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Non-Abelian Representations of Some Sporadic Geometries

Alexander A. Ivanov*

Department of Mathematics, Imperial College 180 Queen’s Gate, London SW7 2BZ, United Kingdom

Dmitrii V. Pasechnik†

RIACA, Department of Mathematics and Informatics, Eindhoven University of Technology, P.O. Box 513, 5600 Eindhoven, The Netherlands

and

Sergey V. Shpectorov‡

Institute for Systems Analysis, 117312, Moscow, Russia

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For a point-line incidence system \( S = (P, L) \) with three points per line we define the universal representation group of \( S \) as

\[
R(S) = \langle z_p, p \in P | z_p^2 - 1 \text{ for } p \in P, z_p z_q z_r - 1 \text{ for } \{p, q, r\} \in L \rangle.
\]

We prove that if \( S \) is the 2-local parabolic geometry of the sporadic simple group \( F_1 \) (the Monster) or \( F_2 \) (the Baby Monster) then \( R(S) \cong F_1 \) or \( 2 \cdot F_2 \), respectively.


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‡ The paper was written while this author held a temporary position at Michigan State University. Currently he is at the Department of Mathematics, The Ohio State University, Columbus, Ohio 43210.
1. INTRODUCTION

We assume that the reader is familiar with basic definitions, results, and notation related to diagram geometries (cf. [Bue, Pas, Tit2]). Suppose we are given a geometry $\mathcal{G}$, such that one of the types occurring in $\mathcal{G}$ is called “points” and some other type is called “lines”. Suppose also that every line in $\mathcal{G}$ is incident to exactly three points. In such a situation it is usual to look for representations (another name is “embeddings”) of $\mathcal{G}$. A representation of $\mathcal{G}$ is a mapping $\phi$ from the point set of $\mathcal{G}$ into the point set of a projective space $\mathcal{P}$ over $GF(2)$ (so that in $\mathcal{P}$ every line is also incident to exactly three points), such that for every triple of points incident to a line, their images under $\phi$ form the point set of a line from $\mathcal{P}$. Let $V$ denote the vector space over $GF(2)$, underlying the projective geometry $\mathcal{P}$. If $p$ is a point of $\mathcal{G}$ then $\phi(p)$ is a one-dimensional subspace in $V$, that is, $\phi(p) = \langle v_p \rangle$ for a uniquely determined non-zero vector $v_p \in V$. In these terms the above condition “lines map onto lines” can be equivalently expressed as follows: whenever $\{p, q, r\}$ is the point set of a line of $\mathcal{G}$, we have $v_p + v_q + v_r = 0$. Let $\mathcal{S} = (P, L)$ be the point-line incidence system associated with $\mathcal{G}$. We will identify a line with the set of points it is incident to. Let $V(\mathcal{S})$ (or $V(\mathcal{G})$) be the vector space over $GF(2)$ given by the following presentation:

$$V(\mathcal{S}) = \langle v_p, p \in P | v_p + v_q + v_r = 0, \text{ for } \{p, q, r\} \in L \rangle.$$ 

If no $v_p$ vanishes in $V(\mathcal{S})$ then the mapping $p \mapsto \langle v_p \rangle$ defines a universal representation of $\mathcal{G}$, from which any other representation can be obtained by a linear mapping. If non-trivial, the space $V(\mathcal{G})$ usually brings about a nice realization of $\mathcal{G}$ and a nice module for $G = Aut \mathcal{G}$.

However, there are some very natural geometries, for which the space $V(\mathcal{G})$ is trivial. Such are the following 2-local geometries of the sporadic simple groups: of the fourth Janko group $J_4$ with the diagram

![Diagram of the Baby Monster group $F_2$ with the diagram](image)

of the Baby Monster group $F_2$ with the diagram

![Diagram of the Monster group $F_1$ with the diagram](image)

(geometries with similar diagrams are called the Petersen type geometries) and of the Monster group $F_1$ with the diagram

![Diagram of the Monster group $F_1$ with the diagram](image)

(geometries with diagrams of this kind are called the tilde type geometries).
In these diagrams the type “points” is depicted by the leftmost node and the type “lines” is depicted by the node next to “points.” Triviality of $V(\mathcal{G})$ for these three geometries was proved in [IS3, IS4]. Nevertheless, these geometries do have “representations.” Suppose $\mathcal{G}$ is one of the geometries and $p$ is a point of $\mathcal{G}$. Then the stabilizer of $p$ in $G = \text{Aut} \mathcal{G}$ has a unique non-trivial central involution $z_p$. Moreover, if $\{p, q, r\}$ is a line then $z_p z_q z_r = 1$. This means that with every point $p$ we can associate a subgroup $\langle z_p \rangle$ of $G$, and then every line naturally defines a Klein four group. Inspired by these examples, we give the following definition. Suppose $\mathcal{S} = (P, L)$ is a point–line incidence system, such that every line is incident to exactly three points. A non-abelian representation of $\mathcal{S}$ is a mapping $p \mapsto \langle z_p \rangle$ from the point set $P$ to the set of subgroups of order 2 of some group $X$ (the word “non-abelian” only stresses that $X$ need not be abelian), such that $\{1, z_p, z_q, z_r\}$ is a Klein four group for every line $\{p, q, r\} \in L$. We will usually identify the subgroup $\langle z_p \rangle$ with its unique non-trivial element $z_p$. The universal representation group of $\mathcal{S}$ is given by the following presentation:

$$R(\mathcal{S}) = \langle z_p, p \in P | z_p^2 = 1 \text{ for } p \in P, \ z_p z_q z_r = 1 \text{ for } \{p, q, r\} \in L \rangle.$$ 

Whenever we have a (non-abelian) representation of $\mathcal{S}$, the group spanned by the images of the points is a quotient of $R(\mathcal{S})$. Note the main difference between the definition of $R(\mathcal{S})$ and the definition of $V(\mathcal{S})$ given earlier. It is implicit in the definition of $V(\mathcal{S})$ that the generating elements $v_p$ commute, whereas the $z_p$ need not commute. Otherwise the definitions are similar to each other and, in fact, under a natural identification we have that

$$V(\mathcal{S}) = R(\mathcal{S})/[R(\mathcal{S}), R(\mathcal{S})].$$

Now we are ready to state the main result of the paper. Let $\mathcal{G}(F_2)$ be the Petersen type geometry belonging to the group $F_2$ and $\mathcal{G}(F_1)$ be the tilde type geometry belonging to $F_1$.

**Theorem 1.** If $\mathcal{G} = \mathcal{G}(F_2)$ or $\mathcal{G}(F_1)$ then $R(\mathcal{G}) \cong 2 \cdot F_2$ or $F_1$, respectively.

It was proved very recently that $J_4$ is the universal representation group of the Petersen type geometry $\mathcal{G}(J_4)$ [IS6].
2. PRELIMINARIES

In Sections 3 and 5 we prove commutativity of the groups $R(\mathcal{G})$ for some low rank P- and T-geometries ("P-", and "T-" are short for "Petersen type" and "tilde type", respectively). The rank 3 P-geometry $\mathcal{G}(M_{22})$ is a point residue in a rank 4 P-geometry $\mathcal{G}(Co_2)$, which is, in its turn, a point residue in $\mathcal{G}(F_2)$. Establishing the commutativity of the groups $R(\mathcal{G}(M_{22}))$ and $R(\mathcal{G}(Co_2))$ constitutes an important step toward the identification of the group $R(\mathcal{G}(F_2))$. For the group $R(\mathcal{G}(F_1))$ we use a different approach and do not need the commutativity of the universal representation groups of the residues $\mathcal{G}(M_{24})$ and $\mathcal{G}(Co_1)$ of $\mathcal{G}(F_1)$. However, as these results can be obtained almost for free, we prove them for the sake of completeness and for possible future references.

First, let us cite a fact, proved in [IS4, Lemma 4.5].

**Lemma 2.1.** Let $\mathcal{G} = \mathcal{G}(Sp_{2n}(2))$ be the $C_n$-geometry of the symplectic group $Sp_{2n}(2)$, $n \geq 2$, then $R(\mathcal{G})$ is abelian.

It is well known [Tit1] that $V(\mathcal{G}(Sp_{2n}(2)))$ (which coincides with $R(\mathcal{G}(Sp_{2n}(2)))$ by the above lemma) has rank $2n + 1$ and is isomorphic to the natural module for $O_{2n+1}(2) = Sp_{2n}(2)$.

Let $\mathcal{P} = (P, L)$ be a point-line incidence system with three points per line, let $R$ be a representation group of $\mathcal{P}$, and let $r_p$ denote the image in $R$ of a point $p \in P$.

Our main tool in Sections 3 and 5 will be the following observation.

**Lemma 2.2.** Suppose that there is a structure of undirected graph $\Sigma$ defined on $P$ such that the following three conditions hold for $A_p$ and $B_p$ denoting the sets of points adjacent or equal and non-adjacent to $p$ in $\Sigma$, respectively.

(a) For every $p \in P$ and $q \in A_p$ we have $[r_p, r_q] = 1$.

(b) For every $p \in P$ the graph on $B_p$ whose edges are defined by the lines $l \in L$ such that $|l \cap B_p| = 2$ is connected.

(c) The complement of $\Sigma$ (that is, the graph in which $p$ is adjacent to all points from $B_p$) is connected.

Then the commutator $R'$ of $R$ is of order at most 2.

**Proof.** Since $R$ is generated by the $r_p$, $R'$ is generated by the elements $z_{pq} = [r_p, r_q]$ for all $p, q \in P$. Since $r_p$ and $r_q$ are involutions, they generate in $R$ a dihedral subgroup of order $2k$ whose characteristic cyclic subgroup $Z_{pq}$ of order $k$ is generated by $z_{pq}$ and is inverted by $r_p$ and $r_q$. By (a) $z_{pq} = 1$ whenever $p$ and $q$ are equal or adjacent in $\Sigma$, so $R'$ is generated by the subgroups $Z_{pq}$ for all $p \in P$ and $q \in B_p$. Let $l = \{x, y, z\}$
be a line defining an edge of the graph in (b), such that \( z \in A_p, x, y \in B_p \). Since \( r_z = r_z r_y \) by definition of the representation group and \([r_p, r_z] = 1\) by (a), we have \( z_{px} = [r_p, r_x] = [r_p, r_z r_y] = [r_p, r_z] r_y = [r_p, r_y] = z_{py}\).

This calculation together with the connectivity from (b) implies that \( Z_{pq} \) is independent of the choice of \( q \in B_p \) and can be denoted by \( Z_p \). Since \( z_{pq} = [r_p, r_q] = [r_q, r_p]^{-1} = z_{qp}^{-1} \), we have \( Z_{pq} = Z_{qp} \). This means that \( Z_p = Z_q \) whenever \( q \in B_p \), i.e., if \( p \) and \( q \) are adjacent in the complement of \( \Sigma \). Since the latter graph is connected by (c), \( Z_p \) is independent of \( p \in P \) and can be denoted by \( Z \). Thus \( R' = Z \) is cyclic. As we observed, for every \( x \in P \) the element \( r_x \) inverts \( Z \). Now if \( l = \{x, y, z\} \) is a line then \( r_x = r_y r_z \) and hence \( r_x \) centralizes \( Z \) as well. Thus \( R' = Z \) must have order at most 2.

Although in all situations to which Lemma 2.2 will be applied in the present paper the commutator subgroup \( R' \) turns out to be trivial, this is not always the case. For instance the hypothesis of the lemma is satisfied for a representation of the \( P \)-geometry associated with \( 3 \cdot \text{Aut } M_{22} \) inside an extraspecial group of order \( 2^{13} \).

In what follows let \( R_p \) denote the subgroup of \( R \) generated by \( r_p \) and all the \( r_q \) for \( q \) collinear to \( p \). Clearly \( r_p \) is in the centre of \( R_p \).

**Lemma 2.3.** Suppose that \( R_p/\langle r_p \rangle \) is abelian for every \( p \in P \). Then for every pair of points \( x, y \) collinear to \( p \) one of the following holds:

(i) \[ [r_x, r_y] = 1; \]

(ii) \( x \) and \( y \) are at distance 2 in the collinearity graph of \( S, [r_x, r_y] = r_p \), and \( p \) is the only point collinear to both \( x \) and \( y \).

The following result is quite obvious.

**Lemma 2.4.** If \( R_p \) is abelian for every point \( p \in P \) then \( [r_x, r_y] = 1 \) whenever \( x \) and \( y \) are points at distance at most 2 in the collinearity graph of \( S \).

Let  \( S \) be a geometry of rank at least three with a diagram of the shape

\[
\text{\includegraphics[width=0.5\textwidth]{diagram.png}}
\]

and let \( S = (P, L) \) be the point-line incidence system associated with \( S \).

**Lemma 2.5.** In the above notation for \( p \in P \) the factor group \( R_p/\langle r_p \rangle \) is a representation group of the residue of \( p \) in \( S \).
3. GEOMETRIES OF THE MATHIEU GROUPS

The point-line system of the rank 3 P-geometry $\mathcal{G}(M_{22})$ can be constructed as follows (see [IS]). Let $\Omega$ be a set consisting of 24 elements called letters on which the Mathieu group $M_{24}$ acts 5-fold transitively preserving a family of 8-element subsets called octads which form the Steiner system $\mathcal{S}$ of type $S(5,8,24)$. For $a, b \in \Omega$ the setwise stabilizer of $\{a, b\}$ in $\text{Aut}\mathcal{S} \cong M_{24}$ is isomorphic to $\text{Aut} M_{22}$. The points of $\mathcal{G}(M_{22})$ are all the 2-element subsets of $\Omega_0 = \Omega \setminus \{a, b\}$. Three points $\{c, d\}, \{e, f\}$, and $\{g, h\}$ form a line whenever $\{a, b, c, d, e, f, g, h\}$ is an octad.

**Lemma 3.1.** The group $R(\mathcal{G}(M_{22}))$ is abelian.

**Proof.** Let us pick an arbitrary point $p = \{c, d\} \subset \Omega_0$ and let $T = \{T_1, \ldots, T_6\}$ be the sextet defined by $T_1 = \{a, b, c, d\}$. The stabilizer of $p$ in $\text{Aut} M_{22}$ has the following orbits on the points other than $p$:

$$\Sigma_1: \{e, f\} \subset \Omega_0 \setminus \{c, d\}, \text{ such that } e, f \in T_i, \text{ for some } 2 \leq i \leq 6$$

(two are the points collinear with $p$):

$$\Sigma_2: \{e, f\} \subset \Omega_0, \text{ such that } |\{c, d\} \cap \{e, f\}| = 1;$$

$$\Sigma_3: \{e, f\} \subset \Omega_0 \setminus \{c, d\}, \text{ such that } e \in T_i, f \in T_j \text{ for some } i \neq j.$$ 

Here $|\Sigma_1| = 30$, $|\Sigma_2| = 40$, and $|\Sigma_3| = 160$. We are going to apply Lemma 2.2. Let us put $A_p = \{p\} \cup \Sigma_1 \cup \Sigma_2$ and $B_p = \Sigma_3$. If $q \in \Sigma_1$ then $q$ is collinear to $p$ and the involutions $z_p$ and $z_q$ commute by the definition of $R(\mathcal{G})$. If $q \in \Sigma_2$ then there is a tetrad $T_i, i \neq 1$, such that $q \subset O = T_1 \cup T_i$. All the 2-element subsets of $O \setminus \{a, b\}$ form a subgeometry in $\mathcal{G}(M_{22})$, and this subgeometry is clearly recognized as the generalized quadrangle of the group $S_6 \cong Sp_4(2)$. By Lemma 2.1, $z_p$ and $z_q$ commute in this case as well. So we have checked the condition (a) of Lemma 2.2. Let us now check the condition (b).

Let $\Gamma$ be the graph on $B_p$, defined as in Lemma 2.2(b). Consider an octad $O$ which contains $a$, $b$, and $c$ and does not contain $d$. As above, the 15 points $\{e, f\} \subset O \setminus \{a, b\}$ form a subgeometry, which is the generalized quadrangle of $S_6 \cong Sp_4(2)$. These 15 points fall into two types: (1) 5 pairs containing $c$, and (2) 10 pairs $\{e, f\} \subset O \setminus \{a, b, c\}$. Points of the first type clearly are not contained in $B_p$. As $O$ intersects any octad $T_1 \cup T_i$ in at most four elements, we see that all the five elements of $O \setminus \{a, b, c\}$ belong to different tetrads $T_i, 2 \leq i \leq 6$. Hence the 10 points of the second type are contained in $B_p$. Each line in the subgeometry (cf. the definition before Lemma 3.1) contains a point of the first type. Therefore, each such octad $O$ defines a Petersen subgraph in $\Gamma$. 
Clearly, every point \( \{e, f\} \in B_p \) is contained in a unique octad \( O \) containing \( a, b, \) and \( c (d \not\in O \) is implied, since \( \{e, f\} \in \Sigma_3 \). It means that such octads define a partition of \( \Gamma \) into 16 Petersen subgraphs. Similarly, the octads, containing \( a, b, \) and \( d, \) and not containing \( c, \) also define a partition of \( \Gamma \) into 16 Petersen subgraphs. Moreover, two Petersen subgraphs from different partitions can intersect each other in at most one point (as the corresponding octads intersect each other in \( a, b, \) and at most 2 further elements). Hence each connected component of \( \Gamma \) contains at least \( 10 \times 10 = 100 \) points. Since \( |B_p| = 160, \) we have shown that \( \Gamma \) is connected.

The condition (c) is easily implied by the fact that \( \text{Aut } M_{22} \) acts primitively on the points of \( \mathcal{S}(M_{22}) \). By Lemma 2.2, the commutator subgroup of \( R = R(\mathcal{S}(M_{22})) \) has order at most 2. Suppose it has order 2. Then the commutator map defines a non-trivial invariant bilinear form on \( V = V(\mathcal{S}(M_{22})) \). On the other hand (compare [IS1]), the \( M_{22} \)-module \( V \) is an indecomposable extension of a 1-dimensional submodule by an irreducible 10-dimensional quotient. The latter quotient is involved in the Golay co-code and its dual is a distinct module involved in the Golay code. So there is no invariant bilinear form on \( V \) and the result follows.

Next we consider the rank 3 T-geometry of the group \( M_{24} \). Once again we will define only the point-line system of this geometry (the geometry itself was constructed in [RS1]). The points of \( \mathcal{S}(M_{24}) \) are all the sextets of the Steiner system \( \mathcal{S} \) of type \( S(5, 8, 24) \). Two sextets \( S = \{S_1, \ldots, S_6\} \) and \( T = \{T_1, \ldots, T_6\} \) are collinear whenever \( |S_i \cap T_j| \) is even for all \( 1 \leq i, j \leq 6 \) (such sextets will be said to have even intersection). If \( O \in \mathcal{S} \) is an octad and \( X, Y \) are two 4-subsets in \( O \), such that \( |X \cap Y| = 2 \), then the sextets defined by \( X \) and \( Y \) are collinear; moreover, any pair of collinear sextets appears in this way. If \( S \neq T \) are collinear sextets then the third sextet on the line through \( S \) and \( T \) is defined by any 4-set \( S_i \Delta T_j, \) where \( S_i \cap T_j \neq \emptyset \) (here \( \Delta \) stays for the symmetric difference operator).

**Lemma 3.2.** The group \( R(\mathcal{S}(M_{24})) \) is abelian.

**Proof.** If we fix any two elements \( a, b \in \mathcal{S} \) then the sextets defined by the 4-sets \( \{a, b, c, d\} \) form the point set of a subgeometry in \( \mathcal{S}(M_{24}) \), isomorphic to \( \mathcal{S}(M_{22}) \). The isomorphism is established by the mapping \( \{a, b, c, d\} \leftrightarrow \{c, d\} \).

By Lemma 3.1, two points belonging to a common subgeometry \( \mathcal{S}(M_{22}) \) define commuting involutions in \( R(\mathcal{S}(M_{24})) \). On the other hand, it is well-known (see [Con]) that any two sextets contain tetrads intersecting in at least two elements. This means that any two points of \( \mathcal{S}(M_{24}) \) are contained in a common \( \mathcal{S}(M_{22}) \)-subgeometry.
If two tetrads belong to a common sextet, then their images in the Golay co-code coincide. This gives a mapping from the point set of $\mathcal{G}(M_{24})$ into the Golay co-code and the images generate a codimension 1 submodule in the co-code. It immediately follows from the definition of the collinearity relation in $\mathcal{G}(M_{24})$ that this is a representation of the geometry. This representation was proved to be universal in [RSm]. So $R(\mathcal{G}(M_{24}))$ is isomorphic to the 11-dimensional irreducible submodule (the even half) of the Golay co-code, spanned by the images of the sextets.

The restriction of the above representation to $\mathcal{G}(M_{22})$ (viewed as a subgeometry of $\mathcal{G}(M_{24})$) is the universal representation of $\mathcal{G}(M_{22})$ (see [IS1]). The images of the points of $\mathcal{G}(M_{24})$ span in the co-code the same 11-dimensional subspace. As an Aut $M_{22}$-module, it has a 1-dimensional submodule (the image of $\langle\{a, b\}\rangle$). The quotient over this submodule is irreducible.

4. TWO MORE INCIDENCE SYSTEMS RELATED TO $M_{22}$

In Section 6 we will need information on universal representations of two other point-line incidence systems associated with the group $M_{22}$.

We start with the definition of the derived graph and derived triple system for a P-geometry as introduced in [IS4].

Let $\mathcal{G}$ be a P-geometry of rank $n$ that is a geometry with the diagram

\[
\begin{array}{ccccccc}
& & & & & P & \\
& 2 & & & & & 1 \\
2 & & & & & 2 & \\
& & & \cdots & & & \\
& & & & & 2 & \\
& & & & & & 2 \\
& & & & & & & 2 \\
& & & & & & & & \\
\end{array}
\]

where the types of elements are assumed to be $1, 2, \ldots, n$ and to increase from the left to the right. The derived graph $\Xi = \Xi(\mathcal{G})$ has the set of elements of type $n$ in $\mathcal{G}$ (corresponding to the rightmost node in the diagram) as vertices with two such elements adjacent if they are incident to a common element of type $n - 1$. Since every element of type $n - 1$ is incident to exactly 2 elements of type $n$, it determines a unique edge of $\Xi$. Note that if $n = 2$ then $\Xi$ is just the Petersen graph. In the general case the valency of $\Xi$ is $2^n - 1$. Let $y$ be an element of type $n - i$, $i \geq 2$ in $\mathcal{G}$. Then the vertices and edges of $\Xi$ incident to $y$ form a subgraph which is the derived graph of the corresponding rank $i$ residual P-geometry. In particular, the elements of type $n - 2$ in $\mathcal{G}$ determine in $\Xi$ a family of Petersen subgraphs. In what follows when talking about Petersen subgraphs in a derived graph we always mean subgraphs from this family.

Now the derived point-line incidence system $\mathcal{D}$ of a P-geometry $\mathcal{G}$ has the vertices of the corresponding derived graph as points and a triple of such vertices form a line if and only if they are adjacent to a common
vertex and are contained in a common Petersen subgraph. The following result is elementary.

**Lemma 4.1.** The universal representation group of the derived system of the rank 2 $P$-geometry $\mathcal{G}$ (that is, of the Petersen graph geometry) is abelian of order $2^4$ and is isomorphic to the orthogonal module for $S_5 \cong \text{Aut } \mathcal{G}$. The images of vertices on a 3-path in the Petersen graph form a basis of the group.

The next result is proved in [IS4, Proposition 6.1].

**Lemma 4.2.** The universal representation group of the derived system of the $P$-geometry $\mathcal{G}(M_{22})$ is abelian of order $2^{10}$ and is isomorphic to an (irreducible) $\text{Aut } M_{22}$-submodule in the Golay code.

The 10-dimensional submodule in the above lemma consists of all the subsets from the Golay code missing both letters from the pair stabilized by $\text{Aut } M_{22}$.

The following result can be either deduced from the proof of Proposition 6.1 in [IS4] or checked directly in the Golay code.

**Lemma 4.3.** Let $\Xi = \Xi(\mathcal{G}(M_{22}))$ be the derived graph of the geometry $\mathcal{G}(M_{22})$ and let $x$ be a vertex of $\Xi$. Then the images of the vertices of $\Xi$ with distance at most 3 from $x$ generate the universal group of the derived system while the images of whose with distance at most 2 generate a subgroup of order $2^7$.

Now let us consider yet another incidence system associated with the Mathieu group $M_{22}$. Namely, let $\mathcal{I} = \mathcal{I}(M_{22})$ be the incidence system whose points are all involutions in $M_{22}$ and lines are the elementary abelian subgroups of order 4 in this group with the natural incidence relation. Clearly such a system $\mathcal{I}(F)$ can be defined for any group $F$ (of even order) and if $F$ is generated by its involutions then $F$ is a factor group of $R(\mathcal{I}(F))$. $F$ naturally acts on $\mathcal{I}(F)$ but not necessarily point- or line-transitively.

The group $M_{22}$ possesses a non-split extension $3 \cdot M_{22}$ by a centre of order 3. With respect to the natural homomorphism of $3 \cdot M_{22}$ onto $M_{22}$, the preimage of every 2-subgroup is isomorphic to the direct product of this subgroup and the centre of order 3. Also, since the extension is non-split, the group $3 \cdot M_{22}$ is generated by its involutions. This immediately implies that $3 \cdot M_{22}$ (as well as $M_{22}$ of course) is a representation group of $\mathcal{I}(M_{22})$.

The following result served in a sense the starting point for the present project.

**Proposition 4.4.** The non-split extension $3 \cdot M_{22}$ is the universal representation group of the incidence system $\mathcal{I}(M_{22})$. 
The available proof of Proposition 4.4 requires intensive computer calculations (cosets enumeration) performed using the group theory language Cayley [Can]. First the group \( R(\mathcal{F}(A_7)) \) was shown to be isomorphic to \( 3 \cdot A_7 \). By the definition \( R(\mathcal{F}(A_7)) \) is a group with 105 generators (the number of involutions in \( A_7 \)) and a certain number of relations. The enumeration of the cosets of an elementary abelian subgroup of order 4 gave the right index. So we have the following result of an independent interest.

**Proposition 4.5.** \( R(\mathcal{F}(A_7)) \cong 3 \cdot A_7 \)

The enumeration of the cosets in \( R(\mathcal{F}(M_{22})) \) (with 1,155 generators indexed by the involutions in \( M_{22} \)) of a subgroup generated by 105 generators corresponding to the involutions from an \( A_7 \)-subgroup gave the index 176 (equal to the index of \( 3 \cdot A_7 \) in \( 3 \cdot M_{22} \)) so we have Proposition 4.4.

Alternatively Proposition 4.4 can be deduced from Proposition 4.5 using a result from [Bau] (see also [BP]) establishing the universal cover of a geometry of \( M_{22} \) having \( A_7 \)-subgroups as maximal parabolics. We believe that it would be very nice to have a computer-free proof of Proposition 4.5.

It is known (compare [Atlas]) that \( M_{22} \) contains exactly two classes of elementary abelian subgroups of order \( 2^4 \) with normalizers \( 2^4.S_5 \) and \( 2^4.A_6 \) (the corresponding normalizers in \( \text{Aut} M_{22} \) are \( 2^5.S_5 \) and \( 2^4.S_6 \)). Every elementary subgroup of order 4 is contained in at least one member of these two classes. So Proposition 4.4 has the following

**Corollary 4.6.** Let \( H \) be a group generated by a set of involutions indexed by the involutions of \( M_{22} \). Suppose that for every elementary subgroup of order \( 2^4 \) in \( M_{22} \) the corresponding 15 generators of \( H \) generate an elementary subgroup of order \( 2^4 \). Then \( H \cong 3 \cdot M_{22} \) or \( M_{22} \).

5. GEOMETRIES OF THE CONWAY GROUPS

The geometries \( \mathcal{F}(Co_1) \) and \( \mathcal{F}(Co_2) \) of Conway’s groups \( Co_1 \) and \( Co_2 \) are defined in terms of the Leech lattice \( \Lambda \) [RSt]. We use the standard terminology for \( \Lambda \) given in [Con]. In particular, we use integer coordinates for the lattice vectors and omit an implicit factor \( 1/\sqrt{8} \) which makes the lattice unimodular. The coordinates are assumed to be indexed by the set \( \Omega \) of letters on which the Steiner system \( \mathcal{S} \) of type \( S(5,8,24) \) is defined. Let \( \Lambda_n = \{ \lambda \in \Lambda | \langle \lambda, \lambda \rangle = 16n \} \) where \( \langle , , \rangle \) denotes the usual inner product.

There is an equivalence relation on \( \Lambda_4 \) with classes of size 48 where two vectors are equivalent if they are equal modulo \( 2\Lambda \). Every equivalence
class forms a coordinate frame. Namely, it determines 24 pairwise orthogonal one-dimensional subspaces in the underlying real space and consists of all vectors in these subspaces with square length 64. The standard frame consists of the vectors with a single non-zero coordinate equal to \( \pm 8 \).

The group \( C_{o1} \), which is the automorphism group of \( \Lambda \) modulo the subgroup of order 2 consisting of scalar transformations, acts faithfully and transitively on the set of coordinate frames. These frames form the point set of the geometry \( \mathcal{F}(C_{o1}) \). Two frames are collinear if and only if the inner product of every vector from one frame with every vector from another frame is divisible by 32. If \( \lambda, \mu \in \Lambda_4 \) are not perpendicular and determine different but collinear frames then the third point on the corresponding line of \( \mathcal{F}(C_{o1}) \) is determined by the vector \( \lambda + \mu \) or \( \lambda - \mu \), whichever belongs to \( \Lambda_4 \).

Let \( p \) be the point of \( \mathcal{F}(C_{o1}) \) corresponding to the standard frame. A vector from a frame collinear to \( p \) has exactly four non-zero coordinates equal to \( \pm 4 \). Two such vectors are in the same coordinate frame if and only if their supports are tetrads from a fixed sextet and the numbers of minus signs are of the same parity. If \( \{p, q, r\} \) is a line of \( \mathcal{F}(C_{o1}) \) then \( q \) and \( r \) correspond to the same sextet but to different parities of signs. In particular, the lines incident to \( p \) are in a natural bijection with the sextets of the Steiner system. Also it is clear that two points in different lines passing through \( p \) are collinear if and only if the corresponding sextets have even intersection. This immediately implies that the residue in \( \mathcal{F}(C_{o1}) \) of a point is isomorphic to \( \mathcal{F}(M_{24}) \).

The stabilizer of \( p \) in \( C_{o1} \) (isomorphic to \( 2^{11}.M_{24} \)) has 6 orbits \( \Sigma_i \), \( 0 \leq i \leq 5 \), on the point set of \( \mathcal{F}(C_{o1}) \). These orbits are described in the table below (compare [Con, Smj]) in which the fourth column shows the form of a vector in the coordinate frame corresponding to a point \( q \) from the orbit. The signs and positions of the non-zero values depend on the Steiner system and are not required.

| the stabilizer of \( p \) and \( q \) | \( |\Sigma_i| \) | Form of a vector |
|----------------------------------|---------|----------------|
| \( \Sigma_0 = \{p\} \)          | 1       | \( (8, 0^{-23}) \) |
| \( \Sigma_1 \)                  | 3542    | \( (4^4, 0^{20}) \) |
| \( \Sigma_2 \)                  | 48576   | \( (6, 2^7, 0^{16}) \) |
| \( \Sigma_3 \)                  | 1457280 | \( (2^4, 2^8, 0^{14}) \) |
| \( \Sigma_4 \)                  | 4145152 | \( (5, 3^2, 1^{21}) \) |
| \( \Sigma_5 \)                  | 2637824 | \( (4, 2^{12}, 0^{11}) \) |

We also present the intersection matrix from [ILLSS] of the orbital graph \( \Sigma \) of valency 3, 542 of \( C_{o1} \) acting on the points of \( \mathcal{F}(C_{o1}) \), that is, of
the collinearity graph of the geometry. The \((i,j)\)th entry in this matrix shows the number of points in \(\Sigma_j\) collinear to a given point from \(\Sigma_i\).

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
3542 & 181 & 35 & 7 & 0 & 0 \\
0 & 480 & 35 & 56 & 21 & 0 \\
0 & 2880 & 1680 & 791 & 630 & 495 \\
0 & 0 & 1792 & 1792 & 1771 & 1760 \\
0 & 0 & 0 & 896 & 1120 & 1287
\end{pmatrix}
\]

It follows particularly from the above matrix that \(\Sigma_2 \cup \Sigma_3\) are the points at distance 2 from \(p\) in \(\Sigma\) while \(\Sigma_4 \cup \Sigma_5\) are those at distance 3.

The following lemma can be checked directly (see also [Smi]).

**Lemma 5.1.** The geometry \(\mathcal{G}(Co_1)\) contains a line, one of whose points is in \(\Sigma_5\) and whose other two defining points are in \(\Sigma_4\).

This lemma implies that \(R(\mathcal{G}(Co_1))\) is generated by the images of points from \(\Sigma_i\) for \(0 \leq i \leq 4\).

The geometry \(\mathcal{G}(Co_2)\) of the second Conway group can be defined as a subgeometry in \(\mathcal{G}(Co_1)\). Choose a vector \(\nu \in \Lambda_2\) and let \(\Lambda_4(\nu)\) be the set of vectors from \(\Lambda_4\) whose inner products with \(\nu\) are all equal to 32. Then the points of \(\mathcal{G}(Co_2)\) are those points of \(\mathcal{G}(Co_1)\) whose coordinate frames contain vectors from \(\Lambda_4(\nu)\). The automorphism group of \(\Lambda\) acts transitively on \(\Lambda_2\) with the stabilizer being isomorphic to \(Co_2\).

Every vector \(\nu\) with two non-zero coordinates both equal to \(\pm 4\) is contained in \(\Lambda_2\) and the \(\mathcal{G}(Co_2)\)-subgeometry determined by such a vector contains the point \(p\) which corresponds to the standard frame. Now the shapes of vectors corresponding to the \(\Sigma_i\) given in the above table implies the following.

**Lemma 5.2.** Let \(q \in \Sigma_i\) for \(0 \leq i \leq 4\), then \(p\) and \(q\) are contained in a common \(\mathcal{G}(Co_2)\)-subgeometry.

Suppose that \(\nu \in \Lambda_2\) determines a \(\mathcal{G}(Co_2)\)-subgeometry in \(\mathcal{G}(Co_1)\) which contains \(p\) and let \(a\) and \(b\) be the non-zero coordinates of \(\nu\) considered as points from \(\mathcal{G}\). Then a line \(l\) incident to \(p\) is contained in \(\mathcal{G}(Co_2)\) if and only if the points \(a\) and \(b\) are in the same tetrad of the sextet corresponding to \(l\). This immediately implies that the residue in \(\mathcal{G}(Co_2)\) of a point is isomorphic to \(\mathcal{G}(M_{22})\). The stabilizer \(G(p)\) of \(p\) in \(G \cong Co_2\) is a maximal subgroup isomorphic to \(2^{10}.\Aut M_{22}\). This stabilizer has 5 orbits \(\Delta_i, 0 \leq i \leq 4\), on the point set of \(\mathcal{G}(Co_2)\) and \(\Delta_i = \Sigma_i \cap \)
$\mathcal{F}(Co_2)$. The intersection matrix of the collinearity graph $\Delta$ of $\mathcal{F}(Co_2)$ is the following.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
462 & 61 & 15 & 7 & 0 \\
0 & 80 & 15 & 28 & 21 \\
0 & 320 & 240 & 203 & 210 \\
0 & 0 & 192 & 224 & 231
\end{pmatrix}
$$

Here $\Delta_2 \cup \Delta_3$ are the points at distance 2 from $p$ in $\Delta$ and $\Delta_4$ are those at distance 3.

**Lemma 5.3.** If $q \in \Delta_i$ then $G(p) \cap G(q)$ is isomorphic to $(2^{10}).\text{Aut} M_{22}.(2^9).2^5.S_5.(2^5).2^4.S_6.(2^4).L_3(2)$ and $L_3(4).2$ for $i = 0, 1, 2, 3$ and 4, respectively. Here when writing $(A).B$ we mean that $A = O_2(G(p)) \cap G(q)$.

The following result is probably well known and a proof can be found in [IS4, Lemma 7.4].

**Lemma 5.4.** Every line of $\mathcal{F}(Co_2)$ intersects $\Delta_4$ in an even number of points (that is, $\{p\} \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$ is a geometric hyperplane) and the subgraph induced by $\Delta_4$ in the collinearity graph of $\mathcal{F}(Co_2)$ is connected.

**Lemma 5.5.** Let $x$ be an element of type 4 in $\mathcal{F} = \mathcal{F}(Co_2)$ and $G(x)$ be the stabilizer of $x$ in $G \cong Co_2$. Then $G(x) \cong (2^{1+6} \times 2^4).L_4(2)$ and it has six orbits $\Psi_i$ on the vertex set of the collinearity graph $\Delta$ of $\mathcal{F}$ with lengths 15, 120, 840, 3,360, 15,360, and 26,880 for $i = 1, 2, 3, 4, 5$ and 6, respectively. A vertex from $\Psi_i$ is adjacent to 7 and 3 vertices from $\Psi_1$ for $i = 2$ and 3, respectively.

**Proof.** Using the GAP [Sch] system we calculate that the inner product of the (primitive) permutation characters of $G$ acting on the cosets of $G(p)$ and $G(x)$ is six which gives the number of orbits. The action of $G(p)/O_2(G(p)) \cong \text{Aut} M_{22}$ on the set of orbits of $O_2(G(p))$ on $\Delta_j$ has by Lemma 5.3 a clear interpretation in terms of the residual geometry $\mathcal{H}$ of $p$ in $\mathcal{F}$, isomorphic to $\mathcal{F}(M_{22})$. Namely, for $j = 1, 2, 3, 4$ the action is the same as: the action on the points of $\mathcal{H}$; on the $S_{43}(2)$-subgeometries; on the elements of the rightmost type, and on the set $\mathcal{F} - \{a, b\}$ (that is, on the point set of the $S(3,6,22)$-Steiner system associated with $\mathcal{H}$), respectively. If $x$ is incident to $p$, then $G(x)$ contains $O_2(G(p))$ and the image in $G(p)/O_2(G(p))$ of $G(x) \cap G(p)$ is the stabilizer of an element of the rightmost type in $\mathcal{H} \cong \mathcal{F}(M_{22})$. By straightforward calculations in $\mathcal{F}(M_{22})$ we obtain the orbit lengths of $G(x) \cap G(p)$ on the vertex set of $\Delta$. Now using the information on the structure of $\Delta$ basically coming from the intersection matrix of $\Delta$ one can determine the fusion of the orbits of $G(x) \cap G(p)$ into the orbits of $G(x)$. ■
Let $\overline{\Lambda}_{24} = \Lambda / 2\Lambda$. The definition of $\mathcal{D}(Co_1)$ immediately implies that it possesses a representation $\phi$ in $\overline{\Lambda}_{24}$. Since $\overline{\Lambda}_{24}$ is an irreducible $GF(2)$-module for $Co_1$, it is clear that the images under $\phi$ of the points of $\mathcal{D}(Co_1)$ generate the module. It was proved independently by a number of people that $\phi$ is the universal representation of $\mathcal{D}(Co_1)$ and a published proof can be found in [Smi].

The action of $Co_1$ on $\overline{\Lambda}_{24}$ preserves on it a unique non-singular quadratic form $Q$ (of plus type) and a unique bilinear form $F$ (the one determined by $Q$). If $p$ is the point of $\mathcal{D}(Co_1)$ corresponding to the standard frame and $q \in \Sigma_i$ then $\phi(p)$ and $\phi(q)$ are orthogonal with respect to $F$ if and only if $i \neq 4$.

The restriction of $\phi$ to the subgeometry $\mathcal{D}(Co_2)$ is of course a representation of the latter. The images of points from $\mathcal{D}(Co_2)$ generate a codimension 1 subspace $\overline{\Lambda}_{23}$ of $\overline{\Lambda}_{24}$ consisting of the vectors orthogonal with respect to $F$ to the image $\overline{\nu} \in \overline{\Lambda}_{24}$ of the vector $\nu \in \Lambda_2$ involved in the definition of $\mathcal{D}(Co_2)$. It was shown in [IS2] and [IS4] that the representation of $\mathcal{D}(Co_2)$ in $\overline{\Lambda}_{23}$ is universal. The group $Co_2$ acting on $\overline{\Lambda}_{23}$ preserves a unique quadratic form and a unique bilinear form. These forms are the restrictions of $Q$ and $F$, respectively, and will be denoted by the same letters. As a $GF(2)$-module for $Co_2$, $\overline{\Lambda}_{23}$ has a one-dimensional submodule $\langle \overline{\nu} \rangle$ which is the radical of $F$ and the corresponding factor module $\overline{\Lambda}_{22} = \overline{\Lambda}_{23}/\langle \overline{\nu} \rangle$ also affords a representation of $\mathcal{D}(Co_2)$. Note that $\overline{\Lambda}_{23}$ is an indecomposable extension of $\overline{\Lambda}_{22}$ by the one-dimensional submodule.

Now we proceed to a consideration of representation groups of $\mathcal{D}(Co_1)$ and $\mathcal{D}(Co_2)$ starting with the latter one.

**Lemma 5.6.** The commutator of $R(\mathcal{D}(Co_2))$ has order at most 2.

**Proof.** Since the point residue of $\mathcal{D}(Co_2)$ is isomorphic to $\mathcal{D}(M_{22})$, by Lemmas 2.5 and 3.1 the factor $R_p / \langle z_p \rangle$ is abelian. It follows from the intersection matrix of the collinearity graph $\Delta$ of $\mathcal{D}(Co_2)$ that any two vertices which are at distance 2 in $\Delta$ are joined by at least 7 paths of length 2. By Lemma 2.3 this means that $R_p$ is abelian. By Lemma 2.4 and the intersection matrix of $\Delta$ we have $[z_p, z_q] = 1$ for all $q \not\in \Delta_4$. Now we are going to apply Lemma 2.2 for $B_p = \Delta_4$. The condition (a) is just established, the condition (b) is proved in Lemma 5.4, and finally (c) follows from the primitivity of the action of $Co_2$ on the vertices of $\Delta$.

**Proposition 5.7.** The group $R(\mathcal{D}(Co_2))$ is abelian and $R(\mathcal{D}(Co_2)) \equiv \overline{\Lambda}_{23}$.

**Proof.** Suppose to the contrary that $R = R(\mathcal{D}(Co_2))$ is non-abelian. Then the commutator $R'$ of $R$ is of order 2 by Lemma 5.6. The factor
group $R/R'$ is the module supporting the universal representation of $\mathcal{G}(Co_2)$ and hence $R/R' \cong \Lambda_{23}$. Finally the power and the commutator maps from $R/R'$ into $R'$ must coincide respectively with the quadratic form $Q$ and the bilinear form $F$ on $\Lambda_{23}$ preserved by $Co_2$. It is easy to see that these conditions determine $R$ uniquely up to isomorphism.

Since $R$ is the universal representation group of $\mathcal{G}(Co_2)$, the automorphism group of the geometry can be realized as a subgroup $H \cong Co_2$ in the automorphism group $A$ of $R$. The group $I$ of inner automorphisms of $R$ is clearly isomorphic to the factor of $\Lambda_{23}$ over the radical of the bilinear form $F$. So $I$ is elementary abelian of order $2^{22}$ and together with $H$ they must generate in $A$ a semidirect product $I\Lambda H \cong 2^{22}Co_2$. The group $O = A/I$ of outer automorphisms of $R$ coincides with the automorphism group of the quadratic form $Q$ on $\Lambda_{23}$, so $O \cong 2^{22}.O_{22}(2)$. Since $Q$ is the unique quadratic form on $\Lambda_{23}$ preserved by $Co_2$, $O$ contains a unique conjugacy class of subgroups isomorphic to $Co_2$ and inducing the specified action on $\Lambda_{23}$. Thus there is a unique conjugacy class of subgroups in the automorphism group $A$ of $R$ which is an extension of the subgroup $I$ of inner automorphisms by $Co_2$. As we observed above this extension must split if $R$ appears as the (universal) representation group of $\mathcal{G}(Co_2)$. But in fact the extension does not split, as follows from the paragraph below.

The Baby Monster group $F_2$ contains an involution $t$ whose centralizer $C$ is isomorphic to $2^{1+22}.Co_2$. Let $Q = O_2(C)$. Consider the unique non-split extension $2 \cdot F_2$ of $F_2$ by a centre of order 2. This extension is the centralizer of an involution in the Monster. Let $\tilde{C}$ and $\tilde{Q}$ be the full preimages in $2 \cdot F_2$ of $C$ and $Q$ respectively. Then the commutator of $\tilde{Q}$ is $\langle t \rangle$ (that is of order 2) and $\tilde{Q}/\langle t \rangle$ as a $GF(2)$-module for $Co_2 \cong \tilde{C}/\tilde{Q}$ is isomorphic to $\Lambda_{23}$. This implies that $\tilde{Q}$ and $R$ are isomorphic. Thus the image of $\tilde{C}$ in the automorphism group of $R$ must coincide with the above mentioned semidirect product $I\Lambda H$. On the other hand, this image is isomorphic to the factor group of $\tilde{C}$ over its centre $Z$ where $Z$ is elementary abelian of order 4. Since the Schur multiplier of $Co_2$ is trivial, the full preimage of $H \cong Co_2$ in $\tilde{C}$ must be isomorphic to the direct product $Z \times H \cong 2^4 \times Co_2$. This implies that both $2 \cdot F_2$ and $F_2$ contain subgroups isomorphic to $Co_2$. But it was proved in [Wil2, Corollary 8.7] that $Co_2$ is not a subgroup in the Baby Monster group. So our starting assumption that $R$ is non-abelian was wrong and the result follows. 

**Proposition 5.8.** The group $R(\mathcal{G}(Co_1))$ is abelian and $R(\mathcal{G}(Co_1)) \cong \Lambda_{24}$.

**Proof.** By Proposition 5.7 if $p$ and $q$ are points in $\mathcal{G}(Co_1)$, then $[r_p, r_q] = 1$ whenever there is a $\mathcal{G}(Co_2)$-subgeometry which contains both $p$ and $q$. By Lemma 5.2 such a subgeometry exists unless $q \in \Sigma_5$. Finally
by Lemma 5.1 if \( q \in \Sigma \) then \( r_q = r_x r_y \) where both \( x \) and \( y \) are not in \( \Sigma_5 \). Hence \( r_p \) commutes with \( r_q \) anyway and the result follows.

The authors are grateful to R. L. Griess who pointed out that the non-splitness of extensions of the extraspecial groups is relevant to the question considered.

We formulate a few more properties of the universal representation module \( \Lambda_{23} \) of \( \mathcal{F}(Co_2) \). Let \( G \cong Co_2 \), \( \mathcal{G} \cong \mathcal{F}(Co_2) \), and \( R = R(\mathcal{G}) \cong \Lambda_{23} \). For a point \( p \in \mathcal{G} \) let \( \langle r_p \rangle \) denote the image of \( p \) under the representation in \( \Lambda_{23} \). As above let \( \Delta_i \), \( 0 \leq i \leq 4 \), be the orbits of the stabilizer \( G(p) \) of \( p \) in \( G \) (isomorphic to \( 2^{10}.\text{Aut } M_{22} \)) on the point set of \( \mathcal{G} \). Let \( R_p \) and \( R^1_p \) be the subgroups in \( R \) generated by the elements \( r_q \) for points \( q \) from \( \Delta_0 \cup \Delta_1 \) and from \( \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \), respectively. The following result can be deduced either from the proof given in [IS4] of the fact that \( \Lambda_{23} \) is the universal abelian representation group of \( \mathcal{G} \) or directly from [Con].

**Lemma 5.9.** In the above notation the following holds:

(i) \( R_p/\langle r_p \rangle \) is the universal representation module (of dimension 11) of the residue \( \mathcal{F}_p \) of \( p \) in \( \mathcal{G} \) (isomorphic to \( \mathcal{F}(M_{22}) \)); the image of \( R_p/\langle r_p \rangle \) in \( \Lambda_{22} \) is 10-dimensional;

(ii) \( \overline{R}_p = R^1_p/R_p \) is the universal derived module for \( \mathcal{F}_p \); \( O_2(G(p)) \) is the kernel of the action of \( G(p) \) on \( \overline{R}_p \); the images of \( r_q \) for \( q \in \Delta_2 \) and \( \Delta_3 \) form orbits with lengths 77 and 330 under the action of \( G(p) \) and \( \overline{R}_p \) is generated by either of these orbits;

(iii) \( R/R^1_p \) has order 2.

Note that \( \overline{R}_p \) and \( O_2(G(p)) \) are isomorphic as \( GF(2) \)-modules for \( \text{Aut } M_{22} \).

It is known [Will1] that \( G \) has a unique orbit of odd length on the non-zero elements of \( \Lambda_{22} \), namely the one consisting of the images of the \( r_p \). In addition, all the suborbits \( \Delta_i \) for \( i \neq 0 \) are of even length. Hence

**Lemma 5.10.** A Sylow 2-subgroup of \( Co_2 \) fixes a unique non-zero vector in \( \Lambda_{22} \) (namely the image of a point of \( \mathcal{F}(Co_2) \)).

This lemma immediately implies the following

**Lemma 5.11.** Let \( H \) be a subgroup in \( G \cong Co_2 \) which contains a Sylow 2-subgroup of \( G \). Then there is a unique minimal non-trivial \( H \)-submodule in \( \Lambda_{22} \).

Since there is a (unique) bilinear form \( F \) on \( \Lambda_{22} \) preserved by \( Co_2 \), the dual statement about maximal \( H \)-submodules holds. Recall that with respect to \( F \) the image of \( p \) is not orthogonal to that of \( q \) if and only if \( q \in \Delta_4 \).
The geometry $\mathcal{S}$ contains a subgeometry $\mathcal{S}$ isomorphic to the $C_3(2)$-geometry associated with $Sp_6(2)$ and stabilized in $G$ by the centralizer of a central involution isomorphic to $2^{1+8}.Sp_6(2)$. Such a subgeometry passing through $p$ contains 30 points from $\Delta_1$ and 32 points from $\Delta_2$. A proof of the following result can also be found in [IS4].

**Lemma 5.12.** Let $\mathcal{S}$ be the $C_3(2)$-subgeometry in $\mathcal{S}$. Then the subspace $R(\mathcal{S})$ generated by $r_q$ for all points $q$ in $\mathcal{S}$ is the universal (seven-dimensional) representation module for $\mathcal{S}$. The image $\overline{R(\mathcal{S})}$ of $R(\mathcal{S})$ in $\overline{\Lambda}_{22}$ is the irreducible six-dimensional natural module for $Sp_6(2)$.

**Lemma 5.13.** Let $\mathcal{S}$ be the $C_3(2)$-subgeometry in $\mathcal{S}$. Let $M$ be the subspace in $\overline{\Lambda}_{23}$ generated by the $r_q$ for all points $q$ collinear to points in the subgeometry. Then $M = \overline{\Lambda}_{23}$.

**Proof.** Let $R(\mathcal{S})$ be as in Lemma 5.12. By that lemma $R(\mathcal{S})$ contains the one-dimensional $G$-submodule in $\overline{\Lambda}_{23}$ and the image $\overline{R(\mathcal{S})}$ of $R(\mathcal{S})$ in $\overline{\Lambda}_{22}$ is six-dimensional irreducible. Let $S \cong 2^{1+8}.Sp_6(2)$ be the stabilizer of $\mathcal{S}$ in $G$. Clearly $M$ is an $S$-submodule. Since $S$ contains a Sylow 2-subgroup of $G$, $\overline{R(\mathcal{S})}$ is the unique minimal $S$-submodule in $\overline{\Lambda}_{22}$. By the remark after Lemma 5.11, if the image of $M$ in $\overline{\Lambda}_{22}$ is a proper subspace of the latter then it is contained in the unique maximal proper $S$-submodule $N$ in $\overline{\Lambda}_{22}$. Moreover, $N$ is the dual of $\overline{R(\mathcal{S})}$ with respect to $F$. This means that $N$ consists of all vectors in $\overline{\Lambda}_{22}$ orthogonal to the images of points from $\mathcal{S}$. So to prove the lemma it is sufficient to show that $M$ contains a vector which is not orthogonal to $r_p$ for some point $p$ from $\mathcal{S}$. Let $q$ be a point from $\Delta_2$ contained in $\mathcal{S}$. The intersection matrix of the collinearity graph of $\mathcal{S}(Co_2)$ given above shows that there is a point $s \in \Delta_4$ collinear to $q$. Then $r_s \in M$ and $r_s$ is not orthogonal to $r_p$ with respect to $\mathcal{S}$, so the result follows.

6. **THE DERIVED GEOMETRY OF $Co_2$**

The final contradiction in the non-existence proof for abelian representations of $\mathcal{S}(F_2)$ in [IS4] comes from the triviality of $V(\mathcal{S})$ for the derived system $\mathcal{S}$ of the geometry $\mathcal{S}(Co_2)$. In this section we start proving the following improvement of that result.

**Proposition 6.1.** The universal representation group of the derived system of the geometry $\mathcal{S}(Co_2)$ is isomorphic to $Co_2$.

In what follows a representation group of the derived system associated with a P-geometry will be called a derived group of that geometry.
Throughout the section \( \mathcal{G} \) denotes the geometry \( \mathcal{G}(Co_2) \), \( \Xi \) the derived graph of \( \mathcal{G} \), \( \mathcal{G} \) the derived system of \( \mathcal{G} \), and \( D = R(\mathcal{G}) \) the universal representation group of \( \mathcal{G} \). The image in \( D \) of a point \( x \) of \( \mathcal{G} \) will be denoted by \( \langle d_x \rangle \).

The valency of \( \Xi \) is 15 and for a vertex \( x \in \Xi \) the set \( \Xi(x) \) of vertices adjacent to \( x \) together with the lines of the derived system contained in \( \Xi(x) \) form a truncation of the three-dimensional projective \( GF(2) \)-space.

The group \( G \cong Co_2 \) acts primitively on the vertex set of \( \Xi \) and the stabilizer \( G(x) \) of a vertex \( x \in \Xi \) is the centralizer in \( G \) of an involution of type \( 2B \) (in the notation of [Atlas]), isomorphic to \( (2^{1+6} \times 2^4).L_4(2) \). So we can (and will) identify the vertex set of \( \Xi \) with the conjugacy class \( K \) of \( 2B \)-involutions in \( G \) so that \( G(x) = C_G(x) \). In these terms \( x \) and \( y \) are adjacent if and only if the product \( x \cdot y \) is from the same class and is contained in \( O_2(G(x)) \cap O_2(G(y)) \). The vertices from \( \Xi(x) \) generate in \( G(x) \) an elementary abelian subgroup of order \( 2^4 \) and this immediately implies the following (Lemma 8.1 in [IS4]).

**Lemma 6.2.** The group \( G \cong Co_2 \) is a derived group of \( \mathcal{G}(Co_2) \).

Thus there is a homomorphism \( \xi: D \to G \) and to prove Proposition 6.1 we have to show that \( \xi \) is actually an isomorphism.

We need certain properties of \( G \) and of its action on \( \Xi \). The information contained in the following three lemmas is well known (cf. [IS4] for a proof). In what follows for a graph \( \Gamma \), a vertex \( x \) in \( \Gamma \), and an integer \( i \) by \( \Gamma_i(x) \) we denote the set of vertices of \( \Gamma \) with distance \( i \) from \( x \) while by \( \Gamma_{\leq i}(x) \) with distance at most \( i \) from \( x \). As usual we write \( \Gamma(x) \) instead of \( \Gamma_1(x) \).

**Lemma 6.3.** Let \( x \) be a vertex of \( \Xi \). Then the vertices from \( \Xi_{\leq 2}(x) \) generate in \( G \) the subgroup \( O_2(G(x)) \cong 2^{1+6} \times 2^4 \). The commutator of this subgroup coincides with \( \langle x \rangle \).

**Lemma 6.4.** Let \( p \) be a point of the geometry \( \mathcal{G} \) that is an element of the leftmost type in the diagram. Then \( p \) is incident to 330 vertices of \( \Xi \) which induce a subgraph \( M(p) \) isomorphic to the derived graph of the residue of \( p \) in \( \mathcal{G} \) isomorphic to \( \mathcal{G}(M_{22}) \). Any pair of vertices with distance at most 3 in \( \Xi \) are contained in \( M(p) \) for some point \( p \).

**Lemma 6.5.** In the above notation the stabilizer \( G(p) \) is isomorphic to \( 2^{10}.\text{Aut } M_{22} \). As a \( GF(2) \)-module for \( \text{Aut } M_{22} \), \( O_2(G(p)) \) is isomorphic to an irreducible submodule in the Golay code. The group \( G(p) \) acting by conjugation on the involutions inside \( O_2(G(p)) \) has three orbits with lengths 77, 330, and 616 consisting of involutions having in \( G \) type \( 2A \), \( 2B \), and \( 2C \), respectively. The involutions of type \( 2B \) correspond to the vertices of \( M(p) \).
The horrific impact of the 2004 tsunami on the Acehnese coast of Sumatra has evoked widespread concern about similarly devastating tsunamis on other populated coasts. One of the most plausible localities for a tsunami of disastrous proportions in the near future is the section of Sumatran coast southeast of the region affected by the 2004 event (Fig. 1). More than 1 million Indonesians live along this length of coast south of the Equator, about twice the pre-2004 population of the west coast of Aceh, 800,000 in Padang, 350,000 in Bengkulu, and tens of thousands more in smaller cities and villages along the coasts of the mainland and the Mentawai Islands.

Paleoseismic data suggest that great earthquakes recur about every 200 to 240 years along the adjacent section of the Sunda megathrust (4). The most recent of these occurred as a closely spaced couplet in 1797 and 1833 (5). Simple calculations of the average recurrence interval, made by dividing the average slip in 1797 (6 m) and 1833 (10–14 m) by the convergence rate of 45 mm/year, yield intervals ranging from 130 to 300 years. Thus, the average interval between great earthquakes (or couplets, as in 1797 and 1833) is nearly equal to the current dormant period. Moreover, calculations show that the ruptures of 2004 and 2005 have brought this section even closer to failure (6, 7). The possible imminence of a great earthquake and tsunami along this section of the Sumatran coast motivate this exploration of the characteristics of past and future tsunamis there.

The permutation character of $G$ on the cosets of $G(p)$ given in [Sch] is equal to $1a + 275a + 2024a + 12650a + 31625b$ and we can calculate that a $2B$-involution fixes 975 points. This and further direct calculations give the following.

**Lemma 6.6.** In the above notation $G(p) \cong 2^{10}$. Aut $M_{22}$ contains exactly $330 + 330 \cdot 8 + 1155 \cdot 16$ involutions of type $2B$.

The next two lemmas can be checked with [Atlas].

**Lemma 6.7.** The group $M_{22}$ contains 1,155 involutions forming a single conjugacy class. The involutions in Aut $M_{22} - M_{22}$ form two conjugacy classes with sizes 330 and 1386.

**Lemma 6.8.** The involutions from the conjugacy class of size 330 in Aut $M_{22}$ are in a bijection with the vertices of the derived graph $\Gamma$ of $\mathcal{G}(M_{22})$. The involution corresponding to $x \in \Gamma$ fixes all the vertices from $\Gamma_{\leq 2}(x)$; the involutions corresponding to the vertices of a Petersen subgraph $\Theta$ in $\Gamma$ generate an elementary abelian subgroup of order $2^5$ whose normalizer coincides with the stabilizer of $\Theta$ in Aut $M_{22}$ (isomorphic to $2^5.S_5$).

We observed in Section 5 that the stabilizer $G(p)$ of a point in $\mathcal{G}$ has five orbits, $\Delta_i$, $0 \leq i \leq 4$, on the point set of $\mathcal{G}$. If $q \in \Delta_1$, then $p$ and $q$ are collinear in $\mathcal{G}$ and the subgraphs $M(p)$ and $M(q)$ have a Petersen subgraph in common. If $q \in \Delta_i$ for $i \geq 2$ then $M(p)$ and $M(q)$ are disjoint.

The centralizer in $G$ of an involution $t$ is isomorphic to $2^{1+8}.Sp_6(2)$, $(2^{1+6} \times 2^4).L_4(2)$, and $2^{10}.Aut S_8$ for $t$ being of type $2A$, $2B$, and $2C$. By Lemma 6.5 this means that there are 63, 15, and 1 points $q$ such that $t \in O_2(q)$ for these three cases. We observed in the previous paragraph that points $p$ and $q$ such that $O_2(G(p)) \cap O_2(G(q))$ contains a $2B$-involution are always collinear. The 63 points corresponding to a $2A$-involution form a $C_3(2)$-subgeometry $\mathcal{S}$. If $p$ is contained in such a subgeometry then 30 of its points are in $\Delta_1$ and 32 are in $\Delta_2$. Moreover, each point from $\Delta_2$ determines a unique subgeometry of this type passing through $p$. Now an easy calculation gives the following.

**Lemma 6.9.** The non-trivial elements of the subgroup $O_2(G(p)) \cap O_2(G(q))$ are

(i) 5 involutions of type $2A$ and 10 involutions of type $2B$ if $q \in \Delta_1$;

(ii) a single involution of type $2A$ if $q \in \Delta_2$. 

The above lemma and Lemma 5.3 imply the following.

**Lemma 6.10.** The image in $G(p)/O_2(G(p)) \cong \text{Aut } M_{22}$ of $O_2(G(p))$ is

(i) elementary abelian of order $2^5$ normalized by $2^5S_5$ and intersecting $M_{22}$ in a subgroup $2^4$ if $q \in \Delta_1$;

(ii) elementary abelian of order $2^4$ normalized by $2^4S_6$ if $q \in \Delta_2$.

Now we start determination of various subgroups in the universal derived group $D$ of $\mathcal{G}$. For a point $p$ of $\mathcal{G}$ the elements $d_x$ for $x \in M(p)$ generate in $D$ a derived group of the residual P-geometry $\mathcal{G}(M_{22})$. By Lemmas 4.2 and 6.5 we have the following

**Lemma 6.11.** The elements $d_x$ for $x \in M(p)$ generate in $D$ an elementary abelian subgroup $D_0(p)$ of order $2^{10}$ which maps isomorphically onto $O_2(G(p))$.

The next result was established in [IS4], Lemma 8.2 (compare Lemma 6.3).

**Lemma 6.12.** For a vertex $x$ of $\Xi$ the elements $d_y$ for $y \in \Xi_{\leq 2}(x)$ generate in $D$ a subgroup which maps isomorphically onto $O_2(G(x)) \cong 2^{1+6} \times 2^4$. In particular, for $u, v \in \Xi_{\leq 2}(x)$ we have $[d_u, d_v] = \langle d_x \rangle$.

If $x$ and $y$ are at distance at most 3 in $\Xi$ then they are contained in $M(p)$ for some point $p$ and $[d_x, d_y] = 1$ by Lemma 6.11. If $x$ and $y$ are at distance 4 then there is a vertex $z$ such that $x, y \in \Xi_{\leq 2}(z)$ and by Lemma 6.12 $[d_x, d_y] \leq \langle d_z \rangle$ and hence we have the following.

**Lemma 6.13.** Let $x$ and $y$ be at distance at most 4 in $\Xi$. Then one of the following holds:

(i) $[d_x, d_y] = 1$;

(ii) the distance between $x$ and $y$ is four, and there is a unique path $(x, u, z, v, y)$ joining them and $[d_x, d_y] = d_z$.

As above let $p$ be a point of $\mathcal{G}$ and $D_0(p)$ be the subgroup generated by all $d_x$ for $x \in M(p)$.

**Lemma 6.14.** Let $u$ be a vertex which is at distance 1 from $M(p)$ in $\Xi$. Then $d_u$ normalizes $D_0(p)$.

**Proof.** Let $x$ be a vertex in $M(p)$ adjacent to $u$. By Lemma 4.3 the elements $d_y$ such that $y \in M(p)$ and the distance from $y$ to $x$ in this subgraph is at most 3 generate $D_0(p)$. Such an element $d_y$ commutes with $d_u$ unless $u$ and $y$ are at distance 4. In the latter case there is a path $(u, x, z, v, y)$ in $\Xi$ such that $z \in M(p)$. By Lemma 6.13 we have $[d_u, d_y] \in \langle d_z \rangle$. Since $d_z \in D_0(p)$ the result follows. □
LEMMA 6.15. In the above notation let \( w \) be at distance 2 from \( M(p) \). Then \( d_w \) normalizes \( D_0(p) \).

Proof. Let \((w,v,x)\) be the shortest path joining \( w \) with \( x \in M(p) \). Then there is a (unique) Petersen subgraph \( \Theta \) in \( \Xi \) which contains this 2-path. Since \( M(p) \) and \( \Theta \) correspond respectively to a hyperplane and a line in the projective space defined on \( \Xi(x) \), the set \( \Theta(x) \) contains a vertex \( u \in M(p) \). Since \( \Theta \) is of diameter 2 there is a 2-path \((w,t,u)\). Now by Lemma 4.1 \( d_w \) is contained in the subgroup generated by \( d_v, d_x, d_u \), and \( d_t \). Since \( d_x, d_u \in D_0(p) \) while \( d_v \) and \( d_t \) normalize \( D_0(p) \) by Lemma 6.14, \( d_w \) normalizes \( D_0(p) \) as well.

It is easy to deduce from the proof of Lemma 6.15 that for a vertex \( w \) which is at distance 2 from \( M(p) \) the element \( d_w \) is the product of two similar elements corresponding to vertices at distance 1 from \( M(p) \).

Let \( M_i(p) \) denote the set of vertices of \( \Xi \) which are at distance \( i \) from \( M(p) \). It follows directly from the structure of \( \Xi \) that \( M_0(p) = M(p) \) contains 330 vertices; \( M_1(p) \) contains \( 330 \times 8 \) vertices, and \( M_2(p) \) contains \( 1,155 \times 16 \) vertices. The vertices from \( M(p) \) generate \( O_2(G(p)) \). The vertices from \( M_1(p) \) and \( M_2(p) \) normalize \( O_2(G(p)) \) by Lemmas 6.14 and 6.15 and hence they are contained in \( G(p) \). Comparing the above with the information in Lemma 6.6 on the number of \( 2B \)-involutions inside \( G(p) \) and the information in Lemma 6.7 on the conjugacy classes of involutions in \( \text{Aut } M_2 \), we obtain the following.

LEMMA 6.16. The involutions of type \( 2B \) in \( G(p) \) are exactly the vertices of \( \Xi \) which are at distance at most 2 from \( M(p) \). Moreover, for \( \overline{G(p)} = G(p)/O_2(G(p)) \approx \text{Aut } M_2 \), we have the following:

(i) the vertices from \( M_0(p) \) correspond to involutions inside \( O_2(G(p)) \);

(ii) the vertices from \( M_1(p) \) map onto the conjugacy class of outer involutions in \( \overline{G(p)} \) of the size 330 and each such outer involution is the image of 8 vertices;

(iii) the vertices from \( M_2(p) \) map onto the conjugacy class of inner involutions in \( \overline{G(p)} \) of size 1,155 and each such inner involution is the image of 16 vertices.

Now Lemma 5.5 together with direct calculations imply the following.

LEMMA 6.17. Let \( p \) and \( x \) be elements of types 1 and 4 in \( \mathcal{F} \cong \mathcal{F}(Co_2) \) where \( x \) is identified with an involution in \( G \cong Co_2 \). Then \( x \in G(p) \) if and only if \( p \) is contained in one of the orbits \( \Psi_1, \Psi_2, \) or \( \Psi_3 \) of the action of \( G(x) \) on the vertex set of the collinearity graph \( \Delta \) of \( \mathcal{F} \).

For \( i = 0, 1, \) or 2 let \( D_i = D_i(p) \) be the subgroup of \( D \) generated by the elements \( d_x \) for \( x \in M_0(p) \cup M_i(p) \). Then \( D_0 = D_0(p) \) is normal in \( D_i \).
for \( i = 1, 2 \) by Lemmas 6.14 and 6.15 and \( D_1 \geq D_2 \) by the remark after Lemma 6.15. Put \( \overline{D}_i = D_i/D_0 \). By the definition \( \overline{D}_1 \) and \( \overline{D}_2 \) are generated by the images in these groups of the elements \( d_x \) for \( x \in M_1(p) \) and \( M_2(p) \), respectively.

**Lemma 6.18.** For \( i = 1 \) or 2 the images in \( \overline{D}_i \) of \( d_x \) and \( d_y \) for \( x, y \in M_i(p) \) coincide if and only if the images in \( G(p)/O_2(G(p)) \) of \( x \) and \( y \) do.

**Proof.** Since \( G \) is a quotient of \( D \) we only need to prove the "if" part of the claim. Namely, in view of Lemma 6.16 we have to show that 8 generators of \( D_1 \) agree modulo \( D_0 \) and 16 generators of \( D_2 \) agree modulo \( D_0 \).

Let \( x \in M(p) \). The mapping \( y \mapsto d_y \) for \( y \in \Xi(x) \) defines a representation of the rank 3 projective \( GF(2) \)-geometry which is the residue of \( x \) in \( \mathcal{G}(Co_2) \). Hence the elements \( d_y \) for \( y \in \Xi(x) \) generate in \( D \) an elementary abelian subgroup of order \( 2^4 \) and the vertices from \( \Xi(x) \cap M(p) \) correspond to a hyperplane in this group. So 8 elements \( d_y \) for \( y \in \Xi(x) - M(p) \) map onto a single generator of \( \overline{D}_1 \).

By Lemma 6.3 the elements \( d_y \) for \( y \in \Xi_{\leq 2}(x) \) generate in \( D \) a subgroup \( A \) of order \( 2^{11} \) and by Lemma 4.3 those for \( y \in \Xi_{\leq 2}(x) \cap M(p) \) generate a subgroup \( B \) of order \( 2^{7} \). Since \( B \) contains \( \langle d_x \rangle = A' \) (compare Lemmas 6.3 and 6.13), \( B \) is normal in \( A \) and \( A/B \) is elementary abelian of order \( 2^4 \). Direct calculations show that \( \Xi_{\leq 2}(x) \cap M_1(p) \) consists of 64 and 112 vertices from \( i = 1 \) and 2, respectively. By the previous paragraph the generators corresponding to the 64 vertices map onto 8 involutions in \( A/B \). This implies that the generators which correspond to the 112 vertices must map onto the remaining 7 involutions in \( A/B \).

The homomorphism \( \xi : D \rightarrow G \) induces homomorphisms \( \xi_1 : \overline{D}_1 \rightarrow \text{Aut} M_{22} \) and \( \xi_2 : \overline{D}_2 \rightarrow M_{22} \). By Lemma 6.18 these homomorphisms are bijective on the set of involutory generators. We are ready to prove that \( \xi_1 \) and \( \xi_2 \) are isomorphisms. We get most of the way there with the following

**Lemma 6.19.** The group \( \overline{D}_2 \) is isomorphic either to \( M_{22} \) or to \( 3 \cdot M_{22} \).

**Proof.** We intend to apply Corollary 4.6. \( \overline{D}_2 \) is generated by the images of \( d_y \) for \( y \in M_2(p) \). By Lemma 6.18 this generating system consists of involutions naturally indexed by the involutions in \( M_{22} \). So all we have to show is that the generators corresponding to involutions from an elementary abelian subgroup of order \( 2^{14} \) in \( M_{22} \) generate in \( \overline{D}_2 \) a group of that order.

For \( i = 1 \) or 2 let \( N_i \) be the subgroup of \( D_2 \) generated by the elements \( d_y \) for \( y \in M_2(p) \cap M(q) \) for \( q \in \Delta_i \) in the notation of Lemma 6.10. Since \( D_0(p) \) and \( D_0(q) \) map isomorphically onto \( O_2(G(p)) \) and \( O_2(G(q)) \)
respectively, the image of \( N_i \) in \( \bar{D}_2 \) is isomorphic to the image of \( G(p) \cap O_2(G(q)) \) in \( G(p) / O_2(G(p)) \) that is isomorphic to an elementary group of order \( 2^4 \) (compare Lemma 6.10). The images are normalized by \( 2^4.S_5 \) and \( 2^4.S_6 \) for \( i = 1 \) and \( 2 \), respectively. This is exactly what we need in order to apply Corollary 4.6.

**Lemma 6.20.** Let \( z \in M_i(p) \). Then \( d_z \) normalizes \( D_2 \).

**Proof.** We know that \( d_z \) normalizes \( D_0 = D_0(p) \). Let \( x \) be (the unique) vertex in \( M(p) \) adjacent to \( z \). Let \( \Theta \) be a Petersen subgraph which is contained in \( M(p) \) and is at distance 1 from \( x \). Since \( \Theta \) is a line of \( \mathcal{G}(Co_2) \) it is incident to 3 points and hence there exists a point \( q \) such that \( M(p) \cap M(q) = \Theta \). Then \( z \in M_i(q) \) and \( d_z \) normalizes \( D_0(q) \) by Lemma 6.15. On the other hand, by Lemma 6.10 the elements \( d_y \) for \( y \in M(q) \cap M_i(p) \) generate in \( D_2 \) a subgroup \( N \) whose image \( \bar{N} \) in \( \bar{D}_2 \) is an elementary subgroup of order \( 2^4 \). Let us specify \( \bar{N} \) as a subgroup of \( \bar{D}_2 \). The group \( \bar{D}_2 \) acts on \( M(p) \) by means of conjugation of the generating involutions of \( D_0 \). This action is isomorphic to \( M_{22} \) and by Lemma 6.19 it is either faithful or has a kernel of order 3. Let \( S \) be the stabilizer of the Petersen subgraph \( \Theta \) in this action. Then \( S \) is isomorphic either to \( 2^4.S_5 \) or \( 3 \times 2^4.S_5 \) and \( \bar{N} = O_2(S) \). So \( d_z \) normalizes the full preimage of \( O_2(S) \) in \( D_2 \). Note that \( O_2(S) \) does not fix the vertex \( x \). There are 28 Petersen subgraphs in \( M(p) \) which are at distance 1 from \( x \) and each of them can be taken for \( \Theta \). In this way we obtain 28 subgroups of order \( 2^4 \) in \( \bar{D}_2 \) normalized by \( d_z \). We claim that these subgroups generate \( \bar{D}_2 \). In fact, on the one hand the subgroup generated is normalized by the stabilizer of \( x \) in \( \bar{D}_2 \) which is a maximal subgroup and on the other hand as mentioned it does not fix \( x \).

Now we see that \( \bar{D}_1 \) normalizes \( \bar{D}_2 \). It is easy to see (compare Lemma 6.15) that in the factor group \( \bar{D}_1 / \bar{D}_2 \) all the generators of \( \bar{D}_1 \) map onto a single involution. So \( \bar{D}_1 \) is either \( \text{Aut } M_{22} \) or \( 3 \cdot \text{Aut } M_{22} \). Also by Lemma 6.15 we see that \( \bar{D}_2 \) acts transitively by conjugation on the generating involutions of \( \bar{D}_1 \). Since there are no conjugacy classes of involutions in \( 3 \cdot \text{Aut } M_{22} \) of size 330, we have finally established that \( \xi \) is an isomorphism.

**Lemma 5.21.** The elements \( d_x \) for \( x \in G(p) \) generate in \( D \) a subgroup which maps isomorphically onto \( G(p) \) under \( \xi \).

In the next section we will prove a general result which will show that Lemma 6.21 is almost all we need to establish Proposition 6.1. What is missing is the following

**Lemma 6.22.** Let \( p \) and \( q \) be collinear points in \( \mathcal{G} \). Then the intersection \( G(p) \cap G(q) \) is generated by the \( 2B \)-involutions contained in the intersection.
Proof. As we observed, $G(p) \cap G(q) \cong [2^{14}].S_5$. By Lemma 6.16 the intersection contains all the vertices of $\Xi$ which are at distance at most 2 both from $M(p)$ and $M(q)$. Since $p$ and $q$ are collinear, these subgraphs intersect in a Petersen subgraph $\Theta$. So it is sufficient to show that the vertices which are at distance at most 2 from $\Theta$ generate $G(p) \cap G(q)$.

First of all, by Lemma 4.1 the vertices from $\Theta$ generate a subgroup of order $2^4$. For every vertex $x \in \Theta$ we can find vertices $y$ and $z$ adjacent to $x$ such that $y \in M(p) - M(q)$ and $z \in M(q) - M(p)$. Then $y$ commutes with the involutions from $M(p)$ and acts on $M(q)$ as the involution corresponding to $x$ in $G(q)/O_2(G(q)) \cong \text{Aut } M_{22}$ (compare Lemma 6.8). The action of $z$ is similar to the roles of $p$ and $q$ being exchanged. By Lemma 6.8 we see that the vertices adjacent to $\Theta$ induce on $M(p) \cup M(q)$ an action isomorphic to $2^5 \times 2^5$.

For a vertex $x$ of $\Xi$ the subgroup $O_2(G(x))$ is non-abelian and hence there exists a path $(u, v, x, b, a)$ in $\Xi$ such that $[u, a] = x \neq 1$ (compare Lemmas 6.3 and 6.13). Since every 2-path in $\Xi$ is contained in a unique Petersen subgraph, we can assume that $u, v, x \in \Theta$ and hence $a$ is at distance at most 2 from $\Theta$. Then $a$ commutes with $\{x\} \cup \Theta$, and particularly preserves $\Theta$ as a whole. On the other hand, $a$ does not commute with $u$ and so acting on $\Theta$ by conjugation $a$ induces the action of a transportation from $S_5 \cong \text{Aut } \Theta$. Since $S_5$ is generated by its transpositions, the vertices of distance 2 from $\Theta$ induce on $\Theta$ its full automorphism group $S_5$.

Now summarizing the above we see that the vertices with distance at most 2 from $\Theta$ induce on $M(p) \cap M(q)$ an action containing $(2^5 \times 2^5).S_5$ and the kernel contains a subgroup of order $2^4$. This proves the lemma. \qed

Now Lemmas 6.21 and 6.22 imply the following:

**Corollary 6.23.** The elements $d_x$ for $x \in G(p) \cap G(q)$ (where $p$ and $q$ are collinear points in $\mathcal{F}$) generate in $D$ a subgroup which maps isomorphically onto $G(p) \cap G(q)$ under $\xi$.

### 7. A Reduction Lemma

In the previous section for every point $p$ of $\mathcal{F} \cong \mathcal{F}(Co_2)$ we reconstructed in the universal derived group $D$ of $\mathcal{F}$ a subgroup which maps isomorphically onto the stabilizer of $p$ in $G$. So we reconstructed in $D$ one parabolic subgroup corresponding to the action of $G$ on $\mathcal{F}$. In this section we prove a lemma which shows that reconstruction in $D$ of other parabolic subgroups from $G$ is quite automatic. We are going to use this lemma several times; first we would like to establish a more general setting.
Suppose $\mathcal{G}$ is a residually connected geometry of rank $n \geq 4$ whose diagram has the shape

$$2 \circ \cdots \circ 2 \circ X \circ 2,$$

and let $G$ be a group acting flag-transitively (not necessarily faithfully) on $\mathcal{G}$ and let the stabilizer of an element of the rightmost type induce on the residual projective space its full automorphism group, namely $L_n(2)$. Suppose also that $\mathcal{G}$ is simply connected which means that it coincides with its universal cover.

**Lemma 7.1.** For $\mathcal{G}$ and $G$ as above, let $K$ be a group and $\phi: K \rightarrow G$ be a homomorphism. Suppose that for every point $p$ of $\mathcal{G}$ there is a subgroup $K(p) \leq K$ which maps under $\phi$ isomorphically onto the stabilizer $G(p)$ of $p$ in $G$; moreover, for collinear points $p, q$ in $\mathcal{G}$ the subgroup $K(p) \cap K(q)$ maps isomorphically onto the intersection $G(p) \cap G(q)$. Under these conditions, if the subgroups $K(p)$ for all points $p$ in $\mathcal{G}$ generate $K$ then $\phi$ is an isomorphism.

**Proof.** As usual we assume that the types in the diagram increase from the left to the right starting with 1 and ending with the rank of the geometry. So the points are elements of type 1.

Let $\Phi$ be a maximal flag in $\mathcal{G}$. Suppose that for every element $a \in \Phi$ we can indicate in $K$ a subgroup $K(a)$ which maps under $\phi$ isomorphically onto the stabilizer $G(a)$ of $a$ in $G$ and for every pair $a, b$ of elements of $\Phi$ the subgroup $K(a) \cap K(b)$ maps isomorphically onto $G(a) \cap G(b)$. If in addition $K$ is generated by the subgroups $K(a)$ for all $a \in \Phi$ then $\phi$ is an isomorphism since $\mathcal{G}$ is assumed to be simply connected. For $i \geq 4$ the left component of the residual of an element of type $i$ is the projective space of rank $i - 1 \geq 3$ which is simply connected. Hence instead of $\Phi$ we can consider a flag of type $\{1, 2, 3\}$. (Note that the fact that the action of $G$ on $\mathcal{G}$ might not be faithful does not make any difference since the kernel is contained in every stabilizer.) The subgroup $K(a)$ for $a$ being a point is given in the hypothesis of the lemma and we have such a subgroup for every point. By manipulation of various of these subgroups we are going to build the required subgroups for elements of types 2 and 3.

Let us first discuss some general properties of the action of $G$ on $\mathcal{G}$. Let $a_1, a_2, a_3$, and $a_4$ be pairwise incident elements in $\mathcal{G}$ of type 1, 2, 3, and 4, respectively. Let $P(a_i)$ denote the set of points (elements of type 1) incident to $a_i$ and let $G(a_i)$ denote the stabilizer of $a_i$ in $G$. Let $G_0(a_i)$ be the kernel of the action of $G(a_i)$ on $P(a_i)$. Then $G_0(a_1) = G(a_1)$ and for $2 \leq i \leq 4$ the action which $G(a_i)$ induces on $P(a_i)$ is isomorphic to $L_i(2)$ and in particular it is doubly transitive. This means that $G(a_i)$ is generated
by $G_0(a_i)$ and $G(a_i) \cap G(p)$ for all $p \in P(a_i)$ Note that if $p$ is incident to $a_i$ for some $i \leq 3$ then $p$ is also incident to $a_4$. Let us fix a point $r \in P(a_4) - P(a_3)$. Then $G(a_i) \cap G(r)$ induces on $P(a_i)$ the same action as $G(a_i)$ does (by our assumption $G(a_4)$ induces $L_4(2)$ on the residual rank 3 projective space). This means that for $i = 1, 2, 3$, $G(a_i)$ is generated by its normal subgroup $G_0(a_i)$ and by the subgroups $G(a_i) \cap G(r) \cap G(p)$ for all $p \in P(a_i)$.

Let us proceed to the reconstruction of the subgroups $K(a_i)$ in $K$. By the hypothesis, for every point $p$ in $\mathcal{S}$ there is a subgroup $K(p)$ in $K$ such that the restriction to $K(p)$ of the homomorphism $\phi: K \to G$ is an isomorphism onto $G(p)$. This restriction will be denoted by $\phi_p$. We know that if $p$ and $q$ are collinear points and if $L \leq G(p) \cap G(q)$ then $\phi_p^{-1}(L) = \phi_q^{-1}(L)$. We define $K(a_i)$ to be the subgroup in $K$ generated by $\phi_p^{-1}(G(a_i) \cap G(p))$ for all points $p \in P(a_i)$, $i = 1, 2, 3$. Under this definition it is clear that $K(a_i)$ is as in the hypothesis and that the restriction to $K(a_i)$ of $\phi$ is a homomorphism onto $G(a_i)$. We are going to prove that it is actually an isomorphism.

We observed that for $p \in P(a_i)$ the subgroup $G(a_i) \cap G(p)$ is generated by $G_0(a_i)$ and $G(a_i) \cap G(r) \cap G(p)$ where $r$ is a fixed point from $P(a_4) - P(a_3)$. This implies that $K(a_i)$ is generated by $\phi_p^{-1}(G_0(a_i))$ and $\phi_p^{-1}(G(a_i) \cap G(r) \cap G(p))$ for all $p \in P(a_i)$. The former of the subgroups is clearly independent of the choice of $p$ and we denote it by $K_0(a_i)$. The latter subgroups normalize $K_0(a_i)$ and, because of the equality $\phi_p^{-1}(G(a_i) \cap G(r) \cap G(p)) = \phi_r^{-1}(G(a_i) \cap G(r) \cap G(p))$, these subgroups taken for all $p \in P(a_i)$ generate in $K$ the subgroup $\phi_r^{-1}(G(a_i) \cap G(r))$. Thus we have shown that $K(a_i)$ is generated by $K_0(a_i)$ and $\phi_r^{-1}(G(a_i) \cap G(r))$. The former of the subgroups is normal in $K(a_i)$ and maps under $\phi$ isomorphically onto $G_0(a_i)$ and the latter maps isomorphically onto $G(a_i) \cap G(r)$ which together with $G_0(a_i)$ generates $G(a_i)$. Finally, $K_0(a_i) \cap \phi_r^{-1}(G(a_i) \cap G(r)) = \phi_r^{-1}(G_0(a_i) \cap G(r))$ maps isomorphically onto $G_0(a_i) \cap G(r)$. This proves that the restriction of $\phi$ to $K(a_i)$ is an isomorphism onto $G(a_i)$.

Now in order to show that $K(a_i) \cap K(a_j)$ maps isomorphically onto $G(a_i) \cap G(a_j)$ for $1 \leq j < i \leq 3$ it is sufficient to show that the index of $K(a_i) \cap K(a_j)$ in $K(a_i)$ is less than or equal to the index of $G(a_i) \cap G(a_j)$ in $G(a_i)$. On the other hand, the subgroup in $K(a_i)$ generated by $K_0(a_i)$ and $\phi_r^{-1}(G(a_i) \cap G(a_j) \cap G(r))$ is contained in $K(a_i)$ and has index in $K(a_i)$ equal to the index of $G(a_i) \cap G(a_j)$ in $G(a_i)$.

Finally, let us discuss why $K$ is generated by the subgroups $K(a_i)$ for $i = 1, 2, 3$. We actually proved that for every element $d$ in $\mathcal{S}$ of type less than 4 there is a subgroup $K(d)$ in $K$ which maps isomorphically onto $G(d)$. By the hypothesis, $K$ is generated by all such subgroups for $d$ being
a point. In fact, if \( \{a, b, c\} \) and \( \{a', b, c\} \) are two flags of type \( \{1, 2, 3\} \) which differ in a single element (of any of the three types) then \( G(a') \) is generated by its intersections with \( G(b) \) and \( G(c) \). This implies that \( K(a') \) is generated by its intersections with \( K(b) \) and \( K(c) \). Now the result follows from the residual connectivity of \( \mathcal{G} \). \[ \square \]

The geometry \( \mathcal{G}(Co_2) \) is simply connected [Shp] so by Lemma 7.1 we see that Proposition 6.1 follows from Lemma 6.21 and Corollary 6.23.

Now we are well prepared to start proving Theorem 1 for the case \( \mathcal{G} \cong \mathcal{G}(F_2) \). In the next section we will review the constructions of \( \mathcal{G}(F_1) \) and \( \mathcal{G}(F_2) \) from which it will follow that \( 2 \cdot F_2 \) is a representation group of \( \mathcal{G}(F_2) \). This means that there is a homomorphism \( \phi: R(\mathcal{G}(F_2)) \to 2 \cdot F_2 \). The geometry \( \mathcal{G}(F_2) \) is simply connected as was proved in [Ivn2] and \( 2 \cdot F_2 \) induces on it an unfaithful flag-transitive action. Because of this, in view of Lemma 7.1 in order to prove Theorem 1 for \( \mathcal{G} \cong \mathcal{G}(F_2) \) all we have to establish is the following.

**Proposition 7.2.** Let \( \mathcal{G} \cong \mathcal{G}(F_2) \), \( B \equiv 2 \cdot F_2 \), and \( R = R(\mathcal{G}) \). For a point \( x \) of \( \mathcal{G} \) let \( b_x \) and \( r_x \) denote the images of \( x \) in \( B \) and \( R \), respectively under the (non-abelian) representation of \( \mathcal{G} \) in these groups. Then

1. for every point \( x \) of \( \mathcal{G} \) the elements \( r_x \) such that \( b_x \in B(x) \) generate in \( R \) a subgroup which maps isomorphically onto \( B(x) \) under the homomorphism of \( R \) onto \( B \);
2. for every two collinear points \( x \) and \( y \) in \( \mathcal{G} \) the intersection \( B(x) \cap B(y) \) is generated by the elements \( b_2 \) contained in the intersection.

8. THE GEOMETRY \( \mathcal{G}(F_2) \)

For our purposes it is most natural to define \( \mathcal{G}(F_2) \) as a subgeometry of \( \mathcal{G}(F_1) \). The Monster group \( F_1 \) contains an involution \( t \) with the centralizer \( C \cong 2^{1+24} \cdot Co_1 \). Let \( E = O_2(C) \). Then \( E \) is extraspecial with centre \( Z = \langle t \rangle \) and \( E/Z \) is isomorphic to \( \overline{\Lambda}_{24} \) as a \( GF(2) \)-module for \( Co_1 \cong C/E \). Clearly the power and the commutator maps associated with \( E \) are the quadratic form \( Q \) and the bilinear form \( F \) preserved by the action of \( Co_1 \) on \( \overline{\Lambda}_{24} \).

The geometry \( \mathcal{G}(Co_1) \) has a representation in \( \overline{\Lambda}_{24} \) which is known (cf. Proposition 5.8 and [Sm]) to be universal even among non-abelian representations. Every element of type \( i \) in \( \mathcal{G}(Co_1) \) for \( 1 \leq i \leq 4 \) is realized by an \( i \)-dimensional subspace in \( \overline{\Lambda}_{24} \). Since all these subspaces are totally singular with respect to \( Q \), their full preimages in \( E \) are elementary abelian groups. Moreover, it is known (cf. [Ivn]) that the normalizer in \( F_1 \) of such a preimage induces its full automorphism group. We define \( \mathcal{G}(F_1) \)
to be a geometry whose elements are all the images under \( F_1 \) of the subgroup \( \langle t \rangle \) and of all the preimages in \( E \) of the subspaces representing the elements of \( \mathcal{G}(C_{O_1}) \); the type of an element is the rank of the corresponding (elementary abelian) subgroup and the incidence relation is defined by the inclusion. By the construction, \( \mathcal{G}(F_1) \) is a tilde geometry of rank 5 in which the residue of a point (that is, of an element of type 1) is isomorphic to \( \mathcal{G}(C_{O_1}) \). Also, we see that the points in \( \mathcal{G}(F_1) \) are naturally identified with the involutions in \( F_1 \) conjugated with \( t \) and the lines are formed by a class of Klein four subgroups with the incidence relation given by inclusion. So we have

**Lemma 8.1.** The Monster group \( F_1 \) is a representation group of the geometry \( \mathcal{G}(F_1) \).

Let \( \nu \) be the vector in the Leech lattice \( \Lambda \) stabilized by the subgroup \( C_{O_2} \) and involved in the definition of \( \mathcal{G}(C_{O_2}) \). Then \( \nu \) corresponds to an isotropic vector in \( \overline{\Lambda}_{24} \) and hence to an elementary abelian subgroup of order four in \( E \). Let \( \nu \) be an involution in this group which is not in the centre of \( E \). Then the centralizer \( B \) of \( \nu \) in \( F_1 \) is a non-split extension \( 2 \cdot F_2 \) of the Baby Monster group by a centre of order 2. The centralizer of \( \nu \) in \( E \) is clearly an index 2 subgroup \( E_\nu \) and \( E_\nu/Z \) is isomorphic to \( \overline{\Lambda}_{23} \) as a \( GF(2) \)-module for \( C_{O_2} \). The elements of the geometry \( \mathcal{G}(C_{O_2}) \) are realized as certain subspaces in \( \Lambda_{23} \) (forming a subgeometry in \( \mathcal{G}(C_{O_1}) \)). The preimages of these subspaces in \( E_\nu \) are elementary abelian and the normalizer in \( B \) of such a preimage still induces its full automorphism group (cf. [Ivn2]). The elements of \( \mathcal{G}(F_2) \) are the images in \( B \) of the subgroup \( \langle t \rangle \) and of all the preimages in \( E_\nu \) of the subspaces in \( \Lambda_{23} \cong E_\nu/\langle t \rangle \) realizing elements of \( \mathcal{G}(C_{O_2}) \). Then \( \mathcal{G}(F_2) \) with respect to the usual type function and the incidence relation becomes a Petersen type geometry in which the residue of a point is isomorphic to \( \mathcal{G}(C_{O_2}) \). Also, it is clear that \( \mathcal{G}(F_2) \) is a subgeometry in \( \mathcal{G}(F_1) \) and the representation of the latter as in Lemma 8.1 restricted to \( \mathcal{G}(F_2) \) gives a representation whose images generate the subgroup \( B \cong 2 \cdot F_2 \).

**Lemma 8.2.** The non-split extension \( 2 \cdot F_2 \) of the Baby Monster group \( F_2 \) is a representation group of the geometry \( \mathcal{G}(F_2) \).

So we can identify the point set of \( \mathcal{G} \) with a conjugacy class of involutions in \( B = 2 \cdot F_2 \) which maps bijectively onto a conjugacy class of (central) involutions in \( F_2 \). Let us study the local structure of the collinearity graph \( \Gamma \) of \( \mathcal{G} \).

We need some properties of the graph \( \Gamma \) and of the action of \( B \) on this graph. These properties are probably well known to specialists but we provide their proofs for the sake of completeness.
For a vertex $x$ of $\Gamma$ (considered both as a point of $\mathcal{S}$ and as an involution in $B$) put $B(x) = C_B(x)$ and $Q(x) = O_2(B(x))$. Then $B(x) \cong 2^{1+23}.Co_2$ and $Q(x)/\langle x \rangle$ is isomorphic to $\overline{\Lambda}_{23}$ as a $GF(2)$-module for $Co_2$. Let $y$ be a point collinear to $x$. Then $y \in Q(x)$ and the image of $y$ in $Q(x)/\langle x \rangle \cong \overline{\Lambda}_{23}$ is a vector representing a point in the residue of $x$, isomorphic to $\mathcal{S}(Co_2)$. We will assume that $y$ maps onto the point $p$ which corresponds to the standard frame of $\Lambda$. As usual $\Gamma(x)$ will denote the set of vertices adjacent to $x$ in $\Gamma$. The lines of $\mathcal{S}$ passing through $x$ define on $\Gamma(x)$ an equivalence relation with classes of size 2 and these classes are naturally indexed by the points in the residue of $x$. In particular we obtain

**Lemma 8.3.** For a vertex $x$ of $\Gamma$ we have $|\Gamma(x)| = 93,150$.

The group $Q(x)$ permutes the vertices of $\Gamma(x)$ inside the equivalence classes while the factor group $B(x)/Q(x)$ induces on the set of classes a primitive action of $Co_2$ on the cosets of its subgroup $2^{1+23}.Aut M_{22}$. The primitivity of the action implies that the elements in any two equivalence classes are permuted by $Q(x)$ independently. So by using notation introduced before Lemma 5.3 we have the following

**Lemma 8.4.** The subgroup $B(x) \cap B(y)$ has exactly five orbits $\Gamma_i$, $0 \leq i \leq 4$, on the set $\Gamma(x) - \{y\}$ so that $z \in \Gamma_i$ if and only if the image of $z$ in $Q(x)/\langle x \rangle \cong \Lambda_{23}$ is contained in $\Delta_i$.

We observed that $B(y) \cap Q(x)$ has index 2 in $Q(x)$ and that the image of $B(x) \cap B(y)$ in $B(x)/Q(x)$ is isomorphic to $2^{1+23}.Aut M_{22}$. Since $Q(x') = \langle x \rangle$, the intersection $Q(x) \cap Q(y)$ is elementary abelian and hence its image in $Q(x)/\langle x \rangle \cong \overline{\Lambda}_{23}$ is totally singular with respect to the bilinear form $F$. This implies that $|Q(x) \cap Q(y)| \leq 2^{13}$ and we will see soon that the bound is attained.

Let us decide which points from $\Gamma(x)$ are adjacent to $y$. The unique vertex $y'$ from $\Gamma_0$, which forms together with $x$ and $y$ a line of $\mathcal{S}$, is clearly adjacent to $y$. Note that Lemma 8.4 implies that for $i \geq 1$ a vertex from $\Gamma_i$ is adjacent to $y'$ if and only if it is adjacent to $y$. Let $z \in \Gamma_1$. Then the images of $y$ and $z$ in $\overline{\Lambda}_{23}$ are collinear points of $\mathcal{S}(Co_2)$ and the construction of $\mathcal{S}$ implies that $x, y$, and $z$ generate in $B$ a subgroup of order $2^3$ which represents an element of type 3 in $\mathcal{S}$. Hence $y$ and $z$ are collinear. So the vertices from $\{x\} \cup \{y\} \cup \{y'\} \cup \Gamma_1$ are contained in $Q(x) \cap Q(y)$. By Lemma 5.9(i) the involutions from $\Gamma_1$ generate a subgroup of order $2^{11}$ modulo the order 4 subgroup $\langle x, y \rangle$. Hence in view of the remark made in the previous paragraph we have the following

**Lemma 8.5.** The subgroup $Q(x) \cap Q(y)$ has order $2^{13}$ and is generated by the involutions from $\{x\} \cup \{y\} \cup \{y'\} \cup \Gamma_1$. 

This lemma together with Lemma 5.9(ii, iii) implies that a point \( z \in \Gamma_i \) for \( i > 1 \) is never contained in \( Q(y) \) and hence it is not adjacent to \( y \). Moreover, since we know the unique bilinear form \( F \) on \( \Lambda_{23} \) preserved by \( Co_2 \) we conclude that \( \Gamma_4 \) is even not contained in \( B(y) \) while \( \Gamma_2 \) and \( \Gamma_3 \) are. So a vertex \( z \in \Gamma_i \) for \( i = 2 \) or \( 3 \) commutes with \( y \) and is at distance 2 from \( y \) in \( \Gamma \). We are going to show that \( \Gamma_2 \) and \( \Gamma_3 \) are in different orbits of \( B(y) \).

**Lemma 8.6.** Let \( z \in \Gamma_i \) for \( i = 2 \) and 3. Then the orbit of \( z \) under \( Q(y) \) has length \( 2^7 \) and \( 2^8 \), respectively.

**Proof.** The image of \( B(x) \cap B(y) \) in \( B(x)/Q(x) \) is naturally identified with the stabilizer \( G(p) \cong 2^{10} \). Aut \( M_{22} \) of the point \( p \) in \( G \cong Co_2 \). Since \( B(x) \cap Q(y) \) has order \( 2^{23} \) and \( Q(x) \cap Q(y) \) has order \( 2^{13} \) by Lemma 8.5, we see that the image of \( B(x) \cap Q(y) \) in \( B(x)/Q(x) \) coincides with \( O_2(G(p)) \). By Lemma 5.3 the orbits of the latter group on \( \Delta_2 \) and \( \Delta_3 \) are of length \( 2^5 \) and \( 2^6 \), respectively. The intersection matrix of the collinearity graph \( \Delta \) of \( \mathcal{S}(Co_2) \) presented in Section 5 shows that for \( q \) in \( \Delta_2 \) or \( \Delta_3 \) there is a vertex \( r \in \Delta_4 \) adjacent to \( q \). Since the relations determined by \( \Delta_2 \) and \( \Delta_3 \) are symmetric, we conclude that there is \( u \in \Gamma_1 \) which does not commute with \( z \) and hence conjugates \( z \) to the third point on the line determined by \( x \) and \( z \). Since \( u \in Q(y) \) the orbits of \( z \) under \( Q(y) \cap B(x) \) have lengths \( 2^6 \) and \( 2^7 \) for the two respective cases. Finally, it is clear that \( y' \) is not adjacent to \( z \), so every element from \( Q(y) - B(x) \) does not stabilize \( z \) and the orbits under \( Q(y) \) are twice as long. 

In view of what we already know the above lemma implies the following.

**Lemma 8.7.** The vertices at distance 2 from \( x \) in \( \Gamma \) commuting with \( x \) form exactly two orbits, say \( \Gamma_1(x) \) and \( \Gamma_2(x) \), under the action of \( B(x) \). The lengths of the orbits of \( Q(x) \) on \( \Gamma_i(x) \) are \( 2^7 \) and \( 2^8 \) for \( i = 1 \) and 2, respectively.

Clearly, every orbit of \( Q(x) \) on \( \Gamma_i(x) \) for \( i = 1 \) or 2 corresponds to a single involution in \( B(x)/Q(x) \cong Co_2 \) and we are going to determine which classes these involutions belong to.

It was implicit in Lemma 8.7 that \( \Gamma_2 \) is contained in \( \Gamma_2^1(y) \) while \( \Gamma_3 \) is contained in \( \Gamma_2^2(y) \). We observed in the proof of Lemma 8.6 that the images of the involutions from \( \Gamma_2 \cup \Gamma_3 \) generate in \( B(y)/Q(y) \) an elementary abelian subgroup of order \( 2^{10} \) normalized by \( 2^{10} \). Aut \( M_{22} \). Now Lemma 5.9(ii) and Lemma 6.5 imply that the images of the involutions from \( \Gamma_2 \) constitute an orbit of length 77 and correspond to involutions of type \( 2.A \) in \( B(y)/Q(y) \cong Co_2 \) and the images of those from \( \Gamma_3 \) constitute an orbit of length 330 and correspond to \( 2.B \)-involutions. Now the straightforward calculations bring us to the following

**Lemma 8.8.** The group \( B(x) \) has two orbits \( \Gamma_1(x) \) and \( \Gamma_2(x) \) on the set of vertices in \( \Gamma_2(x) \) commuting with \( x \). Moreover,
(i) $Q(x)$ has orbits of length $2^7$ on $\Gamma_2^1(x)$, these orbits correspond to involutions of type $2A$ in $Co_2 \cong B(x)/Q(x)$; $|\Gamma_2^1(x)| = 7286400 = 2^7 \cdot [Co_2 : 2^{1+8}.Sp_6(2)]$, and every vertex from $\Gamma_2^1(x)$ is adjacent to 63 vertices in $\Gamma(x)$;

(ii) $Q(x)$ has orbits of length $2^8$ on $\Gamma_2^2(x)$; these orbits correspond to involutions of type $2B$ in $Co_2 \cong B(x)/Q(x)$; $|\Gamma_2^2(x)| = 262310400 = 2^8 \cdot [Co_2 : (2^{1+6} \times 2^4).L_4(2)]$, and every vertex from $\Gamma_2^2(x)$ is adjacent to 15 vertices in $\Gamma(x)$.

By calculating the structure constants of the class product of the central involutions in the Baby Monster group by the GAP system [Sch] we see that there are exactly $1 + 93,150 + 7,286,400 + 262,310,400$ central involutions (naturally identified with the vertices of $\Gamma$) commuting with a given one. By Lemma 8.3 and Lemma 8.8 we obtain the following.

**Lemma 8.9.** Let $x$ and $z$ be vertices of $\Gamma$ which commute as involutions in $B \cong 2 \cdot F_2$, then $z$ is contained in $\{x\} \cup \Gamma(x) \cup \Gamma_2^1(x) \cup \Gamma_2^2(x)$. In particular, the distance between $x$ and $z$ in $\Gamma$ is at most 2.

By Lemma 8.4 and 8.9 we see that $B(x)$ acts transitively on the vertices from $\Gamma_2(x)$ which do not commute with $x$ and we have the following.

**Lemma 8.10.** Two distinct vertices of $\Gamma$ commute if and only if they are either adjacent or at distance two and are joined by more than one path of length 2.

Let $z \in \Gamma_2^2(x)$ and $\overline{z}$ be the image of $z$ in $B(x)/Q(x)$. Then $\overline{z}$ can be considered as an element of the rightmost type in the residue of $x$ in $\mathcal{S}$, isomorphic to $\mathcal{S}(Co_2)$. By Lemma 5.5 we see that the image of $\Gamma(x) \cap \Gamma(z)$ in $Q(x)/\langle x \rangle \cong \overline{\Lambda}_{23}$ is the orbit $\Psi_{1}$ of length 15 of the action of the stabilizer $G(\overline{z})$ of $\overline{z}$ in $G \cong Co_2$. As above let $y$ be the vertex from $\Gamma(x)$ whose image in $Q(x)/\langle x \rangle$ is the point $p$ which corresponds to the standard frame in the Leech lattice. By Lemma 6.17 $\overline{z}$ is contained in $G(p)$ if and only if $p$ is contained in the union $\Psi_1 \cup \Psi_2 \cup \Psi_3$ of the orbits of $G(\overline{z})$ on the point set of $\mathcal{S}(Co_2)$ naturally identified with the orbits of $Q(x)$ on $\Gamma(x)$. Suppose that $\overline{z}$ is contained in $G(p)$. By the last sentence of Lemma 5.5, in view of the paragraph before Lemma 8.5, $y$ is adjacent in $\Gamma$ to at least 3 vertices from $\Gamma(z) \cap \Gamma(x)$ and hence $z$ commutes with $y$ by Lemma 8.10. Now we are ready to prove Proposition 7.2(ii).

**Lemma 8.11.** The vertices of $\Gamma$ commuting with both $x$ and $y$ generate in $B$ the subgroup $B(x) \cap B(y)$.

**Proof.** The vertices $\{x\} \cup \{y\} \cup \bigcup_{i=0}^{3} \Gamma_i$ are contained in $B(y)$ and generate in $B$ the subgroup $B(y) \cap Q(x)$ (which is of index 2 in $Q(x)$). By the paragraph before the lemma if $z \in \Gamma_2^2(x)$ and the image $\overline{z}$ of $z$ in
\[ G = B(x)/Q(x) \cong Co_2 \] is contained in \( G(p) \) then \( z \in B(y) \). By Lemma 6.16 these images generate \( G(p) \cong 2^{10}.\text{Aut} \, M_{22} \), and the result follows.

Now we are going to prove Proposition 7.2(i) and this will complete the proof of Theorem 1 for the geometry \( \mathcal{F}(F_2) \).

Let \( R = R(\mathcal{F}) \) be the universal representation group of \( \mathcal{F} \equiv \mathcal{F}(F_2) \) and for a point \( x \) of \( \mathcal{F} \) let \( r_x \) denote the image of \( x \) in \( R \). Let \( R_x \) be the subgroup in \( R \) generated by the \( r_y \) for all \( y \in \Gamma(x) \). By Lemma 2.5 \( R_x/\langle r_x \rangle \) is a representation group of the residue of \( x \) in \( \mathcal{F} \) (isomorphic to \( \mathcal{F}(Co_2) \)). By Proposition 5.7 \( R_x/\langle r_x \rangle \) must by a factor of \( \Lambda_{23} \) and hence \( R_x \) maps isomorphically onto \( Q(x) \). Now by Lemma 2.3 we see that if \( u \) and \( v \) are vertices of \( \Gamma \) with distance less than 2 then \( r_u \) and \( r_v \) commute and if the distance is 2 then \( [r_u, r_v] \) is contained in \( \langle r_w \rangle \) for very vertex \( w \) adjacent to both \( u \) and \( v \).

Let \( z \in \Gamma_2^i(x) \). We are going to show that \( r_z \) normalizes \( R_x \). By the above paragraph we see that if \( u \) is contained in \( \Gamma(x) \cap \Gamma(z) \) or adjacent to a vertex from this intersection then the commutator \([r_z, r_u]\) is contained in the subgroup generated by the \( r_y \) for \( y \in \Gamma(x) \cap \Gamma(z) \); in particular, the commutator is contained in \( R_x \). So \( r_z \) normalizes the subgroup \( T \) of \( R_x \) generated by the \( r_y \) for all \( y \in \Gamma(x) \) which is at distance at most 1 from \( \Gamma(x) \cap \Gamma(z) \). By Lemma 8.8 the image of \( \Gamma(x) \cap \Gamma(z) \) in \( Q(x)/\langle x \rangle \equiv \Lambda_{23} \) is the point set of a \( C_3(2) \)-subgeometry in \( \mathcal{F}(Co_2) \) stabilized by \( 2^{1+8}.\text{Sp}_6(2) \). By Lemmas 8.4 and 8.5 a vertex from \( \Gamma(x) \) is at distance at most 1 from \( \Gamma(x) \cap \Gamma(z) \) if and only if its image in \( \Lambda_{23} \) is at distance at most 1 from the subgeometry in the collinearity graph \( \Delta \) of \( \mathcal{F}(Co_2) \). Clearly \( r_z \) commute with \( r_z \). So by Lemma 5.13 \( T = R_x \) and hence \( r_z \) normalizes \( R_x \).

By the end of Lemma 5.9(ii) for a vertex \( y \in \Gamma(x) \) and for \( i = 1 \) or 2 the image of \( Q(y) \) in \( B(x)/Q(x) \) is generated by the images of the vertices from \( \Gamma(y) \cap \Gamma_2^i(x) \). This implies that for \( u \in \Gamma_2^z(x) \) the element \( r_u \) is contained in the subgroup of \( R \) generated by the \( r_v \) for all \( v \in \{x\} \cup \Gamma(x) \cup \Gamma_2^z(x) \) and in particular \( r_u \) normalizes \( R_x \) as well.

Let \( R_x^1 \) be the subgroup of \( R \) generated by the \( r_u \) for all points \( u \) commuting with \( x \) in \( B \), that is, from \( \{x\} \cup \Gamma(x) \cup \Gamma_2^1(x) \cup \Gamma_2^2(x) \). Then as we have shown, \( R_x \) is normal in \( R_x^1 \) and the factor group \( \overline{R}_x = R_x^1/R_x \) is generated by the images \( \overline{r}_z \) in this factor group of the elements \( r_z \) for \( z \in \Gamma_2^2(x) \). These generators are indexed by the vertices of the derived graph of the geometry \( \mathcal{F}(Co_2) \), considered as the residue of \( x \) in \( \mathcal{F} \). For every vertex \( y \in \Gamma(x) \), the generators \( \overline{r}_z \) for \( z \) taken from \( \Gamma(y) \cap \Gamma_2^2(x) \) generate by Lemma 5.9(ii) the (universal) derived group of the geometry \( \mathcal{F}(M_{22}) \) which is elementary abelian of order \( 2^{10} \). This means that \( \overline{R}_x \) is a derived group of \( \mathcal{F}(Co_2) \) and hence must be isomorphic to \( Co_2 \) by Proposition 6.1. Thus we have proved that \( R_x \) maps isomorphically onto
\( B(x) \cong 2^{1+23}.\text{Co}_2 \). This proves Proposition 7.2(i) and hence completes the proof of Theorem 1 for the geometry \( \mathcal{G}(F_2) \).

9. THE GEOMETRY \( \mathcal{G}(F_1) \)

In this section we prove Theorem 1 for the geometry \( \mathcal{G}(F_1) \) using its validity for the geometry \( \mathcal{G}(F_2) \) and the fact that the latter geometry is a subgeometry in the former one.

Let \( \mathcal{G} = \mathcal{G}(F_1) \), \( M \cong F_1 \), and \( R = R(\mathcal{G}) \) be the universal representation group of \( \mathcal{G} \). We can identify the point set of \( \mathcal{G} \) with a conjugacy class of \( (2B\text{-type}) \) involutions in \( M \) and there is a homomorphism \( \psi: R \rightarrow M \) under which the set of generations \( r_x \) of \( R \) maps bijectively onto the conjugacy class of \( 2B \)-involutions in \( M \). The \( \mathcal{G}(F_2) \)-subgeometries in \( \mathcal{G} \) are in a bijection with the \( 2A \)-involutions in \( M \), centralized by \( 2 \cdot F_2 \). If \( t \) is such an involution then the points of the subgeometry corresponding to \( t \) commute with \( t \) (as involutions in \( M \)). Note that only some of the \( 2B \)-involutions commuting with \( t \) are contained in the corresponding subgeometry, although they generate the whole centralizer \( C(t) = C_M(t) \cong 2 \cdot F_2 \). Since Theorem 1 is proved for the geometry \( \mathcal{G}(F_2) \), we see that for every \( 2A \)-involution \( t \) in \( M \) there is a subgroup \( R(t) \) in \( R \) which maps isomorphically onto \( C(t) \). We will show that this forces \( \psi \) to be an isomorphism using some geometric arguments which generalize in a certain sense Lemma 7.1.

The Monster graph \( \Theta \) has involutions of type \( 2A \) in \( M \) as vertices and two such involutions are adjacent if their product is again a \( 2A \)-involution. The Monster group has two orbits on the set of triangles of \( \Theta \). The triangles in these orbits are called in [IVn4] short and long, respectively. The involutions form a short triangle product to the identity element and such a triangle is never contained in a complete 4-vertex subgraph in \( \Theta \).

Define \( \mathcal{R}(M) \) to be a rank 4 geometry whose elements of type 1, 2, 3, and 4 are the vertices, the edges, the long triangles, and the complete 4-vertex subgraphs in \( \Theta \), respectively, with the incidence relation given by inclusion. The following result is probably well known and a proof can be found in [IVn4].

**Lemma 9.1.** The geometry \( \mathcal{R}(M) \) is residually connected, and the Monster group \( M \) acts on this geometry flag-transitively. If \( t \) and \( s \) are adjacent vertices of \( \Theta \) then \( C(t) \cap C(s) \) is a perfect group isomorphic to \( 2^2 \cdot 2E_6(2) \).

It is implicit in the above lemma that the stabilizer in \( M \) of an element of type 4 induces the symmetric group \( S_4 \) on the set of vertices incident to this element.

It was shown in [AS] that the fundamental group of the Monster graph is generated by all its triangles and in [IVn4] it was proved that the set of long
triangles is still enough for the generation. This is equivalent to the following

**Lemma 9.2.** The truncation of the geometry $M$ by the elements of type 4 is simply connected.

Let $a_1$, $a_2$, and $a_3$ be pairwise incident elements from $M$ of types 1, 2, and 3, respectively, and let $M(a_i)$ be the stabilizer of $a_i$ in $M$. Then Lemma 9.2 implies the following.

**Lemma 9.3.** Suppose that $R$ is generated by certain of its subgroups $R(a_1)$, $R(a_2)$, and $R(a_3)$ such that $R(a_i)$ maps isomorphically onto $M(a_i)$ under $\psi$ and $R(a_i) \cap R(a_j)$ maps isomorphically onto $M(a_i) \cap M(a_j)$ for all $1 \leq i, j \leq 3$. Then $\psi$ is an isomorphism.

We will reconstruct in $R$ the required subgroups $R(a_i)$ using arguments similar to that in the proof of Lemma 7.1.

Let $t$ and $s$ be adjacent vertices in $\Theta$. Then the corresponding $\mathcal{F}(F_2)$-subgeometries intersect by a truncation of the $E_6(2)$-building [Ivn2], so we have

**Lemma 9.4.** Let $t$ and $s$ be adjacent vertices of $\Theta$. Then the intersection $R(t) \cap R(s)$ maps isomorphically onto $C(t) \cap C(s)$ under $\psi$.

Let $a_4$ be an element of type 4 in $M$ which is incident to the $a_i$ for $1 \leq i \leq 3$ and let $r$ be a point incident to $a_4$ but not to $a_3$. For a vertex $t$ of $\Theta$ we denote by $\psi_t$ the restriction of $\psi$ to $R(t)$ (which is now known to be an isomorphism onto $C(t)$). Let $M_0(a_i)$ be the kernel of the action of $M(a_i)$ on the set of vertices of $\Theta$ incident to $a_i$. We define $R(a_i)$ to be the subgroup in $R$ generated by $\psi_{a_i}^{-1}(M_0(a_i))$ and $\psi_r^{-1}(M(a_i) \cap C(r))$. Then the arguments exactly as in the proof of Lemma 7.1 show that the $R(a_i)$ satisfy the conditions formulated in Lemma 9.3. Hence $\psi$ is an isomorphism and the Monster group $M \cong F_1$ is the universal representation group of $\mathcal{F}(F_1)$.

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**REFERENCES**


A long section of the Sunda megathrust south of the great tsunamigenic earthquakes of 2004 and 2005 is well advanced in its seismic cycle and a plausible candidate for rupture in the next few decades. Our computations of tsunami propagation and inundation yield model flow depths and inundations consistent with sparse historical accounts for the last great earthquakes there, in 1797 and 1833. Numerical model results from plausible future ruptures produce flow depths of several meters and inundation up to several kilometres inland near the most populous coastal cities. Our models of historical and future tsunamis...