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<td>Author(s)</td>
<td>Pasechnik, Dmitrii V.</td>
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The Extensions of the Generalized Quadrangle of Order (3, 9)

Dmitrii V. Pasechnik

It is shown that there is only one extension of GQ(3, 9) namely the one admitting the sporadic simple group McL as a flag-transitive automorphism group. The proof depends on a computer calculation.

1. Introduction and Result

This note continues a series of papers [21, 20, 22, 24, 23] by the author, [8, 9] by Cuypers and [10] by Cuypers, Kasikova and the author, aimed at characterizing sporadic simple groups in a way similar to characterizations of classical groups as automorphism groups of polar spaces. We refer the reader to [22, 23] for further discussion in this direction.

The note may as well be viewed as a contribution to the topic of extended generalized quadrangles of order \((s, t)\) (EGQ\((s, t)\), for short). Investigations in this area were initiated by Buekenhout and Hubaut [4]; see also a survey article by Cameron, Hughes and Pasini [6] and a recent update in Del Fra, Pasechnik and Pasini [12]. One can observe that the difficulties in the problem of classifying the EGQ\((s, t)\) grow rapidly as the parameter \(s\) grows.

The EGQ\((2, t)\) were classified in [4] and in Buekenhout [3]. The EGQ\((3, 1)\) were classified by Blokhuis and Brouwer [1] and by Fisher [11]. The uniqueness of the triangular EGQ\((3, t)\) for \(t = 3\) was shown by the author in [25] (see also Makhnev [16]) and for \(t = 5\) by Makhnev [17].

For the rest of the possible values of \(s\) and \(t\), the only classification results known require assumptions on flag-transitive action of the automorphism group; see, for example, Pasini [26].

The known EGQ\((3, 9)\) can be described via the McLaughlin graph \(\Sigma\), a strongly regular graph on 275 vertices of valence 112 admitting the sporadic simple group McL as an automorphism group, see [2, 7, 19]. The vertices and the maximal cliques of this graph form a EGQ\((3, 9)\). We prove the following.

Theorem 1.1. The extended generalized quadrangle of order \((3, 9)\) is unique.

This result was known to hold under assumptions stronger than ours. Assuming the existence of certain flag-transitive automorphism groups acting on the geometry, it was proved in [4] and, under assumptions weaker than those of [4], in Del Fra, Ghinelli, Meixner and Pasini [13]. The uniqueness of a strongly regular graph with parameters the same as those of the McLaughlin graph was shown in Cameron, Goethals and Seidel [5].

It is well-known that the EGQ\((3, 9)\) admits extensions with automorphism group \(Co_3\). We will consider these extensions in a forthcoming paper.

The proof of the theorem depends upon certain computer calculations performed using a computation group theory system GAP [15] with its shared package GRAPE [28] by Soicher, which in turn uses a program NAUTY [18] by B. McKay.
2. Preliminaries

An incidence system is a pair $\Gamma = (\mathcal{P}, \mathcal{B})$, where $\mathcal{P}$ is a set and $\mathcal{B}$ is a collection of subsets of $\mathcal{P}$, each element of $\mathcal{B}$ being of size greater than one. The incidence between elements of $\mathcal{P}$ (called points) and elements of $\mathcal{B}$ (called blocks) is defined by inclusion. Two points $p, q$ are adjacent (notation $p \perp q$) if they lie in a common block. The set of points adjacent to $p$ is denoted by $p^\perp$. The point graph of $\Gamma$ is a graph with vertex set $\mathcal{P}$, and adjacency as in $\Gamma$. The residue $\Gamma_X$ of a subset $X$ of points is the incidence system of points in $\mathcal{P} - X$ adjacent to each point of $X$, and blocks containing $X$ with $X$ itself removed. We say that $\Gamma$ is connected if its point graph is connected, and that $\Gamma$ is triangular if each triple of pairwise adjacent points lies in a block. In the latter case $\Gamma$ may be reconstructed from its point graph by taking the maximal cliques of $\Gamma$ as the blocks.

Let $X \subseteq \mathcal{P}$. The subsystem of $\Gamma$ induced by $X$ is the incidence system with the set of points $X$ and the set of blocks $\{ B \cap X \mid B \in \mathcal{B}, |B \cap X| \geq 2 \}$. An incidence system having at most one block on any pair of points is called a partial linear space. Blocks of such systems are usually called lines and the adjacency is referred to as the collinearity.

A generalized quadrangle is a partial linear space $\Gamma$ such that for any non-incident point–line pair $(p, B)$ there is exactly one point on $B$ collinear to $p$. If all of the lines are of size $s + 1$, and if there are exactly $t + 1$ lines on any point, $\Gamma$ is said to have order $(s, t)$. (For short, we say that $\Gamma$ is a GQ $(s, t)$.) A standard reference on the subject of GQs is Payne and Thas [27].

A hyperoval in a partial linear space is a subset of points meeting each line in 0 or 2 points. (In [4] such subsets are called local subspaces.) Note that the point graph of the subsystem induced by a hyperoval of a GQ is triangle-free (of valence $t + 1$, if the GQ has $t + 1$ lines on every point).

An extended generalized quadrangle (of order $(s, t)$) is a connected incidence system $\Gamma$ such that all its point residues are generalized quadrangles (of order $(s, t)$). For short, $\Gamma$ is an EGQ (respectively, EGQ$(s, t)$). Observe that the blocks of EGQ$(s, t)$ are of size $s + 2$.

Let $\Gamma$ be an EGQ. The following lemma is well-known; see, for example [6].

\textbf{Lemma 2.1.} Let $(p, B)$ be a non-incident point–block pair of $\Gamma$, and let $\mathcal{B}(p, B)$ be the set of blocks on $p$ intersecting $B$ in more than one point. If $p^\perp \cap B \neq \emptyset$, then $\mathcal{B}(p, B) \neq \emptyset$. Also, $|X \cap B| = 2$ for any $X \in \mathcal{B}(p, B)$. Moreover, $X \cap Y \cap B = \emptyset$ for any $X \neq Y \in \mathcal{B}(p, B)$.

\textbf{Proof.} Let $z \in p^\perp \cap B$. The first assertion holds, since $\Gamma_z$ is a GQ. Indeed, there exists a block $X$ on $p$ and $z$ intersecting $B$ in a point not equal to $z$. Hence $X \in \mathcal{B}(p, B)$. Next, for the same reason, $|X \cap B| = 2$. Finally, let $z \in X \cap Y \cap B$ for $X \neq Y \in \mathcal{B}(p, B)$. Then $B$ contains two different points collinear to $p$ in a generalized quadrangle $\Gamma_z$, a nonsense. \hfill $\square$

The following will be used to show that EGQ$(3, 9)$ is triangular.

\textbf{Lemma 2.2.} Let $\Gamma$ be an EGQ$(3, t)$, let $p$ and $q$ be a pair of adjacent points of $\Gamma$, and let

$$\mathcal{F} = p^\perp - \bigcup_{(p, q) \subseteq B} B.$$

Then $q^\perp \cap \mathcal{F}$ is a hyperoval in the partial linear space $\Lambda$ induced by $\Gamma_p$ on $\mathcal{F}$.
Proof. Observe that \( F \) is the set of points at distance 2 from the point \( q \) in \( \Gamma_p \). In particular, \( \Lambda \) is a partial linear space with line size 3. Let \( q \perp x \in L \in \mathcal{R}(\Lambda) \). By Lemma 2.1, \( |q^+ \cap L| = 2 \). The result now follows.

Lemma 2.3 [4]. Let \( \Gamma \) be a triangular EGQ, \( p, q \in \mathcal{P}, p \not\perp q \). Then \( p^+ \cap q^+ \) is a hyperoval in \( \Gamma_p \).

In view of Lemmas 2.2 and 2.3, it is important to know the hyperovals in \( \Gamma_p \) for each \( p \in \mathcal{P} \). In the next section we present the complete list of the hyperovals in \( GQ(3, 9) \) (note that \( GQ(3, 9) \) is unique up to isomorphism, cf. [27]).

3. Hyperovals in \( GQ(3, 9) \)

It appeared to be necessary to use a computer to find hyperovals in this case. The strategy of the search is the same as in the case of \( GQ(3, 3) \) (see [25]).

Denote \( D = GQ(3, 9) \) and \( G = \text{Aut}(D) \). As we observed in the previous section, every hyperoval \( \Omega \) of \( \Delta \) induces a triangle-free subgraph of valence 10 on the collinearity graph of \( \Delta \). The results of the computer search are as follows. The set of representatives of the orbits of \( G \) on the set of hyperovals consists of the three graphs \( \Omega^1, \Omega^2 \) and \( \Omega^3 \) described below. The lengths of the \( G \)-orbits are 648, 1134 and \( >4000 \), respectively.

1. \( \Omega^1 \) is the Gewirtz graph (see [14, 2]); that is, the unique triangle-free strongly regular graph (of valence 10) on 56 vertices.
2. \( \Omega^2 \) is the incidence graph of the unique symmetric 2-(16, 10, 6) block design admitting an automorphism group of shape \( 2^4 : A_6 \). It is a 32-vertex bipartite distance regular graph of diameter 3.
3. \( \Omega^3 \) is a vertex-, but not edge-transitive graph on 40 vertices: \( |\text{Aut}(\Omega^3)| = 5670 \).

Note that if \( \Omega \subset \Delta \) is a hyperoval, \( v \in \Delta - \Omega \), then the subgraph induced on \( v^+ \cap \Omega \) is isomorphic to \( nK_2 \), \( 0 \leq n \leq 10 \). We describe certain further properties of the hyperovals, also obtained using a computer, in the following lemmas.

Lemma 3.1. Let \( \Omega^j \) be as above, \( j = 1, 2, 3 \).
(i) Let \( \Omega^j \subset \Delta \), \( z \in \Delta - \Omega^j \) and \( \Phi = z^+ \cap \Omega^j \), for \( j = 1, 2, 3 \). Then \( \Phi \neq \emptyset \). Moreover, \( \Phi = 10K_2 \) for \( j = 1 \) and, respectively, \( 4K_2 \) for \( j = 2 \).
(ii) Let \( \Omega^3 \subset \Delta \). There are exactly 56 hyperovals of \( \Delta \) intersecting \( \Omega^3 \) in \( 10K_2 \). All of these 56 hyperovals are isomorphic to \( \Omega^1 \). Let \( \Omega^2 \subset \Delta \). There are no hyperovals of \( \Delta \) intersecting \( \Omega^2 \) in \( 4K_2 \).

Lemma 3.2. Let \( \Xi \) be the graph defined on the set of \( G \)-images of \( \Omega^1 \) such that two vertices \( X, Y \) are adjacent iff \( X \cap Y = 10K_2 \). Then \( \Xi \) has four connected components of size 162 and valence 56 each.

Now we turn to the proof of Theorem 1.1.

4. Final Part of the Proof

Lemma 4.1. Let \( \Gamma \) be a EGQ(3, 9). Then \( \Gamma \) is triangular.

Proof. Assume that \( \Gamma \) is not triangular. This means that there exists a triple \( p, q, x \) of pairwise adjacent points not lying on a block.
By Lemma 2.2, $\Delta = \Gamma_p$ must admit a hyperoval $\Omega$ such that $x \in \Omega$ and any of the points of $\Omega$ is not collinear to $q$ in $\Delta$. However, by Lemma 3.1(i), such an $\Omega$ does not exist. This is a contradiction. 

Thus we can reconstruct $\Gamma$ from its point graph. In what follows we abuse the notation and identify these two objects. Let $u$ and $v$ be two points of $\Gamma$ at distance 2. According to Lemma 2.3, $u^+ \cap v^+$ is a hyperoval of $\Gamma_u$. Also, by Lemma 3.1(i), the diameter of $\Gamma$ is 2.

We consider the possibilities for the hyperovals to appear as $u^+ \cap v^+$ and $u^+ \cap w^+$ for $w \in v^+ + u^+$. It turns out that the only remaining possibility automatically leads to the known example.

Observe that $u^+ \cap x^+ = u^+ \cap y^+$ implies $x = y$ for any two points $x$, $y$ at distance 2 from $u$ in $\Gamma$. Indeed, otherwise $u^+ \cap x^+ = u^+ \cap y^+$ for some $z \in u^+ \cap x^+$, which is well-known to be impossible in $\Gamma_z = \mathrm{GQ}(3, 9)$.

Suppose that $u^+ \cap v^+ = \Omega^1$. The above observation, combined with Lemma 3.1(i) and (ii), yields that $u^+ \cup v^+$ has to be isomorphic to the same subgraph of $\Sigma$. By Lemma 3.2, the connected component containing $v$ of the subgraph of $\Sigma$ induced on the points at distance 2 from $u$ has 162 points. An inspection now shows that these 162 points and the points in $u^+$ are the only points of $\Gamma$ and that $\Gamma = \Sigma$.

Suppose that $u^+ \cap v^+ = \Omega^2$. Then (i) and (ii) of Lemma 3.1 imply a contradiction.

Finally, we may assume that $u^+ \cap x^+ = \Omega^3$ for any $x$ at distance 2 from $u$ in $\Gamma$. Counting in the two ways the edges of $\Gamma$ joining first and the second neighbourhood of $u$, we have that the size of the latter is a non-integer, the final contradiction.

The proof of the theorem is complete.

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DIMITRII V. PASECHNIK*
RIACA/CAN,
419 Kruislaan,
NL-1098 VA Amsterdam,
The Netherlands

* Present address: Department of Mathematics and Informatics, Technical University Eindhoven, PO Box 513, NL-5600 MB Eindhoven, The Netherlands, E-mail: dima@win.tue.nl.