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Affine Extensions of the Petersen Graph and 2-Arc-Transitive Graphs of Girth 5

DMITRII V. PASECHNIK

Let $Q_n$ denote the set of the flag-transitive geometries with the diagram

$$
P \quad \text{AG}(n, 3) \quad 0 \quad 0 \quad 0
$$

We construct examples of finite $Q_n$-geometries for any $n \geq 2$. This allows us to obtain new examples of finite 2-arc-transitive graphs of girth 5, containing Petersen subgraphs. Using coset enumeration, we classify all $Q_2$-geometries.

1. Introduction

Flag-transitive geometries $Q_n$ with the diagram

$$
P \quad \text{AG}(n, 3) \quad 0 \quad 0
$$

are considered in this paper. Examples of finite $Q_n$-geometries are constructed for any $n \geq 2$. In particular, we obtain new examples of finite 2-arc-transitive graphs of girth 5, containing Petersen subgraphs (cf. [4], [5] for an enquiry into this class of graphs). Furthermore, we give a complete classification of $Q_2$-geometries using coset enumeration. In particular, we show that they are all finite.

Eight flag-transitive geometries with the diagram

$$
P \quad L
$$

have been discovered to date (see [4, 7]). All of them arise from simple sporadic or 'almost' sporadic groups.

A. A. Ivanov [4] found a relationship between 2-arc-transitive graphs of girth 5 and flag-transitive geometries with the diagram

$$
P \quad L
$$

In particular, such a geometry can always be constructed starting from a 2-arc-transitive graph of girth 5 without simple 6-circuits, which is not a pentagraph. It follows from the classification of doubly transitive permutation groups that the linear spaces appearing in the stroke

$$
L
$$

in this context are isomorphic either to $PG(n, 2)$ or to $AG(n, 3)$ (cf. [4]). A. A. Ivanov and S. V. Shpectorov are studying the former case intensively (cf., e.g., [6, 7, 9]).

Our terminology concerning diagram geometries is fairly standard, (see, e.g., [1]).
denotes the geometry of vertices and edges of the Petersen graph, and

\[ \text{AG}(n, 3) \]

denotes the point-line geometry of \( \text{AG}(n, 3) \). The types of elements of a geometry \( \mathcal{G} \) are numbered 0, 1, 2 from left to right. The set of elements of \( \mathcal{G} \) is denoted by \( S \), and the set of elements of type \( i \) is denoted by \( S_i \). \( \sigma_i(s) \) denotes the set of elements of type \( i \) incident with \( s \in S \); this set is called the \( i \)-shadow of \( s \). A flag is a set of pairwise incident elements. The type of a flag \( f \) is the set of types of elements belonging to \( f \).

The collinearity graph \( \Gamma = \Gamma(\mathcal{G}) \) is the graph with the vertex-set \( V = V(\Gamma) = S_0 \), where two vertices are joined by an edge if both of them are incident with an element of type 1. \( \Gamma(v) \) (resp. \( \Gamma'(v) \)) denotes the set of vertices lying at the distance (resp. at most) \( i \) from \( v \). If \( X \subseteq V \) and \( F \leq \text{Aut}(\Gamma) \), then \( F(X) \) (resp. \( F\{X\} \)) denotes the pointwise (resp. setwise) stabilizer of \( X \) in \( F \). The complete graph with \( n \) vertices is denoted by \( K_n \). A graph \( \Gamma \) is called \( s \)-transitive if it admits an automorphism group \( G \) acting transitively on the \( s \)-paths, but does not admit a group transitive on \( (s + 1) \)-paths. A group \( G \) as above is said to act \( s \)-transitively on \( \Gamma \).

Let \( G \) be a flag-transitive subgroup of \( \text{Aug}(\mathcal{G}) \), and let \( f \) be a maximal flag of \( \mathcal{G} \). The stabilizer in \( G \) of the flag \( f^i \subseteq f \) is called \( i \)-shadow, and is denoted by \( H_i \). A subgroup \( H_i \), where \( i \) is a type of \( \mathcal{G} \), is called a \textit{maximal} parabolic. The stabilizer of \( f \) is called a Borel subgroup of \( G \), and is denoted by \( B \). The maximal parabolics form the amalgam \( \mathcal{A} = \mathcal{A}(\mathcal{G}) \). \( \mathcal{G} \) can be reconstructed from \( G \) and \( \mathcal{A} \); namely, the elements of type \( i \) are the (left) cosets of \( H_i \) in \( G \), and two elements are incident if they have non-trivial intersection as cosets.

The latter fact gives us a motivation for the following notation: we denote \( \mathcal{G} \) by \( \mathcal{G}(G, \mathcal{A}) \), or by \( \mathcal{G}(G) \) if \( \mathcal{A} \) is evident from the context. The universal group \( F(\mathcal{A}) \) of an amalgam \( \mathcal{A} \) is the group generated by the generators of \( \mathcal{A} \) and by the relations of \( \mathcal{A} \). The universal cover of \( \mathcal{G} = \mathcal{G}(G, \mathcal{A}) \) is \( \mathcal{G}(F(\mathcal{A}), \mathcal{A}) \). Note that the latter fact was proved independently by J. Tits [11], A. Pasini [8] and S. V. Shpectorov (unpubl.).

Our group-theoretic notation coincides with the notation from [2].

The rest of the paper is organized as follows. In Section 2 we consider general properties of \( \Gamma(Q_n) \) and \( \mathcal{A}(Q_n) \). In Section 3 we present the unique example of \( Q_2 \)-geometry known up to now, and construct new examples of \( Q_n \)-geometries and corresponding graphs of girth 5. In Section 4 all \( Q_2 \)-geometries are classified.

2. Some Properties of Amalgams and Collinearity Graphs

Let \( \mathcal{G} \) be a flag-transitive \( Q_n \)-geometry, \( \Gamma = \Gamma(\mathcal{G}), \mathcal{A} = \mathcal{A}(\mathcal{G}) \).

**Lemma 2.1.** \( \Gamma \) is a regular connected graph of valence 3. There is a bijection between the edge-set \( E = E(\Gamma) \) of \( \Gamma \) and the set of elements of type 1 of \( \mathcal{G} \). The elements of type 2 correspond to certain partial subgraphs of \( \Gamma \), isomorphic to the Petersen graph.

We warn the reader that in several cases some partial subgraphs of \( \Gamma \), isomorphic to the Petersen graph, do not correspond to elements of type 2 of \( \mathcal{G} \).

By Lemma 2.1, it is natural to call elements of types 0, 1 and 2 \textit{vertices}, \textit{edges} and \textit{subgraphs} respectively (we will not consider other kinds of subgraphs).

Let \( P \) be the geometry of vertices and edges of the Petersen graph.

**Lemma 2.2.** Let \( T \leq \text{Aut}(P) \) act flag-transitively on \( P \). Then \( T = A_5 \) (or \( S_5 \)). \( T \) acts 2- (or 3-) transitively on \( \Gamma(P) \).
This implies the following:

**Lemma 2.3.** $G$ acts $s$-transitively on $\Gamma$ for some $s \geq 2$.

The following fact is well known.

**Lemma 2.4.** Let $p$ be a point of $A \cong AG(n, 3)$. Then the stabilizer $F$ in $\text{Aut}(A)$ of all lines through $p$ has order 2 and acts semiregularly on the points of $A$ distinct from $p$.

We turn to the structure of $G(v)$, $v \in V(\Gamma)$.

**Lemma 2.5.** For any $v \in V(\Gamma)$ the group $G(\Gamma^\prime(v))$ acts on $\Gamma_2(v)$ either trivially or as a fixed-point-free group of order 2.

**Proof.** Let $1 \neq g \in G(\Gamma^\prime(v))$, $e \in \sigma_1(v)$, $\{v, u\} = \sigma_0(e)$. Observe that $g$ acts trivially on $\sigma_2(v)$. Hence, considering the action of $g$ on $\text{res}(u)$ and applying Lemma 2.4, we have that $g$ acts on $\Gamma_1(u) \setminus \{v\}$ either fixed-point-freely or trivially.

Let us show that the latter implies that $g \in G(\Gamma^\prime(v))$. Indeed, since the automorphism of the Petersen graph fixing a 3-path is trivial, $g$ fixes the subgraphs through $e$ pointwise. Since, for any $p \in \Gamma_1(v)$, there is a subgraph through $p$ and $e$ then, applying Lemma 2.4 again, we have $g \in G(\Gamma^3(p))$. Therefore $g \in G(\Gamma^2(v))$, and we are done.

Next, observe $g \in G(\Sigma^\prime(v))$ for each $\Sigma \in \sigma_3(v)$. Such a non-trivial automorphism of $\Sigma$ is unique and of order 2. Therefore, if $g, g'$ are two elements of $G(\Gamma^\prime(v))$ acting non-trivially on $\Gamma^2(v)$, then $gg'$ acts trivially on $\Sigma$, whence it acts trivially on $\Gamma^2(v)$.

**Lemma 2.6.** $G(\Gamma^2(v))$ is trivial. The order of $G(\Gamma^\prime(v))$ is less or equal to 2.

**Proof.** It is sufficient to prove that $G(\Gamma^2(v)) \leq G(\Gamma^\prime(v))$. Let $u \in \Gamma_1(v)$. Note that $G(\Gamma^2(v)) \leq G(\Gamma^\prime(u)) = H$ acts on $\Gamma_2(u)$ either trivially or as a fixed-point-free group of order 2. On the other hand, since the intersection of $\Gamma_2(u)$ and $\Gamma^2(v)$ is non-empty, $H$ fixes some vertex from $\Gamma_2(u)$. Therefore the above action is trivial. The first statement is proved. The second one now follows from Lemma 2.5.

It is appropriate to depict an amalgam $A$ as follows:

```
      H_2
     /   \
H02   H12
     \   /
      H_0 B H_1
     / \   / \ \\
H01 -- H0 -- H1
```

In general, this scheme does not determine $A$ up to isomorphism. Let $A$ correspond to a flag $\{x_0, x_1, x_2\}$, $x_i \in S_i$, $K = G(\Gamma^\prime(x_0))$.

**Lemma 2.7.**

(i) $[H_0; B] = (3^n - 1)/2$, $[H_02; B] = 3$, $[H_{12}; B] = 2$, $[H_0; H_0] = 3^n$, $[H_1; H_0] = 2$.

(ii) $H_0/K$ contains an elementary abelian normal 3-subgroup acting regularly on $\sigma_1(x_0)$.

The action of $H_0/K$ on $\sigma_1(x_0)$ is doubly transitive.

(iii) $H_2$ acts on $\text{res}(x_2)$ either as $A_5$ or as $S_5$. 

Theorem. Part (i) follows from basic properties of diagram geometries. Part (ii) follows from Lemma 2.3 and the list of doubly transitive groups. Part (iii) follows from Lemma 2.2.

3. Examples of $Q_n$-geometries

3.1. $Q_2$-geometries

First, we consider examples of $Q_2$-geometries. We give information which is necessary for their classification (see Section 4) as well as information of some independent interest, e.g. concerning automorphism groups, (universal) coverings, etc.

Example 1 (due to A. A. Ivanov [4]). We give a construction of the geometry in terms of its collinearity graph $\Gamma$. The construction of $\Gamma$ is due to F. L. Tchuda [10], who has considered the problem of a combinatorial construction for the corresponding primitive permutation representation of the group $L_3(4)$. Let us take as the vertices of $\Gamma$ the unitals $H_2(2^2)$ (equivalent to designs, consisting of the absolute points of a given unitary polarity $\pi$ and non-absolute lines of $\pi$) embedded in the projective plane of order 4: two vertices are adjacent iff the point sets of the corresponding unitals have a one-element intersection. Note that $H_2(2^2) \cong A G(3, 3)$. In what follows we prefer to call these objects affine planes. $\Gamma$ is a primitive distance-transitive graph with the following intersection array:

$$
\begin{array}{ccccccc}
\text{1} & \text{9} & \text{1} & \text{8} & \text{1} & \text{2} & \text{3} \\
\text{9} & \text{1} & \text{8} & \text{1} & \text{72} & \text{6} & \text{144} \\
\text{1} & \text{2} & \text{3} & \text{3} & \text{3} & \text{8} & \text{54} \\
\end{array}
$$

In [4] it is pointed out that the set $\Phi$ of subgraphs of $\Gamma$ isomorphic to the Petersen graph has the following property: the edges and the subgraphs through a fixed vertex form $A G(2, 3)$, with the incidence defined by inclusion. Therefore $V(\Gamma), E(\Gamma)$ and $\Phi$ form a $Q_2$-geometry, which we call $\mathcal{L}$. In fact, the subgraphs in $\Phi$ bijectively correspond to the antiflags (non-incident point-line pairs) of $P G(2, 4)$ in the following way: if $\Xi \in \Phi$ corresponds to the antiflag $(P, l)$, where $P$ is a point and $l$ is a line of $P G(2, 4)$, then an affine plane $A$ is in a vertex of $\mathcal{L}$ iff $l$ contains some line of $A$ consisting of the points, $P_1, P_2, P_3$, say, and each line $(P, P_i)$ of $P G(2, 4)$ intersects $A$ in the unique point $P_i (i = 1, 2, 3)$.

$A u t(\mathcal{L})$ contains the unique minimal flag-transitive subgroup $G \cong L_3(4)$. $\mathcal{A} = \mathcal{A}(G)$ is described by the following scheme:

$$
\begin{array}{c}
A_5 \\
S_3 \\
2^2 \\
3^2: Q_8 \\
2 \\
Q_8 \\
Q_8 \times 2
\end{array}
$$

It can be shown that, up to isomorphism, $F(\mathcal{A})$ is the unique group generated by any amalgam described by this scheme. To find the universal cover $\bar{\mathcal{L}}$ of $\mathcal{L}$, it is sufficient to calculate $\bar{G} = F(\mathcal{A})$. This was accomplished by the author using coset enumeration. In fact, $| \bar{G} : H_0 | = 4480$. Let $Z(H_0) = \langle z \rangle$. The group $\bar{F} = \langle z^G \rangle$ has index 4 in $\bar{G}$. The group $F \leq G$, defined in the same manner, substituting $G$ for $\bar{G}$, coincides with $G$. Hence $\bar{G} = \langle [4] \cdot G \rangle \cdot 2^2$. For further properties of $\bar{F}$ see Section 4.
REMARK 1. Other minimal flag-transitive subgroups $F(\mathcal{A}^i)$, $i = 1, 2$, of $\text{Aut}(\mathcal{L})$ exist in addition to $\mathcal{L}$. The groups $F(\mathcal{A}^i)$, as well as $F(\mathcal{A}) = \mathcal{L}$, are uniquely determined by schemes of their amalgams which differ from the scheme of $\mathcal{A}$ as follows: $H_0(\mathcal{A}^i) = 3^2: 8$, $H_{01}(\mathcal{A}^i) = 8$ and $H_1(\mathcal{A}^i) = \langle x, y \mid x^3, y^2, x^2 = x^{a(0)} \rangle$, where $a(1) = 1$ and $a(2) = 5$.

EXAMPLE 2. Consider the group $G = \text{PGL}_2(9)$ in its natural doubly transitive action on the set $\Omega$, $|\Omega| = 10$. $G$ contains the class $F$ of conjugated subgroups isomorphic to $A_5$, $|F| = 12$. Denote $S_0 = \Omega$, $S_1 = \{\{x, \beta\} \mid x, \beta \in \Omega, x \neq \beta\}$, $S_2 = F$. Define the incidence between elements from $S_0$ and $S_1$ by inclusion, and state that all elements of $S_0$ are incident with all elements of $S_2$. An element $H$ from $S_2$ is incident with an element $\{a, b\}$ from $S_1$ iff $H$ contains an involution with the set $\{a, b\}$ of fixed points. We denote this geometry by $\mathcal{M}$. $G$ acts on $\mathcal{M}$ flag-transitively. $\mathcal{A} = \mathcal{A}(G)$ has the following scheme:

It can be shown that, up to isomorphism, $F(\mathcal{A})$ is the unique group generated by any amalgam determined by this scheme. To find $\mathcal{M}$ it is sufficient to calculate $\mathcal{M} = F(\mathcal{A})$. This was accomplished by the author using coset enumeration. In fact, $|S_0| = |\mathcal{L}| = 10,800$. Let $Z(H_{01}) = \langle z \rangle$. The group $F = \langle z^G \rangle$ has index 2 in $\mathcal{L}$. For further properties of $F$, see Section 4. Observe that $\mathcal{M}$ is the first member of the series of $Q_n$-geometries defined below. It follows from the next subsection that $\mathcal{M}$ has a flag-transitive quotient $\mathcal{M}'$ with $|S_0'| = 3600$, and $G' \cong (\text{PSL}_2(9) \times \text{PSL}_2(9)) \rtimes \text{Aut}(\mathcal{M}')$ acts flag-transitively on $\mathcal{M}'$. Hence $\mathcal{M} \cong (3 \cdot \text{PSL}_2(9) \rtimes 3 \cdot \text{PSL}_2(9)) : 2$, where $\rtimes$ denotes the central product.

REMARK 2. Other minimal flag-transitive subgroups $F(\mathcal{A}^i)$, $i = 1, 2$, of $\text{Aut}(\mathcal{L})$ exist in addition to $\mathcal{L}$. The groups $F(\mathcal{A}^i)$, as well as $F(\mathcal{A}) = \mathcal{L}$, are uniquely determined by schemes of their amalgams, which differ from the scheme of $\mathcal{A}$ in the case $i = 1$ in $H_1 = \langle z, f \mid z^f = z = 1, z^3 = z^5 \rangle$, whereas if $i = 2$ in $H_0 = (3, Q_8)$, $H_{01} = Q_8$ and $H_1 = Q_8: 2$. Note that $\mathcal{A}^2$ can be embedded to the group $M_{10}$ (the point-fixer in the Mathieu group $M_{11}$).

3.2. $Q_n$-geometries

PROPOSITION 3.1. A finite $Q_n$-geometry exists for every $n \geq 2$.

PROOF. The proof is a generalization of the construction of the geometry $\mathcal{M} = \mathfrak{M}(\text{PGL}_2(9))$, given in Section 3.1, for the arbitrary field $GF(3^n)$, $n \geq 2$. Consider the following scheme of amalgam $\mathcal{A}_n$:
It is easy to show that, for any \( n \geq 2 \), this group \( F(M_n) \) is determined uniquely up to isomorphism by any amalgam having the above scheme. Note that \( F(M_2) \cong F(\mathcal{A}(M)) \).

Let us show that \( F(M_n) \) is non-trivial for each \( n > 2 \). We will prove this by constructing a group \( G_n \) such that an amalgam with the scheme \( \mathcal{A}_n \) is realizable inside \( G_n \). We warn the reader that \( G_n \) is a homomorphic image of \( F(\mathcal{A}_n) \) and that \( G_n = F(\mathcal{A}_n) \) need not be hold in general (for instance, we have \( G_2 \neq F(\mathcal{A}_2) \)).

Consider the following elements of the group \( \text{PGL}_2(3^m) \), where \( m = n \) if \( n \) is even, and \( m = 2n \) otherwise:

\[
x = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad z_t = \begin{pmatrix} 0 & \epsilon^t \\ 1 & 0 \end{pmatrix},
\]

where \( \epsilon^2 = -1 \), \( t \in \{1, 3\} \) and \( \omega \) is a primitive element of \( GF(3^m) \).

**Lemma 3.2.** \( \mathcal{A}_n \) can be embedded in \( G = G_n = \langle x, y, b, z_t \rangle \leq \text{PGL}_2(3^m) \) as follows:

\( B = \langle b \rangle \), \( H_0 = \langle x \rangle \), \( H_0 = \langle b, y \rangle \) and \( H_1 = \langle b, z_t \rangle \). We have \( G \cong \text{PGL}_2(3^m) \) if \( n \) is even; otherwise \( G \cong \text{PSL}_2(3^{2n}) \).

**Proof.** It is straightforward to check that the above generators generate the parabolics that we want. It follows from the list of maximal subgroups of \( \text{PSL}_2(q) \) (see [3]) that \( G \) is isomorphic to the group mentioned above. Indeed, \( H_0 \) is contained in the unique maximal subgroup of \( G \), which does not contain \( z_t \).

Finally, if \( n \) is odd it is straightforward to check that \( x, y, b \) and \( z_t \), considered as elements of a projective group, can be presented by matrices with a determinant equal to 1. Indeed, since \( b = x^{3^t-1} \) and \( y \) is the product of \( b \) and a transvection, it suffices to check the above for \( x \) and \( z_t \), i.e. we need only check that \( \det(x) = \omega \) and \( \det(z_t) = -\epsilon^t \) are squares in \( GF(3^{2n}) \). Let \( \alpha \) be a primitive element of \( GF(3^m) \) such that \( \omega = \alpha^{3^m+1} \); hence \( \omega = (\alpha^{3^{(3^t-1)/2}-1})^2 \). Next, \( -\epsilon^t \) is also a square, since \( 3^{2n} - 1 \) is divisible by 8 and \( -\epsilon^t \) has order 4.

Lemma 3.2 also proves Proposition 3.1.

If \( n \) is even we are able to prove that \( G_n \) is not the universal group for \( \mathcal{A}_n \).

**Lemma 3.3.** If \( n \) is even, then \( \mathcal{A}_n \) can be realized inside the group \( X_n \), where \( (L_2(3^n) \times L_2(3^n)) \leq X_n \leq (\text{PGL}_2(3^n) \times \text{PGL}_2(3^n)) \).

**Proof.** Consider the subgroup \( H = \langle (x, x), (y, y), (b, b), (z_1, z_3) \rangle \) in the group \( \text{PGL}_2(3^n) \times \text{PGL}_2(3^n) \). Define a mapping \( \phi: H \rightarrow G \) (\( G \) is defined in Lemma 3.2) by the following rules:

\( \phi((x, x)) = x \), \( \phi((y, y)) = y \), \( \phi((b, b)) = b \) and \( \phi((z_1, z_3)) = z_t \).

Evidently \( G \) is a homomorphic image of \( H \). Let us show that \( G \cong H \). Indeed, consider \( w = z_t y x^{(3^t-1)/2} \). Using the fact that \( w \) is defined over \( GF(9) \), it is easy to check that the order of \( w \) equals 3 if \( t = 1 \) and equals 4 if \( t = 3 \). Hence \( H \) contains some non-trivial elements with trivial projections onto one of the direct multipliers. Using well known facts concerning direct products of simple groups, we deduce that \( H \) contains \( L_2(3^n) \times L_2(3^n) \). Finally, the restriction of \( \phi \) to any parabolic is an isomorphism.

Denote \( Y = X_n \) if \( n \) is even and \( Y = G_n \) otherwise (see Lemma 3.3 (resp. Lemma 3.2) for the definition of \( X_n \) (resp. \( G_n \))).
**Proposition 3.4.** \( \Gamma = \Gamma(\mathcal{G}(Y, \mathcal{A}_n)) \) has girth 5.

**Proof.** Since \( \Gamma \) is not the complete graph, it follows from Lemma 2.3 that \( \Gamma \) is triangle-free. Suppose that \( \mathcal{A}_n \) corresponds to the flag \( \{v, e, \Xi\} \), where \( v \in S_0 \), \( e \in S_1 \) and \( \Xi \in S_2 \). The absence of quadrangles is equivalent to \( |I_2(v)| = 3^n(3^n - 1) \). The latter fact immediately follows from the next lemma.

**Lemma 3.5.** \( H_0 \) acts regularly on \( I_2(v) \).

**Proof.** \( H_0 \) acts transitively on \( I_2(v) \). Hence if suffices to prove that each non-trivial element acts fixed-point-freely on \( I_2(v) \).

For any \( g \in H_0 \) the inclusion \( v^g \in I_2(v) \) holds. Let us show that there exists \( g \in H_2 \) such that \( H_0 \cap H_0^g = 1 \).

**Case of \( n \) even.** Let

\[
g = (z, z^3)(y, y)(z, z^3) = \begin{pmatrix} 1 & 0 \\ \epsilon & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -\epsilon & -1 \end{pmatrix} \end{pmatrix}.
\]

In this case

\[
H_0 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\} \mid a, b \in GF(3^n), a \neq 0 \}
\]

Hence

\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \end{pmatrix}^g = \begin{pmatrix} a + b\epsilon & -b \\ a\epsilon - b - \epsilon & 1 + \epsilon b \end{pmatrix}, \begin{pmatrix} a - b\epsilon & -b \\ a\epsilon - b - \epsilon & 1 + \epsilon b \end{pmatrix} \in H_0
\]

iff \( a\epsilon - b - \epsilon = \epsilon - a\epsilon - b = 0 \), i.e. \( a = 1, b = 0 \).

**Case of \( n \) odd.** Let

\[
g = yz = \begin{pmatrix} 1 & 0 \\ \epsilon & -1 \end{pmatrix}.
\]

In this case

\[
H_0 = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \right\} \mid a, b \in GF(3^n), a \neq 0 \}
\]

So

\[
\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^g = \begin{pmatrix} a + b\epsilon & -b \\ a\epsilon - b - \epsilon & 1 + \epsilon b \end{pmatrix} \in H_0
\]

iff \( a\epsilon - b - \epsilon = 0 \), i.e. \( b = \epsilon(a - 1) \). Hence either \( a = 1, b = 0 \) or \( b \notin GF(3^n) \), a contradiction.

Since \( v^g \in I_2(v) \), then \( Y(v) \cap Y(v^g) = 1 \). \( \square \)

Proposition 3.4 is also proved, since \( |H_0| = |I_2(v)| = 3^n(3^n - 1) \). \( \square \)

**Remark 3.** The graph \( \Gamma = \Gamma(\mathcal{G}(Y, \mathcal{A}_n)) \) admits an automorphism \( f \) fixing \( I_1(v) \) pointwise: namely, if \( n \) is even, \( f \) interchanges the direct multipliers of \( G \); if \( n \) is odd, \( f \) is a field automorphism of order 2.
4. Classification of the $Q_2$-geometries

**Theorem.** $\mathcal{D}$ and $\mathcal{M}$ are the only simply connected $Q_2$-geometries.

**Proof.** Suppose that $\mathcal{G}$ is a simply connected $Q_2$-geometry, $G = \text{Aut}(\mathcal{G})$, \{v, e_1, \Xi\} is a flag of $\mathcal{G}$, where $v \in S_0$, $e_1 \in S_1$ and $\Xi \in S_2$, and $\mathcal{A} = \mathcal{A}(\mathcal{G})$ is the corresponding amalgam.

Enumeration of the possible amalgams $\mathcal{A}$, as in, e.g., [9], is not so useful in our case since there are many non-isomorphic amalgams associated with isomorphic geometries (cf. Remarks 1 and 2).

We have chosen a different method as follows:

**Step 1:** construction of a certain subgroup $F \leq G$. The group $F$ need not be flag-transitive. However, it is vertex- and edge-transitive on the collinearity graph of $\mathcal{G}$.

**Step 2:** enumeration of the possible presentations $\mathcal{F}_i$ of $F$ such that the defining relations of $\mathcal{F}_i$ follow from relations of $\mathcal{F}_1$.

**Step 3:** using coset enumeration, identification of the graphs $\Gamma(\mathcal{F}_i)$ defined by $\mathcal{F}_i$ with the collinearity graphs of either $\mathcal{D}$ or $\mathcal{M}$. Since the analogue of the group $F$ inside $F(\mathcal{A}(\mathcal{D}))$ or $F(\mathcal{A}(\mathcal{M}))$ is a homomorphic image of one of the $\mathcal{F}_i$, $i = 1, 2$, this finishes the proof.

**Step 1.** The following well known technical lemma improves Lemma 2.4 in our particular case.

**Lemma 4.1.** Let $A = AG(2, 3)$, and let $T \leq \text{Aut}(A)$ act flag-transitively on $A$. Then for any point $p$ of $A$ the subgroup $T_p \leq T$, stabilizing all lines through $p$, has order 2 and acts semiregularly on the other points of $A$. If $p'$ is a point distinct from $p$ then $(T_p, T_{p'}) \cong S_3$.

Let $\sigma_0(e_1) = \{v, v'\}$. Let $Z$ be the kernel of action of $H_{01}$ on $\sigma_2(e_1)$.

**Lemma 4.2.** If $G(\Gamma^i(v)) \neq 1$, then $Z = \langle G(\Gamma^i(v)), G(\Gamma^i(v')) \rangle \cong 2^2$. Otherwise $Z \cong 2$. In both cases $Z$ contains the unique involution $y = y(e_1)$ acting semiregularly on both $\sigma_1(v) \setminus \{e_1\}$ and $\sigma_1(v') \setminus \{e_1\}$.

**Proof.** If $K = G(\Gamma^i(v)) = 1$, then the statement follows from Lemma 4.1. According to the notation of that lemma, $T_{e_1} = \langle y \rangle = Z$ (e_1 is a point of res(v) $\cong AG(2, 3)$). Otherwise let $K = G(\Gamma^i(v)) = \langle x \rangle$, $K' = G(\Gamma^i(v')) = \langle x' \rangle$, $x, x' \in H_{01}$ are involutions generating $K$ and $K'$ (cf. Lemma 2.6). Evidently, both $K$ and $K'$ are subgroups of $Z$. Moreover, we have $K \neq K'$, since, e.g., can consider $\Xi \in \sigma_2(e_1)$, restricting $Z$ on $\sigma_0(\Xi)$, and see that it is so. $y = xx'$ acts trivially on $\sigma_2(e_1)$, as both $x$ and $x'$ do so. Hence $\langle y \rangle$ acts on res(v) as $T_{e_1}$ from Lemma 4.1. Therefore $x$ and $x'$ commute, and $y^2 = 1$. We have proved that $Z \cong \langle K, K' \rangle \cong 2^2$.

Note that the action of $y$ is as required. Finally, suppose that there exists $z \in Z \setminus \langle K, K' \rangle$. If the action of $z$ on the sets under consideration is not the same as the action of $y$, then $z \in K$ or $z \in K'$, a contradiction. Hence we have both $z = y(\text{mod } K)$ and $z = y(\text{mod } K')$. This forces $z = yx$ and $z = xy'$. Hence $x = x'$, a contradiction. □

In what follows the involution $y(e)$ for $e \in S_1$ will be denoted by $e$.

Define a group $F = \langle e \mid e \in S_1 \rangle \leq G$. The following useful lemma is evident.

**Lemma 4.3.** Let $\mathcal{O}$ be an orbit of a subgroup $H$ of $G$ on $S_1$. Then the set $\{e \mid e \in \mathcal{O}\}$ is also an orbit of $H$, acting on it by conjugation.
It is worth noting that this does not imply, in general, that there is a bijection between $\mathcal{O}$ and \{e \mid e \in \mathcal{O}\}.

**Step 2.** Let $e_2 \in \sigma_1(v) \cap (\sigma_1(\mathcal{E}_1) \setminus \{e_1\})$.

**Lemma 4.4.** $S = \langle e_1, e_2 \rangle \leq H_{02}$ acts faithfully on both $\text{res}(\mathcal{E}_1)$ and $\text{res}(v)$. In particular, $S \cong S_3$.

**Proof.** Since $\mathcal{E}_i = \mathcal{E}_1$, $i = 1, 2$, then $S \cong H_{02}$. If $K = 1$ then, by Lemma 4.1, we are done. Moreover, by that lemma we are done if $S \cap K = 1$. Otherwise, i.e., $1 \neq K$ is normal in $S$, we have $S/K = S_3$. On the other hand, since $e_1$ and $e_2$ are conjugated in $H_{02}$, $S$ acts on $\text{res}(\mathcal{E}_1)$ as $S_3$. By Lemmas 2.5 and 2.6, $S$ acts on $\text{res}(\mathcal{E}_1)$ as $S_3 \times 2$, a contradiction. \[\square\]

For $\Omega \in \sigma_2(e_1)$ define $O_{\Omega} = \{w \mid w \in \sigma_1(\Omega), d_{\Omega}(\sigma_0(e_1), \sigma_0(w)) = 2\}$, where $d_{\Omega}$ denotes the distance in the subgraph $\Omega$.

**Lemma 4.5.** Let $X = X(\mathcal{E}_1) = \langle e \mid e \in \sigma_1(\mathcal{E}_1) \rangle$. Then $X = \langle e_1, e_2, e_3 \rangle = A_5 \leq H_2$, where $e_1$ and $e_2$ are as in Lemma 4.4 and $e_3 \in O_{\mathcal{E}_1}$.

**Proof.** Since $\mathcal{E}_i = \mathcal{E}_1$ for any $e \in \sigma_1(\mathcal{E})$, we have $X \leq H_2$. It is easy to prove that $\langle e_1, e_2, e_3 \rangle$ acts vertex- and edge-transitively on $\text{res}(\mathcal{E})$. Hence, by Lemmas 4.3 and 4.4, we have $X = \langle e_1, e_2, e_3 \rangle$. Suppose that $e = e_{01}$. Since $e_3$ stabilizes $e_1$, by the bijection between $\sigma_1(\mathcal{E})$ and $e_1'$, using Lemma 4.3 we have $e = e_1$, i.e. $(e_1 e_3)^2 = 1$. The relation $(e_2 e_3)^2 = 1$ is equivalent to $(e_2 e_3)^2 = e_3$, which holds by Lemma 4.3 and the fact that the image of $e_2$ under the action of $(e_2 e_3)^2 = e_3$. By Lemma 4.4 we have $(e_2 e_3)^3 = 1$. We have proved that $X$ is a homomorphic image of the Coxeter group $A_5 \times 2$.

On the other hand, the action of $X$ on $\text{res}(\mathcal{E}_1)$ is similar to the action of $A_5$. To complete the proof it suffices to prove that $(e_1 e_2 e_3)^2 = 1$, or, equivalently $e_3 = (e_1 e_3)^{e_1 e_2 e_3}$. The latter follows from $e = e_1 e_2 e_3 \in e_1'$, and we turn to the proof of this containment.

Let $O_{\mathcal{E}_1} = \{e_3, e_3'\}$. It is easy to see that $\langle e_1, e_3, e_3' \rangle$ is isomorphic to either $2^2$ or $2^3$. In the former case we are done. Let us show that the latter case cannot happen, i.e. $y = e_1 e_2 e_3' = 1$. Since $y$ acts trivially on $\text{res}(\mathcal{E}_1)$, $y$ acts faithfully on $\sigma_2(e_1)$. The latter action coincides with the action of $z = e_3 e_3'$. Consider this more precisely. Since $|\sigma_2(e_1)| = 4$ and $z$ stabilizes $\mathcal{E}_1$, $z$ stabilizes some other subgraph $\mathcal{E}$ from $\sigma_2(e_1)$. As $e_3$ and $e_3'$ stabilize $\mathcal{E}_1$, one of them acts on $\sigma_2(e_1)$ trivially, but if both of them act trivially then $y = 1$. So, if $y \neq 1$ then $e_3$ and $e_3'$ are not conjugated in $H_1$. To rule out this case we will use an argument of a 'global' nature.

We will show that $\mathcal{A}$ contains a subamalgam $\mathcal{A}'$, such that $F(\mathcal{A}) = F(\mathcal{A}')$: $\langle y \rangle$ and $F(\mathcal{A}')$ acts flag-transitively on $\mathcal{A}$. We will see that this implies $y = 1$. Denote by $L$ the kernel of the action of $H_2$ on $\text{res}(\mathcal{E}_1)$. Since a non-trivial $K$ interchanges $e_3$ and $e_3'$, $K$ is trivial owing to the non-conjugacy of $e_3$ and $e_3'$ in $H_1$. Consequently, $L$ acts faithfully on $\text{res}(v, e_1)$. This, in particular, implies that $L \leq S_3$, i.e. either $L = \langle y \rangle$ or $L \cong 3: \langle y \rangle$. Hence $H_2 \cong A_5 \times L$. Since $X$ is normal in $H_2$ and $X \cap L = \langle y \rangle$, we have $L = \langle y \rangle$. Next, $H_{01} = T \cdot L$, where $T \cong Q_8$ acts flag-transitively on $\text{res}(v)$. We have proved that $\mathcal{A}$ has
the following scheme:

```
A_5 x L  
\downarrow \downarrow \downarrow
S_3 x L  \quad 2^2 x L
\uparrow \quad \quad \downarrow \downarrow
T: L  \quad T: (L x 2)
```

To see \( \mathcal{A}' \) on this picture, delete \( L \). Of course, \( F(\mathcal{A}) = F(\mathcal{A}') = \langle y \rangle \). On the other hand, \( F(\mathcal{A}') \) contains \( \{ e \mid e \in \mathcal{S}_1 \} \). Thus \( y = 1 \), i.e. \((e_1 e_2 e_3)^5 = 1\).

**Lemma 4.6.** Let \( Y = \langle e \mid e \in \sigma_1(v) \rangle \). Then \( Y = \langle e_1, e_2, e_3 \rangle \cong \mathbb{Z}_2^2 : \mathbb{Z}_2 \leq H_0 \), where \( e_1 \) and \( e_2 \) are as in Lemma 4.4 and \( e_3 \in \sigma_1(v) \setminus \sigma_1(\mathcal{S}_1) \).

**Proof.** Observe \( Y \leq H_0 \). Using Lemma 4.4 and the fact that \( \text{res}(v) \) can be generated by a triangle, we obtain \( Y = \langle e_1, e_2, e_4 \rangle \). Again by Lemma 4.4, \((e_i e_j)^3 = 1\) for \( i, j \in \{1, 2, 4\}, \ i \neq j \). It suffices to show that \( y = [e_1 e_2, e_1 e_4] = 1 \). This relation holds in the factor group of \( \langle Y, K \rangle \) by \( K \); hence \( y \in K \). Therefore \( y \) commutes with the generators of \( Y \). So, if \( y \neq 1 \) the order of \( z = e_1 e_2 y \) equals 6. Since \((e_1 e_4)^{e_2 e_3} = z \), we have \( y = 1 \).

Let \( \sigma_2(e) = \{ \Xi_1, \Xi_2, \Xi_3, \Xi_4 \} \). Note that \( e_3 \) acts on \( \sigma_2(e) \) (non-faithfully, in general). Hence, without loss of generality, we can choose \( \Xi_2 \in \sigma_2(e) \) such that \( \mathbb{Z}_2 \approx \Xi_2 \) and \( e_4 \in \sigma_1(\Xi_2) \).

**Lemma 4.7.** Let \( U = \langle e \mid e \in O_{\Xi}, i = 1, \ldots, 4 \rangle \). Then \( U = \langle e_1, e_3, e_5 \rangle \leq H_1 \), where \( e_3 \) is as in Lemma 4.5 and \( e_5 \in O_{\Xi} \). Moreover, \( e_5 \) can be chosen such that either:
(i) \( e_5 e_3 = 1 \) and \( U \cong \mathbb{Z}_2 \); or
(ii) \( e_1 = (e_3 e_5)^2, e_2^{e_5 e_3} = e_4 \) and \( U \cong D_8 \).

**Proof.** It is easily seen that the generators of \( U \) stabilize \( e_1 \) (indeed, any of them interchanges the two vertices of \( e_1 \)). Hence we have \( U \leq H_1 \). On the other hand, \( H_{01} \) acts transitively on \( \{O_{\Xi}, i = 1, \ldots, 4\} \). Hence, by Lemma 4.3, \( H_{01} \) acts on the generators of \( U \) by conjugation and has at most two orbits in this action. It follows from Lemma 4.5 that \( e_1 \in U \) (indeed, if \( \{e_3, e_3'\} = O_{\Xi} \), then \( (e_2, e_2') \cong \mathbb{Z}_2^2 \) and \( e_1 = e_3 e_3' \)). Hence, as a generating set for \( U \) we can take \( e_1 \) and one element from each of \( O_{\Xi}, i = 1, \ldots, 4 \). First, let us prove a technical statement.

**Statement 1.** Let \( \Xi, \Xi' \in \sigma_2(e), e \in O_{\Xi}, e' \in O_{\Xi'} \) and the actions of \( e \) and \( e' \) on \( \sigma_2(e_1) \) coincide. Then either \( e = e' \) or \( e = e'e_1 \).

**Proof.** Consider \( t = ee' \). Evidently, \( t \) acts trivially on \( \sigma_2(e_1) \), i.e. \( t \in Z \) (see the definition of \( Z \) immediately before Lemma 4.2). If \( t = 1 \) or \( e_1 \) we are done. Without loss of generality, by Lemma 4.2 \( (t) = G(\Gamma^q(v)) \). Then \( (t e_1) = G(\Gamma^q(v')) \). On the other hand, since \( e \) interchanges \( v \) and \( v' \), we have \( t e_1 = e t e = t \). Hence \( e_1 = 1 \), a contradiction.

It follows from the flag-transitivity that if for some \( i = 1, \ldots, 4 \) and \( e \in O_{\Xi_i} \) \((i = 1, \ldots, 4) \) the action of \( e \) on \( \sigma_2(e_1) \) is trivial, then the same holds for any \( i = 1, \ldots, 4 \) and \( e \in O_{\Xi_i} \). Hence, by Statement 1, the first case of our lemma holds.
Thus we can assume that all elements from \{e \mid e \in O_{\Xi}, i = 1, \ldots, 4\} act on \(\sigma_2(e_i)\) non-trivially.

Next, we prove the following:

**Statement 2.** \(U\) acts on \(\sigma_2(e_i)\) as a 2-group.

**Proof.** Suppose false. This implies that \(U\) acts (at least) doubly transitively on \(\sigma_2(e_i)\). Let \(e \in O_{\Xi_1} \cup O_{\Xi_2}\). Since there exists an involution \(h \in U\) interchanging \(\Xi_1\) and \(\Xi_2\), then either \(e = e_3^h\) or \(e_1 e_3 = e_3^h\). Hence \(e \in \{e_1, e_3\}\). Next, if \(e_5\) (recall that \(e_5 \in O_{\Xi_4}\)) stabilizes \(\Xi_1\) (or \(\Xi_2\)), then owing the existence of an involution \(h^t \in U\) interchanging \(\Xi_3\) and \(\Xi_1\) (or \(\Xi_2\)) we obtain that either \(e_3 = e_3^h\) or \(e_3 e_4 = e_3^h\). Hence, by Statement 1 applied to \(e_3\) and \(e\), the fixed points of \(e_3\) and \(e_4\) in their action on \(\sigma_2(e_i)\) coincide, contradiction. Similarly to the beginning of this proof, we obtain that if \(e' \in O_{\Xi_5} \cup O_{\Xi_4}\) then \(e' \in \{e_1, e_3\}\). Thus \(U = \langle e_1, e_3, e_5 \rangle\) and does not act doubly transitively on \(\sigma_2(e_i)\), a contradiction.

Let \(e \in O_{\Xi_1} \cup O_{\Xi_2}\). By Statement 2, the fixed points of \(e_3\) and \(e\) coincide in their action on \(\sigma_2(e_i)\). Applying Statement 1, we obtain \(e \in \{e_1, e_3\}\). Again by Statement 2, \(e_5\) fixes \(\Xi_4\). Similarly to the case of involutions corresponding to \(\Xi_1\) and \(\Xi_2\), for \(e' \in O_{\Xi_5} \cup O_{\Xi_4}\) we obtain that \(e' \in \{e_1, e_3\}\). Thus \(U = \langle e_1, e_3, e_5 \rangle\). Denoting \(t = e_3 e_5\) and observing \(t \in H_{01}\), we obtain \(t^2 = e_1\). Thus \(U = D_8\). Finally, either \(e_2 = e_4\) and 2) holds, or \(e_2 = e_4\). In the latter case we can consider, instead of \(e_5\), the other member \(e_3\) of \(O_{\Xi_2}\). Substituting \(e_5\) for \(e_4\) we obtain \(e_2 = e_4\). Thus (ii) holds as well.

Let \(e \in O_{\Xi_1} \cup O_{\Xi_2}\). By Statement 2, the fixed points of \(e_3\) and \(e\) coincide in their action on \(\sigma_2(e_i)\). Applying Statement 1, we obtain \(e \in \{e_1, e_3\}\). Again by Statement 2, \(e_5\) fixes \(\Xi_4\). Similarly to the case of involutions corresponding to \(\Xi_1\) and \(\Xi_2\), for \(e' \in O_{\Xi_5} \cup O_{\Xi_4}\) we obtain that \(e' \in \{e_1, e_3\}\). Thus \(U = \langle e_1, e_3, e_5 \rangle\). Denoting \(t = e_3 e_5\) and observing \(t \in H_{01}\), we obtain \(t^2 = e_1\). Thus \(U = D_8\). Finally, either \(e_2 = e_4\) and 2) holds, or \(e_2 = e_4\). In the latter case we can consider, instead of \(e_5\), the other member \(e_3\) of \(O_{\Xi_2}\). Substituting \(e_5\) for \(e_4\) we obtain \(e_2 = e_4\). Thus (ii) holds as well.

Now we are able to write down the possible presentations \(\hat{F}_i\) for \(F\), such that their defining relations follow from relations of \(\mathcal{A}\).

**Lemma 4.8.** \(\hat{F}_i = \langle \text{involutions e}_1, \ldots, e_5 \mid (e_1 e_2)^3 = (e_1 e_4)^3 = (e_2 e_4)^3 = [e_1 e_2, e_1 e_3] = (e_1 e_3)^3 = (e_1 e_2 e_3)^3 = 1, (e_4 e_3)^3 = (e_1 e_4 e_3)^3 = (e_1 e_3)^3 = (e_1 e_4 e_3)^3 = (e_1 e_4 e_3)^3 = 1, (e_5 e_4 e_3 e_5)^3 = 1, (e_5 e_4 e_3 e_5)^3 = 1 \rangle, \) where \(\mathcal{R}_1 = \{e_3 \in \Xi_3\}, \mathcal{R}_2 = \{(e_3 e_5)^2 = e_1, e_2 e_5 e_3 = e_4\}\).

**Proof.** By Lemmas 4.5, 4.6 and the connectness of \(G\), we obtain \(F = \langle e_1, \ldots, e_5 \rangle\). The first four relations hold in \(Y\), which is defined in Lemma 4.6, while the next three relations hold in \(X(\Xi)\), which is defined in Lemma 4.5. Note that by Lemma 4.7 for some \(e \in O_{\Xi}\) we have \(e e_3 = 1\). Since \(e_4 \in \sigma_1(\Xi_2), \) Lemma 4.5, applied to \(\Xi_2\), gives us the next two relations. Without loss of generality, we can suppose that \(e_4 e_3 \in \sigma_1(\Xi_2) \cap \sigma_1(\Xi_4)\), whereas \(e_4 e_3 \in \sigma_1(\Xi_4) \cap \sigma_1(\Xi_2)\). For \(\Xi_3\) and \(\Xi_4\), the involution \(e_5\) plays the same role as \(e_3\) plays for \(\Xi_1\) and \(\Xi_2\). Hence the next three relations hold in \(X(\Xi_3)\), and two last but one hold in \(X(\Xi_4)\). Finally, \(\mathcal{R}_i, i = 1, 2\), are taken from Lemma 4.7 (i) and (ii) respectively.

**Step 3.** Observe that \(F\) acts vertex- and edge-transitively on \(G\). Hence \(G\) is a 2-orbit of \(F\) in its action on the cosets of \(F(v) < F\) in \(F\). Therefore, to complete the proof of the Theorem, it suffices to show the faithfulness of the presentations of \(G\) given in Lemma 4.8, since \(\Gamma(\hat{F})\) and \(\Gamma(\mathcal{A})\) uniquely determine \(\mathcal{F}\) and \(\mathcal{A}\) respectively. As shown above, \(V(\hat{F}) = \langle e_1, e_2, e_4, e_3, e_5 \rangle \cong F(v)\). Using a coset enumeration, we obtain \(|\hat{F}_1 : V(\hat{F})| = 4480\) and \(|\hat{F}_2 : V(\hat{F})| = 10\,080\). Recalling the description of the corresponding subgroups in the groups of automorphisms of the known \(Q_2\)-geometries (see Section 3.1), we find \(\hat{V}(\hat{F}) = \hat{V}(\hat{F}_i), i = 1, 2\) and \(\Gamma(\hat{F}_1) = \Gamma(\hat{F}_2) = \Gamma(\hat{F}_2) = \Gamma(\mathcal{A})\).

\(\square\)
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