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Geometric Characterization of Graphs from the Suzuki Chain

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Let $\Sigma_0, \dots, \Sigma_6$ be the graphs from the Suzuki chain [12]. We classify connected locally Σ_i graphs Θ_{i+1} for $i = 3, 4, 5$. If $i = 3, 4$ then Θ_{i+1} is isomorphic to Σ_{i+1} , whereas Θ_6 is isomorphic either to Σ_6 or to its 3-fold antipodal cover $3\Sigma_6$.

1. INTRODUCTION AND RESULTS

Recently there has been extensive study of the geometries of classical groups such as point–line systems with a fixed local structure. This approach has led to many beautiful results. We refer the reader to a recent paper of Cohen and Shult [14] for a brief survey.

This paper is an attempt to look at the Suzuki chain sporadic simple groups from the aforementioned point of view.

It is necessary to mention considerable activity in this direction under some additional assumptions. The first idea is to assume some group action on the geometry (usually, a diagram geometry). Such an approach leads to the study of some presentations for sporadic groups, and sometimes to proving the faithfulness of these presentations: for instance, see Ivanov [7], Shpectorov [9] and Soicher [10]. Another direction is via assuming some global property of the geometry, e.g. assuming that the related graph is strongly regular or distance-regular: see the book by Brouwer *et al.* [1], which contains many results, and many references to results of this kind. See also Cameron *et al.* [3]. It is worth mentioning that here we do not assume either any group action or any global properties.

Note that graphs with a constant neighbourhood often arise as collinearity graphs of diagram geometries (see Buekenhout [2]). Neumaier [8] observed that the graphs from a Suzuki chain produce nice flag-transitive diagram geometries for the related groups.

Throughout the paper we consider undirected graphs without loops and multiple edges. Given a graph Γ , let us denote the set of vertices by $V = V\Gamma$, and the set of edges by $E = E\Gamma$. Let $X \subseteq V\Gamma$. We denote by $\langle X \rangle = \langle X \rangle_\Gamma$ the subgraph Ξ of Γ induced by X (i.e. $V\Xi = X$, $E\Xi = \{(u, v) \in E\Gamma \mid u, v \in X\}$). Given two graphs Γ and Δ , the graph $\Gamma \cup \Delta$ (resp. the graph $\Gamma \cap \Delta$) is the graph with the vertex set $V\Gamma \cup V\Delta$ (resp. $V\Gamma \cap V\Delta$) and the edge set $E\Gamma \cup E\Delta$ (resp. $E\Gamma \cap E\Delta$). Given $v \in V\Gamma$, we denote $\Gamma_i(v) = \langle \{x \in V\Gamma \mid x \text{ at distance } i \text{ from } v\} \rangle$, and $\Gamma_1(v) = \Gamma(v)$. Furthermore, $\Gamma(X) = \bigcap_{x \in X} \Gamma(x)$. To simplify the notation we use $\Gamma(v_1, \dots, v_k)$ instead of $\Gamma(\{v_1, \dots, v_k\})$ and $u \in \Gamma_i(\dots)$ instead of $u \in V\Gamma_i(\dots)$. As usual, $v = v(\Gamma) = |V\Gamma|$, $k = k(\Gamma) = v(\Gamma(x))$ and $\lambda = \lambda(x, y) = \lambda(\Gamma) = v(\Gamma(x, y))$, where $x \in V\Gamma$ and $y \in \Gamma(x)$. Let $y \in \Gamma_2(x)$. We denote $\mu = \mu(x, y) = \mu(\Gamma) = v(\Gamma(x, y))$. Of course, we use k , λ and μ if it makes sense, i.e. if those numbers are independent of the particular choice of the corresponding vertices. If Δ is a (proper) subgraph of Γ we denote this fact as $\Delta \subseteq \Gamma$ (resp. $\Delta \subset \Gamma$).

We denote the complete n -vertex graph by K_n , the complete multipartite graph with m parts of equal size n by $K_{n \times m}$, the circuit of length n by C_n , and the empty graph by \emptyset .

$\text{Aut}(\Gamma)$ denotes the automorphism group of Γ . Our group-theoretic notation is as in [5].

A *strongly regular* graph Γ is a connected regular graph of valency k , such that $\lambda = \lambda(\Gamma) = \lambda(u, v)$ and $\mu = \mu(\Gamma) = \mu(u, w) > 0$ are independent of the choice of $u \in V\Gamma$, $v \in \Gamma(u)$ and $w \in V\Gamma \setminus (\{u\} \cup \Gamma(u))$. Note that $\mu > 0$ implies that the diameter of Γ is less than or equal to two.

Let Γ, Δ be two graphs. We say that Γ is *locally* Δ if $\Gamma(v) \simeq \Delta$ for any $v \in V(\Gamma)$.

Let $\Gamma, \bar{\Gamma}$ be two graphs. We say that Γ is a *cover* of $\bar{\Gamma}$ if there exists a mapping φ from $V\Gamma$ to $V\bar{\Gamma}$ which maps edges to edges and for any $v \in V\Gamma$ the restriction of φ to $\Gamma(v)$ is an isomorphism onto $\bar{\Gamma}(\varphi(v))$. Note that, since the latter assumption in the definition of cover is usually omitted, our definition of cover is not particularly obvious.

Suppose that we have a chain of graphs $\Sigma_1, \dots, \Sigma_n$, such that Σ_i is locally Σ_{i-1} , $i = 2, \dots, n$. Then, for any complete k -vertex subgraph Y of Σ_m , $1 < k < m \leq n$, we have $\Sigma_m(VY) \simeq \Sigma_{m-k}$.

The *Suzuki chain* [12] consists of the following graphs $\Sigma_1, \dots, \Sigma_6$.

Σ_1 is the empty four-vertex graph.

Σ_2 is the incidence graph of the (unique) 2 – $(7, 4, 2)$ design.

$\Sigma_3, \dots, \Sigma_6$ are strongly regular with parameters (v, k, λ, μ) equal to $(36, 14, 4, 6)$, $(100, 36, 14, 12)$, $(416, 100, 36, 20)$ and $(1782, 416, 100, 96)$ respectively. The automorphism groups of Σ_i are $S_4, PGL_2(7), G_2(2), J_2:2, G_2(4):2$ and $Suz:2$ respectively.

Note that $\Sigma_i, i = 2, \dots, 6$ is locally Σ_{i-1} . Besides Σ_6 , there exists another connected locally Σ_5 graph $3\Sigma_6$, that is, the 3-fold antipodal cover of Σ_6 . $3\Sigma_6$ is a distance-transitive graph (see [1]) with the intersection array $\{416, 315, 64, 1; 1, 32, 315, 416\}$. It has been constructed by Soicher [10] (see also [11]).

THEOREM. *Let Θ_{i+1} be a connected locally Σ_i graph, $i = 3, 4, 5$. Then $\Theta_4 \simeq \Sigma_4$, $\Theta_5 \simeq \Sigma_5$ and $\Theta_6 \simeq \Sigma_6$ or $3\Sigma_6$.*

Note that under the additional assumption that $\text{Aut}(\Theta_{i+1})$ is transitive on the ordered k -cliques of Θ_{i+1} for any $k \leq i$, this statement is proved in [10] using a coset enumeration.

REMARK. Results from [10] and our Theorem immediately lead to a computer-free proof of the faithfulness of the well-known presentations for groups from a Suzuki chain, given in [5]: namely, if G_n is given by the presentation

$$\left\langle \begin{array}{c} \circ \text{---} \circ \text{---} \dots \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \\ g_n \quad g_{n-1} \quad g_3 \quad g_2 \quad g_1 \quad g_0 \quad a \end{array} \quad \left| \begin{array}{l} (g_1 g_2)^4 = a, \\ (g_0 g_1 g_2 g_3)^8 = 1 \end{array} \right. \right\rangle,$$

then $G_4 \simeq J_2:2$, $G_5 \simeq G_2(4):2$ and $G_6 \simeq 3Suz:2$.

Our results may be also viewed as a characterization of the corresponding diagram geometries, cf. [8], [2].

2. PROOF OF THE THEOREM

In an attempt to simplify the reading of Section 2, we give a sketch of the proof here.

Let $\Gamma = \Theta_i$. Pick $u \in V\Gamma$ and look at $\Gamma(u, v)$ for each $v \in \Gamma_2(u)$. First, we establish the characterization of $\Gamma(u, v)$ in terms of their local structure (Lemma 1(1)). Then we classify the connected components of the possible candidates for $\Gamma(u, v)$ (Proposition 3). Next, we list all the possible groupings of $\Gamma(u, v)$ from the connected components (Lemma 5). This list and Lemma 1(2) allows us to determine the edges between $\Gamma(u)$ and $\Gamma_2(u)$. Using Proposition 2 we express the adjacency in $\Gamma_2(u)$ in terms of

intersections of $\Gamma(u, v)$ -subgraphs (Section 2.3 ($i = 4, 5$), Section 2.4 ($i = 6$); Lemmas 6 and 7 are also used). In cases $i = 4, 5$ we are done by Lemma 4, which states that $\Gamma(u, v)$ are ‘hyperplanes’ of $\Gamma(v)$. This also holds in a subcase (Case $c(\Omega) = 3$) of the case $i = 6$. In the remaining subcase (Case $c(\Omega) = 1$, since Case $c(\Omega) = 2$ immediately leads to a contradiction) of the case $i = 6$ we show that Γ is a (unique) covering of Σ_6 , and hence $\Gamma \cong 3\Sigma_6$.

2.1. Suzuki’s construction

We include the necessary information about the construction of Σ_i , given by Suzuki [12], $i = 3, \dots, 6$ (see also [5]).

Let $\Delta = \Sigma_{i-1}$, $\Gamma = \Sigma_i$ and $H = \text{Aut}(\Delta)$ for $i = 3, \dots, 6$. In fact, Γ can be defined in terms of Δ and H .

Let ∞ be an extra symbol. Let S be the conjugacy class of 2-subgroups $Z = \langle z \rangle$ of H , where z is a 2A-involution of H , $i = 3, 4, 5$. In the remaining case $i = 6$ S is the conjugacy class of 2^2 -subgroups $Z = \langle x, y \rangle$ of H , such that $[H : N_H(Z)] = 1365$ and x, y are 2A-involutions of H . Then $V\Gamma = \{\infty\} \cup V\Delta \cup S$.

$E\Gamma$ is exactly as follows. The vertex ∞ is adjacent to each vertex from $V\Delta$, two vertices from $V\Delta$ are adjacent if they are adjacent as vertices of Δ , a vertex $x \in S$ is adjacent to a vertex $v \in V\Delta$ if a non-trivial element of x , considered as a subgroup of H , fixes v , two vertices $x, y \in S$ are adjacent if x, y , considered as subgroups of H , do not commute, but there exists $z \in S$ commuting with both of them.

2.2. Neighbourhood of two vertices at distance two

We start with a simple general fact. Let Δ be a connected graph satisfying the following property:

(*) For any $u \in V\Delta$ and $v \in V\Delta \setminus (V\Delta(u) \cup \{u\})$:

- (1) the subgraph $\Delta(u, v)$ is isomorphic to some M_Δ , the isomorphism type of which is independent of the particular choice of u, v ;
- (2) for $w \in V\Delta \setminus (V\Delta(u) \cup \{u\})$, $w \neq v$, $\Delta(u, v) \neq \Delta(u, w)$.

LEMMA 1. *Let Γ be locally Δ graph, where Δ satisfies (*). For any $u \in V\Gamma$ and $v \in \Gamma_2(u)$ the following holds:*

- (1) $\Gamma(u, v)$ is locally M_Δ ;
- (2) for $w \in \Gamma_2(u)$, $w \neq v$, we have $\Gamma(u, v) \neq \Gamma(u, w)$.

PROOF. By (1) of (*), we have $\Gamma(u, v, y) \cong M_\Delta$ for any $y \in \Gamma(u, v)$. The first claim is proved. By way of contradiction, let $w \in \Gamma_2(u)$, $w \neq v$, $\Gamma(u, v) = \Gamma(u, w)$. Pick $y \in \Gamma(u, v)$. Looking at the common neighbourhood of u, v and u, w in $\Gamma(y) \cong \Delta$, we obtain a contradiction. \square

Note that $\Delta = \Sigma_i$ ($i = 3, 4, 5$) satisfies (*) above. Indeed, the stabilizer of $u \in V\Delta$ in $\text{Aut}(\Delta)$ acts transitively and primitively on $\Delta_2(u)$. The transitivity implies (1) of (*), while the primitivity implies (2) of (*). Thus Lemma 1 holds for locally Σ_i graphs Γ for any $i = 3, 4, 5$. In particular, for any $u \in V\Gamma$ and $v \in \Gamma_2(u)$ the subgraph $\Gamma(u, v)$ is a locally M_{Σ_i} subgraph of $\Gamma(u) \cong \Sigma_i$ ($i = 3, 4, 5$). We turn to the classification of the locally M_{Σ_i} subgraphs of Σ_i ($i = 3, 4, 5$).

First, we need a technical statement. Note that in fact we will prove slightly more than we will really need. For a graph Γ and $g \in \text{Aut}(\Gamma)$ we denote the subgraph induced by the vertices fixed by g as $\text{Fix}(g) = \text{Fix}_\Gamma(g)$.

PROPOSITION 2. Let g be a 2A-involution in $\text{Aut}(\Sigma_i)$. Then:

if $i = 3$ there exist 6 2A-involutions $h \in \text{Aut}(\Sigma_3)$ with $\text{Fix}(g) \cap \text{Fix}(h) \simeq$	$K_{4 \times 2}$,
24	$K_{2 \times 2}$,
32	K_3 ;
if $i = 4$ there exist 10 2A-involutions $h \in \text{Aut}(\Sigma_4)$ with $\text{Fix}(g) \cap \text{Fix}(h) \simeq$	$K_{4 \times 3}$,
80	$K_{2 \times 3}$,
160	K_4 ,
64	\emptyset ;
if $i = 5$ there exist 20 2A-involutions $h \in \text{Aut}(\Sigma_5)$ with $\text{Fix}(g) \cap \text{Fix}(h) \simeq$	$K_{4 \times 4}$,
320	$K_{2 \times 4}$,
1024	K_5 .

PROOF. We follow the notation of Section 2.1. Denote $\Gamma = \Sigma_i$, $\Omega = \text{Fix}(g)$ and $G = \text{Aut}(\Gamma)$. Pick $\infty \in V\Omega$, and denote $\Delta = \Gamma(\infty)$. It follows from the Suzuki's construction given in Section 2.1 that Ω is locally M_Γ . It should be mentioned that $\Omega_2(\infty) = \{x \in g^{\text{Aut}(\Delta)} \mid (xg)^2 = 1\}$. Let us denote by Ξ_h the subgraph $\Omega \cap \text{Fix}(h)$ for $h \in g^G$.

Case $i = 3$. There exist two distinct subgraphs Ξ of Ω isomorphic to $K_{4 \times 2}$ and containing ∞ . It is easy to check that Ξ determines exactly two $h \in g^G$ such that $\Xi_h = \Xi$. Hence there exist exactly $2 \cdot 2v(\Omega)/v(K_{4 \times 2}) = 6$ involutions $h \in g^G$ such that $\Xi_h \simeq K_{4 \times 2}$.

It is well-known that the action of G on g^G coincides with the action on the set of points of a generalized hexagon \mathbf{H} of order $(2, 2)$, and that the corresponding subdegrees are 1, 6, 24, 32 according to the distance in the collinearity graph of \mathbf{H} . Hence if $h' \in g^G$ corresponds to the subdegree 24 then there exists $h \in g^G$ such that $\Xi_h \simeq \text{Fix}(h) \cap \text{Fix}(h') \simeq K_{4 \times 2}$. Hence $v(\Xi_{h'}) \geq 4$. On the other hand, it is easy to find elements $h', h'' \in g^G$ with $\Xi_{h'} \simeq K_{2 \times 2}$ and $\Xi_{h''} \simeq K_3$. The proof for $i = 3$ is complete.

Case $i = 4$. By induction, there exist 6 elements $h_j \in g^G$ such that $\Xi_{h_j}(\infty) \simeq K_{4 \times 2}$. Pick $y = h_1$ and $u \in V\Xi_y(\infty)$. Since $v(\Xi_y(u)) \geq 1 + k(K_{4 \times 2})$, we have $\Xi_y(u) \simeq K_{4 \times 2}$. Hence a connected component Y of Ξ_y is locally $K_{4 \times 2}$. Thus $Y \simeq K_{4 \times 3}$ (cf. Proposition 1.15 in [1]). It can easily be shown that for any $w \in V\Omega$ the subgraph $\Omega(w) \cap Y$ is non-empty. Hence $\Xi_y = Y$. Thus there exist exactly $6v(\Omega)/v(K_{4 \times 3}) = 10$ involutions $h \in g^G$ such that $\Xi_h \simeq K_{4 \times 3}$.

There exist 24 elements $h'_j \in g^G$ such that $\Xi_{h'_j}(\infty) \simeq K_{2 \times 2}$. Pick $y = h'_1$ and $u \in V\Xi_y(\infty)$. Since $\Xi_y(u)$ contains a 2-path, we have $\Xi_y(u) \simeq K_{2 \times 2}$ or $K_{4 \times 2}$. The latter case is impossible. Indeed, as we have already proved, if the latter case holds, we have $\Xi_y \simeq K_{3 \times 4}$. Thus $\Xi_y(\infty) \simeq K_{2 \times 4}$, a contradiction. Hence a connected component Y of Ξ_y is locally $K_{2 \times 2}$. Thus $Y \simeq K_{2 \times 3}$. Therefore a connected component of Ξ_y distinct from Y (if such a component exists at all) is a singleton $s \in V\Xi_y$. Looking at the neighbourhood of s , we obtain a contradiction, since $\Xi_y(s) = \emptyset$. Hence $\Xi_y = Y$. Thus there exist exactly $24v(\Omega)/v(K_{2 \times 3}) = 80$ elements $h' \in g^G$ such that $\Xi_{h'} \simeq K_{2 \times 3}$.

There exist 32 elements $h''_j \in g^G$ such that $\Xi_{h''_j}(\infty) \simeq K_3$. Pick $y = h''_1$ and $u \in V\Xi_y(\infty)$. Similarly to the previous case, we obtain that a connected component Y of Ξ_y is locally K_3 . Thus $Y \simeq K_4$. Again, looking at Ξ_y as at a subgraph of Ω we obtain $\Xi_y = Y$. Thus there exist exactly $32v(\Omega)/v(K_4) = 160$ elements $h'' \in g^G$ such that $\Xi_{h''} \simeq K_4$.

Clearly, for the other $h \in g^G$ we have $\Xi_h = \emptyset$. The proof in the case $i = 4$ is complete.

Case $i = 5$. The argument runs parallel to the argument for the case $i = 4$. □

PROPOSITION 3. Any connected locally M_{Σ_i} subgraph of Σ_i coincides with the subgraph $\text{Fix}(g)$ for a 2A-involution $g \in \text{Aut}(\Sigma_i)$ ($i = 3, 4, 5$).

PROOF. Denote $\Gamma = \Sigma_i$. Let Ω be a connected locally M_Γ subgraph of Γ , $i = 3, 4, 5$. We will frequently use the construction of Γ given in Section 2.1, as well as the corresponding notation. Without loss of generality, $\infty \in V\Omega$.

Case $i = 3$. Recall that Δ is the incidence graph of the unique $2-(7, 4, 2)$ design. Since the latter design is complementary to $\Pi = PG(2, 2)$, the equivalent description of Δ (and Γ) can be given in terms of Π : namely, the vertices of Δ are the points and the lines of Π , and a point p is adjacent to a line l if p does not belong to l .

For each $2A$ -involution v of H there exists a unique flag (= incident point–line pair) (p, l) such that v leaves all points on l and all lines through p fixed. Thus it is easy to check that the subgraph $M_\Gamma = \Gamma(\infty, v)$ is isomorphic to $K_1 \cup K_1 \cup C_4$.

It is easy to check that any (generated) subgraph of Δ of the latter shape is $\Gamma(\infty, v)$ for some $v \in \Gamma_2(\infty)$. Indeed, Δ has the remarkable property that any 2-path in Δ lies in a unique C_4 (i.e. Δ is a *rectagraph* (see [1])). Moreover, $\text{Aut}(\Delta)$ acts transitively on the C_4 -subgraphs of Δ . Finally, it is straightforward to check that there exist exactly two vertices of Δ which are non-adjacent to any vertex of a subgraph Ξ of Δ that is isomorphic to C_4 .

Thus, without loss of generality, we may assume that $\Gamma(\infty, v) \subset \Omega$ for some $v \in \Gamma_2(\infty)$. Let Ξ be the C_4 -subgraph of $\Omega(\infty)$, $u \in V\Xi$. Since $\Gamma(u) \cong \Delta$ is a rectagraph, a quick look at $\Xi(u)$ gives us $v \in V\Omega$. Next, v is adjacent to each vertex in $\Omega(\infty)$. Hence any $w \in \Gamma_2(\infty) \cap \Gamma(v)$ does not belong to $V\Omega$.

Let $x \in \Omega(\infty) \setminus V\Xi$. The unique C_4 -subgraph of $\Omega(x)$ lies in $\Gamma_2(\infty)$. On the other hand, $\Gamma(x, y) \subset \Gamma_2(\infty)$. Hence for $Q = \Gamma_2(\infty) \cap \Gamma_2(v) \cap \Gamma(x)$ we have $v(Q) = 4$. Thus $Q \subset \Omega$.

Consider v as a $2A$ -involution of H . It is easy to see, using [5], that v is a $2A$ -involution of $\text{Aut}(\Gamma)$. Therefore its fixed vertices are $\{\infty\} \cup \{v\} \cup V\Gamma(\infty, v) \cup VQ \subseteq V\Omega$. Since the subgraph $\text{Fix}(v)$ induced by the vertices fixed by v is connected and locally M_Γ , we have equality in the latter inclusion. This completes the proof in the case $i = 3$.

Case $i = 4$. By induction, any subgraph of Σ_3 (isomorphic to $\Gamma(\infty, v)$ for some $v \in \Gamma_2(\infty)$) is in fact $\text{Fix}(g)$ for some $2A$ -involution $g \in H$. Hence, without loss of generality, we may assume that $\Gamma(\infty, v) \subset \Omega$.

Let $u \in \Omega(\infty)$, $v \in \Omega_2(\infty) \cap \Omega(u)$. Remembering the structure of $\Omega(u)$, we have $v(\Omega(\infty, u, v)) \geq 4$. Hence, by Proposition 2, we have $v(\Omega(\infty, v)) \geq 8$. So $v(\Omega_2(\infty)) \leq 7$. Let Q be the set of all possible candidates to $V\Omega_2(\infty)$. Since $\lambda(\Omega) = 6$, there are precisely 60 edges of $E\Omega$ going from $V\Omega(\infty)$ to $V\Omega_2(\infty)$. On the other hand, precisely 60 edges are going from Q to $V\Omega(\infty)$. Hence $Q = V\Omega_2(\infty)$. In particular, a $2A$ -involution $w \in H$ belongs to Q . Next, w is a $2A$ -involution of $\text{Aut}(\Gamma)$. Therefore $\Omega = \text{Fix}(w)$.

Case $i = 5$. The argument runs parallel to the argument in the previous cases. \square

We already know the connected components of locally M_Γ subgraphs Ω . Let us study the relationship between Ω and the vertices outside $V\Omega$.

LEMMA 4. Let Ω be connected locally M_Γ , $v \in V\Gamma \setminus V\Omega$, $\Gamma(v) \cap \Omega \neq \emptyset$, where $\Gamma = \Sigma_i$ ($i = 3, 4, 5$). Then $\Gamma(v) \cap \Omega \cong K_{2 \times (i-1)}$. If $i = 3, 4$ then for any $v \in V\Gamma \setminus V\Omega$ the subgraph $\Gamma(v) \cap \Omega \neq \emptyset$. In the remaining case $i = 5$ for any $v \in V\Gamma \setminus V\Omega$ either the subgraph $\Gamma(v) \cap \Omega$ is non-empty or v lies in a uniquely determined connected locally M_Γ subgraph.

PROOF. If $i = 3, 4$ then the centralizer of a $2A$ -involution in $\text{Aut}(\Gamma)$ has two orbits on $V\Gamma$ (look at the scalar product of permutation characters, given in [5]). Hence the

isomorphism type of $\Xi = \Xi_i = \Gamma(v) \cap \Omega$ is independent on the choice of $v \in V\Gamma \setminus V\Omega$. Next,

$$v(\Xi)(v(\Gamma) - v(\Omega)) = v(\Omega)(k(\Gamma) - k(\Omega)).$$

Thus $v(\Xi) = 4$ or 6 , according to $i = 3$ or 4 .

If $i = 3$ it is easy to check that $k(\Xi) = 2$. Indeed, pick a vertex $u \in V\Xi$. Look at $\Delta = \Gamma(u)$. Recall that $V\Omega(u)$ is the set of fixed points of a 2A-involution $g \in \text{Aut}(\Delta)$. $C_{\text{Aut}(\Delta)}(g)$ has one orbit on $V\Delta \setminus V\Omega(u)$. Performing the same calculation as above, we have $k(\Xi) = v(\Omega(u) \cap \Delta(v)) = 2$. The proof in the case $i = 3$ is complete.

By induction, if $i = 4$ then Ξ is locally $C_4 = K_{2 \times 2}$. Such a graph Ξ is unique and isomorphic to $K_{2 \times 3}$.

Consider Σ_5 as the subgraph $\Delta = \Gamma(\infty)$ of $\Gamma = \Sigma_6$, as in Section 2.1. For a vertex $u \in \Gamma_2(\infty)$ the group $H = N_{\text{Aut}(\Delta)}(u)$ (recall that u is a 2^2 -group of $\text{Aut}(\Delta)$) has two orbits on Δ . By Section 2.1 and Proposition 3, $\Gamma(\infty, u)$ is the disjoint union of three copies of Ω . This enables us to calculate, for $v \in V\Gamma(\infty) \setminus V\Gamma(u)$, the number n of edges coming from v to $\Gamma(\infty, u)$: namely, $n = 24$. There exists $g \in H(v)$ of order 3 permuting cyclically the connected components of $\Gamma(\infty, u)$. Hence $v(\Xi) = n/3 = 8$. On the other hand, Ξ is locally $K_{2 \times 3}$. Such a graph is isomorphic to $K_{2 \times 4}$.

Finally, the last statement of the lemma follows directly from the aforementioned consideration of Σ_5 as a subgraph of Σ_6 . \square

Lemma 4 immediately implies the following:

LEMMA 5. *Let Ω be locally M_{Σ_i} ($i = 3, 4, 5$). If $i = 3, 4$ then Ω is connected. If $i = 5$ then there exists a unique equivalence relation φ with the class size three on the set of connected locally M_{Σ_i} subgraphs such that all connected components of Ω lie in one class of φ .* \square

2.3. The final part of proof for $i = 3$ and 4

Let $\Delta = \Sigma_i$, Γ be a connected locally Δ graph. Let us identify Δ with $\Gamma(\infty)$ for $\infty \in V\Gamma$. Since Δ satisfies (*), by the classification of the locally M_Δ subgraphs of Δ given in Section 2.2, we have that $\mu(\Gamma)$ equals the number of the fixed points of some 2A-involution $g \in H = \text{Aut}(\Delta)$. Since

$$v(\Gamma_2(\infty)) = k(\Gamma) \cdot (k(\Gamma) - k(\Delta) - 1) / \mu(\Gamma),$$

$v(\Gamma_2(\infty)) = |g^H|$. Since Δ satisfies (*), by Lemma 1 for a locally M_Δ subgraph Ω of Δ there exists a unique $v = v_\Omega \in \Gamma_2(\infty)$ such that $\Omega = \Gamma(\infty, v)$ and for two distinct locally M_Δ subgraphs Ω, Ω' we have $v_\Omega \neq v_{\Omega'}$. Thus we have proved that we do not have any choice determining the edges from $\Gamma(\infty)$ to $\Gamma_2(\infty)$.

Let $v \in \Gamma_2(\infty)$, $u \in \Gamma(v) \cap \Gamma(\infty)$. It follows that Lemma 4, applied to $\Gamma(v) \simeq \Delta$, that $\Gamma(\infty, v, u) \simeq K_{2 \times (i-1)}$ (in particular, Γ has diameter 2). Next, $v(\Gamma_2(\infty) \cap \Gamma(v)) = 24$ if $i = 3$ (80 if $i = 4$). On the other hand, by Proposition 2 if $i = 3$ there exist exactly 24 (80 if $i = 4$) locally M_Δ subgraphs of Δ intersecting $\Gamma(\infty, v)$ in a subgraph of the shape $K_{2 \times (i-1)}$. Thus the adjacency in $\Gamma_2(\infty)$ is also uniquely determined.

The proof of Theorem is complete for $i = 3, 4$.

2.4. The final part of proof for $i = 5$

Let $\Delta = \Sigma_5$, Γ be a connected locally Δ graph. First we prove further lemmas about locally M_Δ subgraphs of Δ . Denote $G = \text{Aut}(\Delta)$. Let Ω be a connected locally M_Δ

subgraph of Δ . Define the graph Λ with $V\Lambda = \Omega^G$, $E\Lambda = \{(\Omega', \Omega'') \mid \Omega', \Omega'' \in \Omega^G, \Omega' \cap \Omega'' \simeq K_{2 \times 4}\}$.

LEMMA 6. *The graph Λ is a connected graph of valency $k(\Lambda) = 320$, on which G acts vertex- and edge-transitively. For each $v \in V\Lambda$ and $u \in \Lambda_2(v)$ we have $k(\Lambda(v, u)) \leq 32$.*

PROOF. The vertex-transitivity is clear. $k(\Lambda) = 320$ follows immediately from Proposition 2. Let $v \in V\Lambda$. Consider the action of the stabilizer F of Ω in $G(v)$ on the set of 80 locally $M_{\Delta(\infty)}$ subgraphs $\Xi_j(v)$ of $\Delta(v)$ such that $\Omega(v) \cap \Xi_j(v) \simeq K_{2 \times 3}$ (cf. the case $i = 4$ of Proposition 2). This action is transitive, since $\Xi_j(v)$ ($j = 1, \dots, 80$) constitute a suborbit of $J_2: 2$ in its primitive action of degree 315. By Proposition 3 each $\Xi_j(v)$ lifts to a unique locally M_Δ subgraph Ξ_j of Δ . By Proposition 2 we have $\Xi_j \cap \Omega \simeq K_{2 \times 4}$. Hence F acts transitively on the Ξ_j ($j = 1, \dots, 80$). Since the stabilizer of Ω in G acts transitively on $V\Omega$, the edge-transitivity is proved.

The unique imprimitivity system of G on $V\Lambda$ is given by the equivalence relation φ defined in Lemma 5 (cf. [5]). Hence Λ is connected.

The last statement follows from the fact that Λ appears as the subgraph $\Gamma_2(\infty)$ of the graph $\Gamma \simeq 3\Sigma_6$ and that $\mu(\Gamma) = 32$. \square

Let us denote by $\bar{\Omega}$ the φ -equivalence class of Ω . Define the graph $\bar{\Lambda}$ by $V\bar{\Lambda} = \bar{\Omega}^G$ and $E\bar{\Lambda} = \{(\bar{\Omega}', \bar{\Omega}'') \mid \bar{\Omega}', \bar{\Omega}'' \in \bar{\Omega}^G, \bar{\Omega}' \cap \bar{\Omega}'' \simeq K_{2 \times 4}\}$.

LEMMA 7. *For each $(\bar{\Omega}', \bar{\Omega}'') \in E\bar{\Lambda}$, we have $\bar{\Omega}' \cap \bar{\Omega}'' \simeq K_{2 \times 4} \cup K_{2 \times 4} \cup K_{2 \times 4}$. The graph $\bar{\Lambda}$ is a connected graph of valency $k(\bar{\Lambda}) = 320$.*

PROOF. Let S be the set of $\Xi \in \Omega^G$ satisfying $\Xi \cap Y \simeq K_{2 \times 4}$ for some Y φ -equivalent to Ω . Since Ω does not contain a subgraph $K_{2 \times 4} \cup K_{2 \times 4}$, there is a unique such subgraph Y for each Ξ of S . Hence the stabilizer F of Ω in G acts on S transitively. The subdegrees of G on $\bar{\Omega}^G$ are 1, 20, 320, 1024. Since $|S| = 960$, $S = \bigcup_{\Xi \in S} \bar{\Xi}$. Hence we have proved the first two claims of the lemma. The last one follows from the primitivity of G on $\bar{\Omega}^G$. \square

Let us identify Δ with $\Gamma(\infty)$ for $\infty \in V\Gamma$. Choose a vertex $v \in \Gamma_2(\infty)$. Denote $\Omega = \Gamma(\infty, v)$. According to the results of Section 2.2, we know the connected components of Ω up to conjugacy. By Lemma 5 we have several possibilities for number of the components $c(\Omega)$ of Ω .

Case $c(\Omega) = 3$. We have $\Omega = \bar{\Theta}$ for a connected locally M_Δ subgraph Θ of Δ . It follows from Lemmas 4 and 5 that the subgraph $\Xi(v) = \langle V\Gamma(v) \setminus V\Omega \rangle$ lies in $\Gamma_2(\infty)$. Clearly, $v(\Xi(v)) = v(\Delta) - v(\Omega) = 320$. Let $u \in \Xi(v)$. Then, by Lemma 4, $\Gamma(\infty, v, u) \simeq K_{2 \times 4} \cup K_{2 \times 4} \cup K_{2 \times 4}$, where the connected components of this subgraph lie in distinct connected components of Ω . Hence $\Gamma(\infty, u) = \bar{\Theta}'$ for a connected locally M_Δ subgraph Θ' of Δ . By Lemma 7 a connected component Ξ of $\Gamma_2(\infty)$ containing v is isomorphic to $\bar{\Lambda}$. Counting in two ways the number of edges from $\Gamma(\infty)$ to $\Gamma_2(\infty)$, we obtain $\Gamma_2(\infty) = \Xi$. Thus $\Gamma \simeq \Sigma_6$ and the proof in this case is complete.

Case $c(\Omega) = 1$. Consider $u \in \Gamma(v) \cap \Gamma_2(\infty)$. By Lemma 4, $\Gamma(\infty, v, u) = K_{2 \times 4}$. Therefore $\Gamma(\infty, u)$ is connected. Look at the connected component Y of $\Gamma_2(\infty)$ containing v . By Lemmas 6 and 4, we have $k(Y) \geq 320$, and Y contains Λ as a (possibly non-generated) subgraph. Calculating in two ways the number of edges coming from $\Gamma(\infty)$ to $\Gamma_2(\infty)$, we obtain that $\Gamma_2(\infty) = Y$.

Let us prove that $Y = \Lambda$, i.e. if $(u, v) \in EY$ then $\Gamma(\infty, u, v) \cong K_{2 \times 4}$. Suppose the contrary for some $(u, v) \in EY$. We know that $\Gamma(v, u) \cong \Sigma_4$. Let us determine to which layers of Γ with respect to ∞ the vertices of $\Gamma(v, u)$ do belong. By Lemma 4, $\Gamma(\infty, u, v) = \emptyset$. Next, u lies in a uniquely determined locally M_Δ subgraph Ω' of $\Gamma(v)$: namely, $\Omega' \in \bar{\Omega}$ (cf. Lemma 5). Dually, v lies in such a subgraph of $\Gamma(u)$. Therefore the number of vertices of $\Gamma(v, u)$ belonging to $\Gamma_2(\infty)$ cannot exceed $k(\Omega') = 20$. Thus, $\Gamma(v, u)$ has at least 80 vertices at the distance 2 from ∞ . Let w be a vertex of this set. It is easy to check that $\Gamma(\infty, v, w) \cong \Gamma(\infty, u, w) \cong K_{2 \times 4}$. One can consider (v, w, u) as a 2-path in Λ . There are at least 80 such paths from v to u , which contradicts the last part of Lemma 6. Thus we are done.

Let $w \in \Gamma_3(\infty) \cap \Gamma(v)$. Denote $\Theta = \Gamma(w)$.

LEMMA 8. *The subgraph $\Xi = \Gamma_2(\infty) \cap \Theta$ equals $\Theta_2(t)$ for some $t \in V(\Theta)$.*

PROOF. From the above we know that $w \in VA$ for a connected locally M_Δ subgraph $A \subset \Gamma_3(\infty) \cap \Gamma(v)$. Thus $k(\Xi) = v(\Theta(v)) - k(A) = 80$. By Proposition 3, for any $u \in V\Xi$ the subgraph $\bar{\Xi}(u) = \Theta(u) \cap \Gamma_3(\infty)$ equals $\Theta(u, t^u)$ for some $t^u \in \Theta_2(u)$. This faces us with the problem of classifying the subgraphs of Θ that satisfy the local property above.

Denote $t = t^v$. It suffices to show that $\Pi^u = \bar{\Xi}(u) \cap \Theta(u, t) \setminus \bar{\Xi}(v)$ is non-empty for any $u \in \Xi(v)$. Indeed, since the pointwise stabilizer of v and t in $\text{Aut}(\Theta)$ acts transitively on $\Theta(t) \setminus \bar{\Xi}(v)$, the sets Π^u ($u \in \Xi(v)$) cover it, i.e. $\Theta(t) \cap V\Xi = \emptyset$.

Pick $u \in \Xi(v)$. Denote $\Pi = \Pi^u$. By Lemma 4, $\Phi = \Theta(v, t, u) = \Theta(v) \cap \bar{\Xi}(u)$ is isomorphic to $K_{2 \times 3}$. Pick $(x, y) \in E\Phi$. Consider $\Psi = \Theta(u, x, y) \cong \Sigma_2$. The subgraph $\Phi(x, y)$ of Ψ consists of two non-adjacent vertices a and b . We are interesting in ways of completing $\{a, b\}$ to the subgraph of fixed points of a $2A$ -involution of $\text{Aut}(\Psi)$ (one such way gives $\Theta(u, t, x, y)$). Since avb is a 2-path in Ψ , the involutions fixing both of a and b commute. Therefore if t' gives us some $\Theta(u, t', x, y)$ -subgraph that we are interested in, $\Theta(u, t, x, y) \cap \Theta(u, t', x, y)$ contains $\{a, b\}$ properly. Since $\Pi \subseteq \Theta(u, t, x, y) \cap \Theta(u, t', x, y) \setminus \{a, b\}$, we are done. \square

Next, let $w' \in \Theta(t)$. By Lemma 8, $\Gamma(w') \cap \Gamma_2(\infty) = \Gamma(w') \cap \Gamma_2(t')$ for some $t' \in \Gamma(w')$. Suppose that $t \neq t'$. Since $t' \in V\Theta$, we then have $\Theta(w, t') \cap \Theta_2(t) \neq \emptyset$, a contradiction. Thus $t = t'$.

Thus $\Gamma(t) \subseteq \Gamma_3(\infty)$. Therefore $t \in \Gamma_4(\infty)$. By standard calculations, we can verify that Γ is distance-regular, with the same intersection array as $3\Sigma_6$.

Note that we have shown that any non-trivial connected cover of Σ_6 is distance-regular with the same intersection array as $3\Sigma_6$. Therefore any (in our sense, see Section 1) non-trivial connected cover of Σ_6 is isomorphic to $3\Sigma_6$. Hence, in order to complete the proof, it suffices to show that Γ is a cover of Σ_6 .

Since $\Gamma_3(\infty) \cong \Sigma_5 \cup \Sigma_5$, to be at the distance 4 in Γ is an equivalent relation on $V\Gamma$. Let $t \in \Gamma_4(\infty)$. Then for any $x \in \Gamma(\infty)$ there exists a unique $y \in \Gamma(t)$ such that $x \in \Gamma_4(y)$. Hence our relation is well-defined on $E\Gamma$. Thus we may define the quotient graph $\bar{\Gamma}$, the vertices (resp. edges) of which are equivalence classes of vertices (resp. edges) of Γ . Clearly, for any $v \in V\Gamma$ the restriction of our quotient mapping to $\Gamma(v)$ is an isomorphism. Hence $\bar{\Gamma}$ is locally Σ_5 . Therefore $\bar{\Gamma} \cong \Sigma_6$. The proof in this case is now complete.

Case $c(\Omega) = 2$. It follows from the above that in this case $v(\Gamma(\infty, u)) = 64$ for any $u \in \Gamma_2(\infty)$. Now the standard calculation of the edges coming from $\Gamma(\infty)$ to $\Gamma_2(\infty)$ shows that $v(\Gamma_2(\infty))$ is not an integer, a contradiction.

The proof of the Theorem is complete.

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