X_x = X \{x\} for x = a, b, c, d. Thus,

γ_3(X) = γ_2(⟨X_a⟩) + γ_2(⟨X_b⟩) + γ_2(⟨X_c⟩) + γ_2(⟨X_d⟩)

We need to prove that

γ_2(⟨X_a⟩) + γ_2(⟨X_b⟩) + γ_2(⟨X_c⟩) + γ_2(⟨X_d⟩) = 0. \quad (7)

We can assume that O*:\{u^2\} is the conic of St(Γ^2) represented by the equation x_4^2 + x_4 x_5 = 0. Thus, the following matrices represent the elements of A^*_{O^*}:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & b_1 & b_2 \\
0 & 0 & 0 & b_{1,1} & b_{1,2} \\
0 & 0 & 0 & b_{2,1} & b_{2,2}
\end{pmatrix}
\]

where b_1^2 = b_{2,1} b_{1,1}, b_2^2 = b_{1,2} b_{2,2} and b_{1,1} b_{2,2} + b_{2,1} b_{1,2} = 1. Therefore A^*_{O^*} \simeq SL(2, q) acts doubly transitively on O*:\{u^2\} and there is an element g ∈ Aut(Γ_q) that maps a and b onto (0, 0, 0, 0, 1) and (0, 0, 0, 1, 0) respectively and fixes u^2 elementwise. In particular, g fixes the points (X_a) ∩ u^2, (X_b) ∩ u^2, (X_c) ∩ u^2 and (X_d) ∩ u^2.
Thus, we can assume without loss that \( a = (0, 0, 0, 0, 1) \) and \( b = (0, 0, 0, 1, 0) \). Then \( c = (c_1, c_2, 1, r, r^{-1}) \) and \( d = (d_1, d_2, 1, s, s^{-1}) \) for some \( c_1, c_2, d_1, d_2, r, s \in GF(q) \), with \( r, s \neq 0 \) and \( r \neq s \). It is straightforward to check that
\[
\begin{align*}
(X_a) \cap u^\infty &= (sd_1 + rc_1, sd_2 + rc_2, s + r, 0, 0), \\
(X_b) \cap u^\infty &= (rd_1 + sc_1, rd_2 + sc_2, r + s, 0, 0), \\
(X_c) \cap u^\infty &= (d_1, d_2, 1, 0, 0), \\
(X_d) \cap u^\infty &= (c_1, c_2, 1, 0, 0).
\end{align*}
\]
Therefore the following hold:
\[
\begin{align*}
\gamma_2(X_a) &= \varepsilon (sd_1 + rc_1)(sd_2 + rc_2) + (sd_1rc_1)^2 + (sd_2 + rc_2)^2 \\
&\quad (s + r)^2, \\
\gamma_2(X_b) &= \varepsilon (rd_1 + sc_1)(rd_2 + sc_2) + (rd_1 + sc_1)^2 + (rd_2 + sc_2)^2 \\
&\quad (r + s)^2, \\
\gamma_2(X_c) &= \varepsilon d_1d_2 + d_1^2 + d_2^2, \\
\gamma_2(X_d) &= \varepsilon c_1c_2 + c_1^2 + c_2^2.
\end{align*}
\]
A straightforward computation now yields (7).

**Theorem 18.** The 2-cocycle \( \gamma_2 \) defines a \( q/2 \)-fold cover \( \tilde{\Gamma}_q \) of \( \Gamma_q \).

**Proof.** We have \( (a, b, c) \cap u^\infty \in O \) for every triple \( \{a, b, c\} \) of coplanar points of \( \Gamma_q \). Hence \( \gamma_2 \) satisfies (6a) of Section 3.1.

Let \( a = (0, 0, 0, 0, 1), b = (0, 0, 0, 1, 0) \) and \( c = (0, t, 1, 1, 1) \), with \( t \) any element of \( GF(q) \). Then \( \gamma_2(\{a, b, c\}) = t^2 \). This makes it clear that \( \gamma_2 \) also satisfies (6b) of Section 3.1.

Therefore \( \gamma_2 \) defines an \( |A| \)-fold cover \( \tilde{\Gamma}_q \) of \( \Gamma_q \), by Lemma 16. As \( |A| = q/2 \), \( \tilde{\Gamma}_q \) is a \( q/2 \)-fold cover of \( \Gamma_q \).

**Corollary 19.** We have \( Y_1 \simeq \tilde{\Gamma}_q \).

**Proof.** The universal cover \( Y_1 \) of \( \Gamma_q \) is a double cover of \( \Gamma_q \). Hence \( Y_1 \simeq \tilde{\Gamma}_q \).

### 4. Problems

1. Let \( \Gamma_q \) be regular and let \( \tilde{\Gamma}_q \) be its \( q/2 \)-fold cover constructed in Theorem 18. Is \( \tilde{\Gamma}_q \) simply connected?

When \( q = 4 \) the answer is affirmative by Corollary 19. The following informations on the point-graph of \( \tilde{\Gamma}_q \) might be useful to attack the general case: the diameter is 3 and ‘being at distance 3’ is an equivalence relation; every 3-clique is contained in a plane. (We omit the proofs of these claims.)

Note that the simple connectedness of \( \tilde{\Gamma}_q \) can also be deduced from the above informations. Indeed, let \( \Gamma \) be a cover of \( \tilde{\Gamma}_q \). The diameter of \( \Gamma \) is 3, by general results on diameters of extended generalized quadrangles [6, Corollary 3.5 and Theorem 3.16] and because \( \tilde{\Gamma}_4 \) has diameter 3. Assume that \( \Gamma \neq \tilde{\Gamma}_4 \). Then some path of \( \Gamma \) of length 3 is mapped onto a 3-clique of \( \tilde{\Gamma}_4 \), which cannot be contained in any plane of \( \tilde{\Gamma}_4 \). However, every 3-clique of \( \tilde{\Gamma}_4 \) belongs to a plane—a contradiction. Therefore \( \Gamma = \tilde{\Gamma}_4 \).

2. What about covers of \( \Gamma_q \) when \( \Gamma_q \) is non-regular?
(3) Prove that no flag-transitive proper quotients exist of $\Gamma_q$ when $q > 2$. (Note that $\Gamma_2$ admits just one flag-transitive proper quotient. It is flat.)

(4) Are the extensions of $T^\circ_2(O)_q$ mentioned in Section 1.1.5 quotients of some of the geometries $\Gamma_q$ constructed in this paper?

(5) Is $Y_2$ (Section 1.1.6) a member of some infinite family of extended generalized quadrangles?

(6) Find a geometric construction of the extended generalized quadrangle $Y_3$.

(7) Are there any extensions of some of the non-classical generalized quadrangles of order $(q, q)$ or $(q, q^2)$?

(8) Are there any extensions of the dual of $AS(q)$ when $q > 3$? (When $q = 3$ the dual of $AS(q)$ is $H_3(2^2)$. The items (10) and (11) of the table of Section 1.1.3 are in fact extensions of $H_3(2^2)$.)

ACKNOWLEDGEMENT

We wish to thank S. Yoshiara for many corrections and improvements suggested to an earlier version of this paper.

ADDED IN PROOF. S. Yoshiara has recently obtained an affirmative answer to Problem (5). An affirmative answer to Problem (8) has also been obtained by the third author of this paper, a few months ago.

REFERENCES

1. A. Blokhuis and A. Brouwer, Locally 4-by-4 grid graphs, J. Graph. Theory, 13 (1989), 229–244.
11. M. De Soete and J. Thas, A characterization theorem for the generalized quadrangle $T^\circ_2(O)$ of order $(s, s + 2)$, Ars Combin., 17 (1984), 225–242.

Received 12 November 1995 and accepted in revised form 8 March 1996

ALBERTO DEL FRA
Faculty of Engineering, University of L’Aquila,
Monteluco, I-67040 Poggio Reale, L’Aquila, Italy

DMITRII V. PASECHNIK
RIACA/CAN,
Kruidlaren 419, 1098 VA Amsterdam, The Netherlands

AND

ANTONIO PASINI
Faculty of Engineering, University of Siena,
Via Roma 56, I-53100 Siena, Italy