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# Extending Polar Spaces of Rank at Least 3*,†

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It is shown that the extended polar space for the sporadic Fischer group $Fi_{22}$ is the only extended polar space which has more than two extended planes on a block and is not isomorphic to a quotient of an affine polar space over $GF(2)$. New examples of $EGQ(4, 1)$ and $EGQ(4, 2)$ are presented as well. 

## 1. INTRODUCTION AND RESULTS

This paper is a continuation of an earlier work [19] by the author, where, among other results, the $Fi_{22}$-extended polar space was characterized by a stricter assumption involving certain 3- and 4-vertex configurations in the point graph. We also prove certain conjectures made in [6] on the extendability of polar spaces. Our main statement was known to be true under additional assumptions on the existence of a flag-transitive automorphism group acting on the geometry, see e.g. [6, 11, 17, 22].

A connected incidence system $\Gamma = \Gamma(\mathscr{P}, \mathscr{B})$ is an *extended polar space* (respectively *extended (projective) plane*) if its point residues are finite, thick, nondegenerate polar spaces (respectively finite nondegenerate projective planes).

We say that an extended polar space $\Gamma$ admits *extended planes* if there exists a nonempty set $\Sigma = \Sigma(\Gamma)$ of subsets of the set $\mathscr{P}$ of points of $\Gamma$ such that $\{x\} \cup \pi \in \Sigma$ whenever $\pi$ is a plane in the residue $\Gamma_x$ of the point $x$ and for any $\Xi \in \Sigma$ the incidence system of points and blocks of $\Gamma$ on $\Xi$ is an extended projective plane.

THEOREM 1.1. *Let $\Gamma$ be an extended polar space admitting extended planes. Then exactly one of the following holds.*

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(i) *$\Gamma$ is a standard quotient of an affine polar space over $GF(2)$.*

(ii) *$\Gamma$ is the $Fi_{22}$-extended polar space.*

(iii) *There are exactly two extended planes on each block, and $\Gamma$ is an extended $Q_5^+(4)$-polar space.*

*Remark.* The geometries in case (i) are all known, see [6, 4] (also [12, 8, 9] for various generalizations). Namely, let $\Gamma$ be an affine polar space over $GF(2)$, that is the geometry induced on the complement of a geometric hyperplane in a nondegenerate polar space over $GF(2)$. $\Gamma$ admits a (proper) standard quotient if and only if the diameter of the point graph on $\Gamma$ is 3. In the latter case the point graph is a two-fold antipodal cover of a complete graph. Here being at maximal distance is an equivalence relation with the classes of size 2, and the quotient is defined on the equivalence classes of objects (that is, points, lines, etc.).

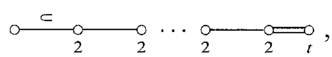
In case (iii), however, no examples are known at all.

*Remark.* It is easy to see that $\Gamma$ has the structure of a geometry with the following diagram: (see [5] for the notion of diagram geometry)

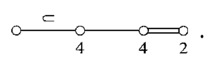
The types of the elements are, from left to right, as follows: points, pairs of adjacent points, blocks, extended projective 2-spaces, ..., extended projective $n-1$-spaces, where $n$ is the rank of the polar space we are extending. Note that the elements corresponding to the extended $j$-subspaces for $j>2$ should be recovered as certain subsets of points, along the lines of the proof of Lemma 2.2, where extended 3-spaces are reconstructed.

The diagrams corresponding to the cases of the Theorem are as follows:

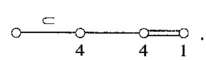
(i)



(ii)



(iii)



Note that $t \in \{1, 2, 4\}$.

*Remark.* Let $\Gamma$ be an extended polar space with point residues of rank at least 3. If each triple of pairwise adjacent points of $\Gamma$ lies in a block (that is, $\Gamma$ is *triangular*) then, by [6], $\Gamma$ admits extended planes. However, since the proper standard quotients of affine polar spaces are not triangular, the triangularity assumption is stronger than the assumption on existence of extended planes.

Since in case (iii) no examples are known, it seems quite natural to ask the following.

QUESTION. *Does there exist an extenson $\Gamma$ of the $Q_5^+(4)$-polar space $\Delta$?*

It appears that the technique employed in the paper cannot cope with this question. Indeed, $\Delta$ admits hyperovals, cf. Proposition 3.1, and obvious counting tricks do not rule thme out. By comparing the orders of the automorphism groups of the hyperovals (see Section 5) to the one of $\Delta$, one sees that the number of hyperovals in $\Delta$ is huge. It makes attempts to reconstruct $\Gamma$ using a computer particularly difficult. In this respect, it would be quite interesting to find a computer-free construction of the hyperovals of $\Delta$.

*Hyperovals and extensions of generalized quadrangles.* A *hyperoval* $\Omega$ of a polar space $\Pi$ is a set of points of $\Pi$ such that each line of $\Pi$ meets it in 0 or 2 points. (In [6] such objects are called *local subspaces.*) They play an important role in extensions of $\Pi$. As it was observed in [6], a hyperoval $\Omega$ in a polar space $\Pi$ of rank $r$ is a triangular extended polar space of residual rank $r-1$. The structure of an extended polar space on $\Omega$ can be seen within the subgraph of the point graph of $\Pi$ induced by $\Omega$.

In particular, as happens in the case we shall be mainly concerned with (that is $r=3$, $\Pi$ is over $GF(4)$), $\Omega$ is an extended generalized quadrangle of order $(4, t)$ ($EGQ(4, t)$, for short), where $t \in \{1, 2, 4, 8, 16\}$. $EGQ(s, t)$ are rather interesting in themselves, in particular a number of finite simple groups, including sporadic ones, act on them as flag-transitive automorphism groups, see e.g. [7]. As a by-product of our investigation of hyperovals we find new examples of $EGQ(4, t)$, $t=1, 2$. However, these examples do not admit flag-transitive automorphism groups. For $t=2$ our example is the only known example of an extension of a classical generalized quadrangle of order $(s, t)$, $t>1$, which does not admit a flag-transitive automorphism group.

These examples are discussed in greater detail in the last section of the paper.

Note that the proof of Theorem 1.1 depends upon calculations using the computer algebra system GAP [16] and its package for computations with graphs and groups GRAPE [20].

## 2. PRELIMINARIES

An *incidence system* is a pair $\Gamma = \Gamma(\mathscr{P}, \mathscr{B})$, where $\mathscr{P}$ is a set (of *points*) and $\mathscr{B}$ is a set of subsets of $\mathscr{P}$ (called *blocks*), each element of $\mathscr{B}$ is of size greater than one and the incidence between points and blocks is defined by inclusion. Two points $p, q$ are said to be *adjacent* (notation $p \perp q$) if they lie in a common block. The set of points adjacent to $p$ is denoted by $p^{\perp}$. The *point graph* of $\Gamma$ is a graph with vertex set $\mathscr{P}$, and adjacency the same as in $\Gamma$. The *residue* $\Gamma_p$ of a point $p$ is the incidence system of points adjacent to $p$ excluding $p$ itself, and blocks on $p$ with the point $p$ removed. We say that $\Gamma$ is *connected* if its point graph is connected.

From now on $\Gamma$ is an extended polar space of residual rank at least 3 admitting extended planes.

LEMMA 2.1. *The point residues of $\Gamma$ are isomorphic.*

*Proof.* Let $\Phi$ be an extended plane of $\Gamma$ and $x \in \Phi$. Then $\Gamma_x$ is a polar space of rank at least 3. Its isomorphism type is uniquely determined by the isomorphism type of the polar space of blocks and extended planes on $xy$, where $y \in x^{\perp} - \{x\}$, and incidence is by inclusion. Hence $\Gamma_x \cong \Gamma_y$, and by connectivity of $\Gamma$ the result follows. ∎

LEMMA 2.2. *Let $\Delta = \Gamma_x$, $x \in \mathscr{P}$. Then $\Delta$ is either*

(i) *a polar space over $GF(2)$, or*

(ii) *a rank 3 polar space over $GF(4)$.*

*Proof.* Note that $\Delta$ admits an embedding into a projective space over $GF(q)$, cf. e.g. Tits [21]. In particular the planes of $\Delta$ are of a prime power order $q$. Since the only projective planes of such an order which are extendable are of order 2 or 4 (cf. [13, 10]), $\Delta$ is defined over $GF(2)$ or $GF(4)$. It remains to check that in the latter case the rank of $\Delta$ is 3.

Assume to the contrary that $\Delta$ is defined over $GF(4)$ and has rank at least 4. Let $X = \{x\} \cup \tau$, where $\tau$ is the set of points of a 3-dimensional singular projective subspace of $\Delta$. We claim that $\Gamma(X, \mathscr{B}_X)$ is an extended $PG(3, 4)$, where $\mathscr{B}_X = \{B \in \mathscr{B} \mid B \subset X\}$. It suffices to show that, given three points $a, b, c \in X$, there exists a unique $B \in \mathscr{B}_X$ containing them. There exists a plane $\pi \subset \tau$ containing $a$, $b$ and $c$. Then $\Pi = \{x\} \cup \pi \in S$ is an extended plane, so there exists $B \in \mathscr{B}_\Pi$ containing $a$, $b$ and $c$. The uniqueness of $B$ follows from the fact that $\Gamma_a$ is a polar space and $B - \{a\}$ is a line of it. Thus $\Gamma(X, \mathscr{B}_X)$ is an extended $PG(3, 4)$. On the other hand, according to [13] (see also [10]), $PG(3, 4)$ is *not* extendable. This contradiction implies (ii). ∎

It was shown in [6, 4] that in case (i) of Lemma 2.2, $\Gamma$ is a standard (perhaps improper) quotient of an affine polar over $GF(2)$, and in fact all such objects are known. Thus we assume from now on that $\Delta = \Gamma_x$ is a rank 3 polar space over $GF(4)$.

LEMMA 2.3. *Let $C$ be a block of $\Gamma$, $r$ a point of $\Gamma$, and suppose that there is no extended plane on $r$ and $C$. Then $|C \cap r^{\perp}| = 0, 2$ or 4.*

*Proof.* Assume that $C \cap r^{\perp}$ is not empty. Let $x \in C \cap r^{\perp}$. Then there exists a unique point $x' \in C$ such that $\{x'\} = C \cap r^{\perp_{\Gamma_x}}$. So there exists a block $B^x$ containing $r$, $x$ and $x'$. Clearly $C \cap B^x = \{x, x'\}$. Since obviously $B^{x'} = B^x$, there is an equivalence relation on $C \cap r^\perp$ with classes $\{x, x'\}, \ldots$. Thus $|C \cap r^\perp|$ is even. It remains to show that $C \not\subseteq r^\perp$.

Let $\{p, p', q, q'\} \subseteq C \cap r^\perp$, and let $\Pi$ be an extended plane containing $C$. Then $\Pi$ contains a unique block $C^q$ such that $q, q' \in C^q$ and $C^q - \{q\} = \Pi \cap r^{\perp_{\Gamma_q}}$. Also, $\Pi$ contains a unique block $C^p$ such that $p, p' \in C^p$ and $C^p - \{p\} = \Pi \cap r^{\perp_{\Gamma_p}}$. Note that $|C^p \cap C^q|$ is 0 or 2. Assume first that the latter holds, this is $\{y, y'\} = C^p \cap C^q$. Then the three extended planes $\Pi$, $\langle C^p, r\rangle$ and $\langle C^q, r\rangle$ form a triangle in the generalized quadrangle of blocks and extended planes containing $yy'$, a contradiction. So $C^p \cap C^q = \varnothing$.

Assume $\{x, x'\} = r^\perp \cap C - \{p, p', q, q'\}$. Then $\Pi$ contains a unique block $C^x$ such that $x, x' \in C^x$ and $C^x - \{x\} = \Pi \cap r^{\perp_{\Gamma_x}}$. Repeating the argument above with $C^x$ in place of $C^p$, one has that $C^x \cap C^q = \varnothing$. Similarly, $C^x \cap C^p = \varnothing$. So the extended plane $\Pi$ admits three blocks with trivial pairwise intersections, the situation well-known to be impossible. This is a contradiction. ∎

Note that if $\Gamma$ is triangular then the case $|p^\perp \cap C| = 4$ does not occur.

PROPOSITION 2.4. *There exists $p \in \mathcal{P}$ such that $\Delta = \Gamma_p$ contains a hyperoval $\Omega$. Moreover, if $\Gamma$ is not triangular then $\Omega \cap r^{\perp_\Delta} = \varnothing$ for some $r \in p^\perp - \{p\}$.*

*Proof.* If $\Gamma$ is triangular then the statement is well-known, cf. [6]. So we assume that $\Gamma$ is not triangular. There exist points $p, q, r$ such that $p \perp q \perp r \perp p$, but there is no extended plane containing all of them. Note that this implies $d_{\Gamma_p}(q, r) = d_{\Gamma_q}(p, r) = d_{\Gamma_r}(p, q) = 2$. In what follows we call such a triple $pqr$ of points a *bad triangle*. Let $C$ be a block on $pq$. By Lemma 2.3, $C \cap r^\perp = \{p, p', q, q'\}$, where $p', q'$ are such that there exist blocks containing $\{p, p', r\}$, respectively $\{q, q', r\}$. Now we are able to define a hyperoval $\Omega$ in $\Gamma_p$. For each block $B$ on $p$ such that $|B \cap r^\perp| = 4$, set $z, z' \in \Omega$, where $B \cap r^\perp = \{p, p_1, z, z'\}$ and there exist blocks containing $\{r, p, p_1\}$ and $\{r, z, z'\}$. Note that the triangles $prz$ and $prz'$ are bad.

The remaining possibilities for $B$ are $|B \cap r^\perp| = 2$ or 6 and we set $B \cap \Omega = \varnothing$. We shall check that our definition of $\Omega$ is correct. If $|B \cap r^\perp| = 2$ or 6 then all the triangles $prw$, where $w \in B \cap r^\perp$, are good. So there is no block $B'$ such that $|B' \cap r^\perp| = 4$ and $w \in B'$. We are done. This proves the first part of the proposition. The second one follows from the construction of $\Omega$. ∎

The following observation will be useful in determining the hyperovals of $\Delta$.

LEMMA 2.5. *Let $\Pi$ be a subspace of $\Delta$. Then $\Pi \cap \Omega$ is a hyperoval of $\Pi$ whenever $\Omega$ is a hyperoval of $\Delta$.*

## 3. Classification of the Hyperovals

Let $\Delta$ be a rank 3 nondegenerate polar space over $GF(4)$. Then $\Delta$ is one of the following polar spaces: $Q_5^+(4)$, $S_5(4)$, $Q_7^-(4)$, $H_5(4)$ or $H_6(4)$. We say that two hyperovals of $\Delta$ are of the same *type* if one can be mapped onto the other by an automorphism of $\Delta$.

Proposition 3.1. *The hyperovals $\Omega$ in $\Delta$ are as follows.*

(i) $\Delta \cong Q_5^+(4)$: *two types, one has* 72 *points, the other has* 96 *points.*

(ii) $\Delta \cong H_5(4)$: *two types, one has* 126 *points, the other has* 162 *points. Each point of $\Delta$ is collinear with a point of $\Omega$.*

(iii) $\Delta \cong S_5(4)$, $Q_7^-(4)$ *or* $H_6(4)$. *No hyperovals.*

*Proof.* (i) and (ii) are results of a computer backtrack search. Following Lemma 2.5, we first find all the hyperovals (up to type) of $x^\perp \subset \Delta$ containing $x \in \mathscr{P}(\Delta)$. Then we try to extend each of them to hyperovals of $\Delta$.

(iii) For $\Delta \cong S_5(4)$ we use a computer and the fact that $Q_5^+(4) \subset S_4(4)$, which allow us, by Lemma 2.5, to start the search from a hyperoval of $Q_5^+(4)$. It turns out that none of the hyperovals of $Q_5^+(4)$ are extendable to a hyperoval in $\Delta$.

For $\Delta \cong Q_7^-(4)$, the Proposition follows immediately from Lemma 2.5 and the fact that $S_5(4) \subset Q_7^-(4)$ does not admit hyperovals.

For $\Delta \cong H_6(4)$ we are able to give a proof which is computer-free, apart from using part (ii) of the Proposition. Let $\Omega$ be a hyperoval of $\Delta$, and let $\mathscr{S}$ be the set of the $H_5(4)$-subspaces of $\Delta$. We claim that $\Phi \cap \Omega \neq \varnothing$ for any $\Phi \in \mathscr{S}$.

Let $\Phi_0 \in \mathscr{S}$ intersect $\Omega$ nontrivially (such $\Phi_0$ clearly exists, since there is an element of $\mathscr{S}$ on any point of $\Delta$). Each $\Phi_1 \in \mathscr{S}$ intersecting $\Phi_0$ in a hyperplane with a deep point $p$ (that is, in the set $p^{\perp_{\Phi_1}}$, $p \in \mathscr{P}(\Phi_1)$) satisfies $\Phi_1 \cap \Omega \neq \varnothing$, as well. Indeed, by the second part of (ii), $p^{\perp_{\Phi_0}} = p^{\perp_{\Phi_1}} \subset \Phi_1$ intersects $\Omega$ nontrivially, so $\Phi_1$ intersects $\Omega$ nontrivially.

Finally, note that the graph defined on $\mathscr{S}$ such that two vertices are adjacent iff the corresponding subspaces intersect in a hyperplane with deep point is connected. In particular, there is a path in this graph from $\Phi_0$ to $\Phi$. Hence $\Phi \cap \Omega \neq \varnothing$.

So each of the 2752 elements $\Phi$ of $\mathscr{S}$ corresponds to a subhyperoval of $\Omega$, that is, to a nonempty set $\Phi \cap \Omega$. Moreover, different elements of $\mathscr{S}$ correspond to different subhyperovals. There are 704 elements of $\mathscr{S}$ on each point of $\Delta$. So there is the same number of subhyperovals on each point of $\Omega$. By (ii), each of those subhyperovals has either 126 or 162 points. Counting in two ways the number of incident point-subhyperoval pairs, we have

$$704\,|\Omega| = 162a + 126(2752 - a), \tag{1}$$

where $a$ denotes the number of 162-point subhyperovals of $\Omega$.

Next, note that there are 176 elements of $\mathscr{S}$ on any pair of noncollinear points of $\Delta$. Hence each pair of noncollinear points of $\Omega$ is contained in the same number of subhyperovals. Counting in two ways the number of the pairs "pair of noncollinear points within a subhyperoval", we have

$$176\,|\Omega|\,(|\Omega| - 166) = 162 \cdot 116a + 126 \cdot 80(2752 - a). \tag{2}$$

Since the system of equations (1)–(2) does not have any nonnegative integral solutions, $\Omega$ does not exist. ∎

## 4. Extensions of $H_5(4)$

Here we complete the proof of Theorem 1.1. It follows from Proposition 3.1 and the following result.

Proposition 4.1. *Let $\Gamma$ be an extension of $\Delta \cong H_5(4)$. Then $\Gamma$ is the extended polar space for $Fi_{22}$.*

*Proof.* It follows from Propositions 3.1(ii) and 2.4 that $\Gamma$ is triangular. So $\Gamma$ can be recovered from its point graph, cf. [6]. For simplicity, we denote the point graph of $\Gamma$ by $\Gamma$.

For a point $u \in \mathscr{P}(\Gamma)$, denote by $\Gamma_2(u)$ the set of points at distance 2 from $u$. For each $x \in u^\perp - \{u\}$, $|x^\perp \cap \Gamma_2(u)| = 512$. For each $v \in \Gamma_2(u)$, $|u^\perp \cap v^\perp| = 126$ or 162, cf. Proposition 3.1(ii). Counting in two ways the number of edges between $u^\perp$ and $\Gamma_2(u)$, we see that there exists $v \in \Gamma_2(u)$ such that $|u^\perp \cap v^\perp| = 126$. Indeed, otherwise, for any $x \in u^\perp - \{u\}$,

$$|\Gamma_2(u)| = |u^\perp - \{u\}| \cdot |x^\perp \cap \Gamma_2(u)|/162 = 693 \cdot 512/162,$$

which is a non-integer.

Let $w \in v^\perp - u^\perp - \{v\}$. The subgraph induced by $\{u, v, w\}^\perp$ is isomorphic to the 2-clique extension $\Phi$ of the collinearity graph of $GQ(2, 2)$, cf. [19, Lemma 2.8(i)]. Thus the hyperovals $u^\perp \cap v^\perp$ and $u^\perp \cap w^\perp$ must intersect in a subgraph isomorphic to $\Phi$. A computer search shows that $|u^\perp \cap w^\perp| \neq 162$. By [19, Lemma 2.10], $\{u, v, w\}^\perp = \{u, v, w'\}^\perp$ implies $w = w'$ for $w' \in v^\perp - u^\perp$.

Let $\Xi$ be the graph defined on the 126-point hyperovals of $\Delta$, such that two vertices $\Omega$, $\Omega'$ are adjacent iff $\Omega \cap \Omega' \cong \Phi$. By [19, Lemma 2.11], $\Xi$ has three connected components $\Xi'$, $\Xi''$ and $\Xi'''$, each of size 1408 and valence 567. They are permuted by automorphisms of $\Delta$. Since $|v^\perp - u^\perp - \{v\}| = 567$, the connected component $\Lambda$ containing $v$ of the subgraph of $\Gamma$ induced on $\Gamma_2(u)$ is a cover of $\Xi'$. By counting the edges between $u^\perp$ and $\Gamma_2(u)$, the index of this cover is at most 2. Since $\mu(\Xi') = 216$, a number bigger than the size of a hyperoval of $\Delta$, we find that $\Xi' \not\cong \Lambda$. So $\Lambda$ is a 2-fold cover of $\Xi'$.

Continuing as in [19, Sect. 2.3] (that is by observing that $\Gamma$ is a triple graph and recovering the 3-transposition group associated which it), it follows that $\Gamma$ is the example related to $Fi_{22}$.

Alternatively, note that we have shown that $|\Gamma(x, y)| = 126$ for any two vertices $x$, $y$ at distance 2. Since all the hyperovals of $H_5(4)$ of size 126 are of the same type (cf. Proposition 3.1(ii)), they must be isomorphic to the particular type of the hyperovals appearing in the $Fi_{22}$-example. Therefore the condition on $C_4$-subgraphs of $\Gamma$ in [19, Theorem 1.1] holds, and so $\Gamma$ is indeed the $Fi_{22}$-example.

## 5. New $EGQ(4, 1)$ and $EGQ(4, 2)$

Here we describe new extended generalized quadrangles constructed as hyperovals of a polar space $\Delta$. A computer-free proof of the existence of the 162-point $EGQ(4, 2)$ is given. We slightly abuse notation by sometimes identifying the hyperoval with the respective $EGQ$ and/or with the point graph of this $EGQ$.

$\Delta = Q_5^+(4)$. There are two types of hyperovals. They give two non-isomorphic $EGQ(4, 1)$. The first one, with 72 points, is isomorphic to the one in [7, Example 9.15(iii)]. Its automorphism group is of order 28800 and it acts flag-transitively. The second one, with 96 points, is apparently new. Its automorphism group is of order 3200 and it has two orbits on points of length 16 and 80, respectively. The distribution diagrams with respect to a point (see [3]) are different for points from different orbits. The diameter of the point graph is 3.

$\Delta = H_5(4)$. There are two types of hyperovals. They give two non-isomorphic $EGQ(4, 2)$. The first one, with 126 points, is well-known. See e.g. [7, Example 9.9(b)(ii)], [19].

A hyperoval $\Pi$ of the second type has 162 points. The automorphism group of $\Pi$ is isomorphic to $(3^4 : S_6) \cdot 2$, and it is not flag-transitive (namely, there are two edge orbits). We show its existence without a computer.

LEMMA 5.1 (A. E. Brouwer and H. Cuypers). *There exists a pair of* 126-*point hyperovals* $\Xi$ *and* $\Xi'$ *intersecting in a* 45-*point Fischer subspace* *of $\Xi$. The symmetric difference $\Pi$ of $\Xi$ and $\Xi'$ is a* 162*-point hyperoval of $\Delta$.*

*Proof.* The first part of the statement easily follows from [19, Lemma 2.3(2)]. Let us turn to the second part.

Let $\Xi$ be a 126-point hyperoval of $\Delta$. It has Fischer subspaces $S$ of size 45 (in the natural orthogonal description of $\Xi$, where the points are $(+)$-type points of the 6-dimensional $GF(3)$-space equipped with a non-degenerate symmetric bilinear form with discriminant 1, adjacency coincides with perpendicularity, and each $S$ consists of the points orthogonal to a given isotropic point). Pick one such $S$. There are exactly three 126-point hyperovals intersecting in $S$, see [19, Lemma 2.3]. Pick any two of them, say $\Xi, \Xi'$. We claim that the subgraph $\Pi$ induced on the symmetric difference of the pointsets of $\Xi$ and $\Xi'$ is a hyperoval.

Let $x \in \Xi - S$, and let $l$ be a line of $\Delta$ on $x$. Let $\{x, y\} = l \cap \Xi$. Assume first that $y \in S$. Then $l \cap \Xi' = \{x', y\}$, so $l \cap \Pi = \{x, x'\}$, as required.

It remains to show that in the case $y \in \Xi - S$ we have $l \cap \Xi' = \emptyset$. By [18, Lemma 2.1(iv)], the subgraph of the point graph of $\Delta$ induced on $x^{\perp} \cap S$ is isomorphic to the point graph $\Upsilon$ of $GQ(2, 2)$. By [19, Lemma 2.8(1)], the subgraph of the point graph of $\Delta$ induced by $x^{\perp} \cap \Xi'$ is isomorphic to the 2-clique extension of $\Upsilon$. Thus if a line on $x$ intersects $\Xi'$ then it intersects $S$, and we are in the case already considered. ∎

Note that the 126-point hyperovals and the sets $S$ described above are Fischer subspaces of $\Delta$, and they are related to near subhexagons of $H_5(4)$-dual polar space, see [2]. The proof of the lemma given by Andries Brouwer used a technique developed in [2]. The proof just given seems to be a streamlined version of Brouwer's proof.

Next, we give two other descriptions of $\Pi$. The first one is the author's description communicated to Andres Brouwer. Let $\Lambda$ be the incidence graph of a $PG(6, 6, 2)$ defined on [3, p. 373] (Van Lint–Schrijver $PG$). Note that $\Lambda$ is not distance transitive. Let $u$ be a vertex of $\Lambda$. $\Lambda_3(u)$ is split into the two orbits $\Lambda_3^1(u)$ and $\Lambda_3^2(u)$ of lengths 60 and 15 respectively. $\Lambda_4(u)$ is split into the two orbits $\Lambda_4^1(u)$ and $\Lambda_4^2(u)$ of lengths 20 and 30 respectively. To obtain the point graph of $\Pi$, one has to choose as the set $\Pi(u)$ the union of $\Lambda_3^2(u)$ and $\Lambda_4^2(u)$, and then apply $Aut(\Lambda)$ to get all the remaining edges.

Andres Brouwer (personal communication) also gave the following elegant description of $\Pi$. Let

$$f(x, y) = x_1 y_1^2 + \cdots + x_6 y_6^2$$

be the Hermitian form defining $\Delta$. Then, without loss of generality, the points of $\Pi$ are the points of $\Delta$ such that the product of the coordinates lies in $GF(4) \backslash GF(2)$.

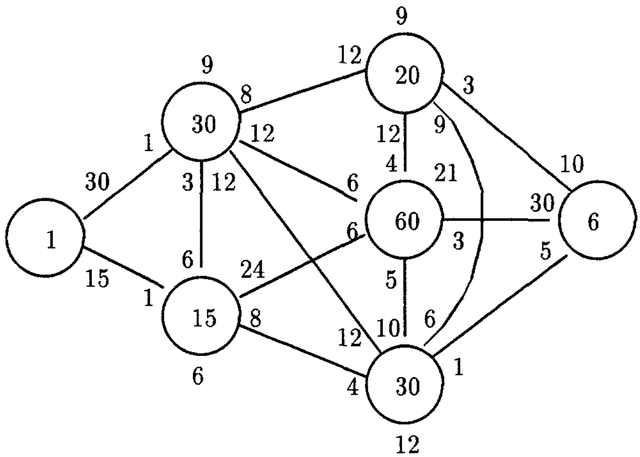



FIG. 1. The distribution diagram of the 162-point $EGQ(4, 2)$.

*Remark.* Steven Linton (personal communication) computed, using his vector enumeration program, which implements his algorithms presented in [15], that the fundamental group of the 162-point $EGQ(4, 2)$ is perfect. Hence the index of any its proper covers is at least 60, the order of the smallest perfect group. On the other hand, upper bounds on the number of points in $EGQ(s, t)$ given in [7] show that such big covers do not exist. Thus the 162-point $EGQ(4, 2)$ does not have any proper covers.

Note that the 126-point $EGQ(4, 2)$ has the triple cover, which is the only proper cover, see [1, 17, 7].

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