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Constructions and bounds on linear error-block codes

San Ling · Ferruh Özbudak

Abstract We obtain new bounds on the parameters and we give new constructions of linear error-block codes. We obtain a Gilbert–Varshamov type construction. Using our bounds and constructions we obtain some infinite families of optimal linear error-block codes over $\mathbb{F}_2$. We also study the asymptotic of linear error-block codes. We define the real valued function $\alpha_{q,m,a}(\delta)$, which is an analog of the important real valued function $\alpha_q(\delta)$ in the asymptotic theory of classical linear error-correcting codes. We obtain both Gilbert–Varshamov and algebraic geometry type lower bounds on $\alpha_{q,m,a}(\delta)$. We compare these lower bounds in graphs.

Keywords Linear error-block codes · Gilbert–Varshamov type construction · Optimal linear error-block codes · Asymptotic of linear error-block codes · Gilbert–Varshamov bound · Algebraic geometry type bound

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1 Introduction

Let q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. A *partition*, π, of a positive integer n is given by $n = l_1m_1 + l_2m_2 + \cdots + t_tm_t$, where $m_1 > m_2 > \cdots > m_t \geq 1$ and $l_1, \ldots, l_t \geq 1$. This partition is denoted as

$$\pi = [m_1]^{l_1}[m_2]^{l_2} \cdots [m_t]^{l_t}. \tag{1.1}$$

Let $s, n_1, \ldots, n_s$ be the integers given by π as

$$s = l_1 + \cdots + l_t, \quad n_1 = \cdots = n_{l_1} = m_1, \ \cdots \ , n_{l_1+\cdots+l_{t-1}+1} = \cdots = n_s = m_t.$$

Moreover let V_i be the $\mathbb{F}_q$-linear space $\mathbb{F}_q^{n_i}$ for $1 \leq i \leq s$. We have $\mathbb{F}_q^n = \oplus_{i=1}^s V_i$.

For $\boldsymbol{u} = (\boldsymbol{u}_1, \ldots, \boldsymbol{u}_s) \in \mathbb{F}_q^n = \oplus_{i=1}^s V_i$, the π-weight $w_\pi(\boldsymbol{u})$ of $\boldsymbol{u}$ is defined by

$$w_\pi(\boldsymbol{u}) = |\{1 \leq i \leq s : \boldsymbol{u}_i \neq \boldsymbol{0} \in V_i\}|.$$

An $\mathbb{F}_q$-linear subspace C of $\mathbb{F}_q^n$ is called an $[n, k, d]$ *linear error-block code of type* π over $\mathbb{F}_q$ (or a q-ary $[n, k, d]$ linear error-block code of type π) if $\dim_{\mathbb{F}_q}(C) = k$ and

$$d = \min\{w_\pi(\boldsymbol{c}) : \boldsymbol{0} \neq \boldsymbol{c} \in C\}.$$

Here n and d are called the *length* and the *minimum π-weight* of the linear error-block code C. Note that the *minimum π-distance* $d_\pi(C)$ of C, which is $\min\{w_\pi(\boldsymbol{c}-\boldsymbol{c}') : \boldsymbol{c}, \boldsymbol{c}' \in C, \ \boldsymbol{c} \neq \boldsymbol{c}'\}$, is equal to the minimum π-weight of C. If $\pi = [1]^n$, then C is a classical linear error-correcting code over $\mathbb{F}_q$.

Linear error-block codes have applications in experimental designs [4]. Recently some algebraic aspects of linear error-block codes have been studied in [3]. There is another generalization of classical error-correcting codes initiated in [7] and developed, for example, in [1] and [5]. We refer the reader to [3] and the references therein for further information on linear error-block codes and their applications. In the concluding section of [3], a few open problems on error-block codes are given, including, for example, the search for new constructions of nice error-block codes as well as the determination of the parameters of "optimal" error-block codes, etc.

The work in this paper is to some extent motivated by some of the open problems mentioned in [3]. We obtain some new bounds on the parameters and give new constructions of linear error-block codes. We also study the asymptotic of linear error-block codes. While some of the results are analogs of certain properties of classical error-correcting codes, the proofs often require ideas and techniques beyond those used for classical error-correcting codes. More explicitly, our contributions are summarized as follows:

(i) Theorem 3.1 is a new construction of linear error-block codes, using an argument similar to the Gilbert–Varshamov type argument for classical linear error-correcting codes. In this construction, it is necessary to control the relations of different columns in a block of a generating matrix. This situation does not arise in the case of classical linear error-correcting codes, where there are no blocks with more than one column. Some of the simple and preliminary results of Sect. 2 are used in this construction. As an example, we construct an optimal linear error-block code (see Definition 2.8) using Theorem 3.1, which cannot be obtained from [3].

(ii) A general algebraic geometry construction of linear error-block codes was presented in [3,Theorem 5.3], where the construction was used only for genus 0 curves (see [3,Theorem 5.4 and the example of Sect. 5]). In the direction of finding generalizations to error-block codes of new versions of algebraic geometry codes such as XNL codes (see [11]), it is important to demonstrate the usefulness of higher degree places from algebraic curves. In Sect. 3, we construct an optimal linear error-block code using degree two places of an elliptic curve, which cannot be obtained using [3,Theorem 5.4].

(iii) Regarding optimality, only Singleton and Hamming type bounds were proved in [3,Theorem 2.1]. In Corollaries 2.3 and 2.6 (see also Remarks 2.4 and 2.7), we obtain an effective method to apply extensive tables and bounds of classical linear error-correction codes in order to obtain bounds and to decide optimality for linear error-block codes. These bounds are tighter than the earlier known bounds in some cases. Using this method and our constructions mentioned in items (i) and (ii) above, several infinite families of optimal linear error-block codes are obtained.

(iv) After presenting a finite Gilbert–Varshamov type construction and having algebraic geometry type constructions for linear error-block codes, it is natural to consider the asymptotic as the length of the code approaches infinity. The asymptotic of linear error-block codes is more complicated than that of linear classical error-correcting codes, since apart from the ratios of the dimension and the minimum distance to the length, one needs to consider also the "ratio of the partitions" to the length as well. For simplicity, we only consider the asymptotic of a sequence $\{C(i)\}_{i=1}^{\infty}$ of linear error-block codes over $\mathbb{F}_q$ such that, for a fixed integer $m > 0$, their sequence of partitions $\{\pi(i)\}_{i=1}^{\infty}$ is in the form $\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}$, where $n(i) = ms_m(i) + s_1(i)$ and $\lim_{i\to\infty} n(i) = \infty$. This subclass is rich enough to reveal the differences in the asymptotic of linear error-correcting codes and that of linear error-block codes. For such a sequence of linear error-block codes, we also assume the existence of a real number a such that $\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a$, which is not needed for classical linear error-correcting codes. We introduce a real valued function $\alpha_{q,m,a}(\delta)$ (analog of $\alpha_q(\delta)$ for classical linear error-correcting codes) and obtain two asymptotic lower bounds for $\alpha_{q,m,a}(\delta)$ in Theorems 4.5 and 4.10. In Theorem 4.5, the asymptotic Gilbert–Varshamov type bound depends on a new parameter μ which comes from an optimization argument comparing the effects of the different block sizes in the proof of Theorem 4.5. This has no analog in the case of classical linear error-correcting codes. Theorem 4.10 uses an algebraic geometry type construction together with a concatenation to derive an algebraic geometry type lower bound. Therefore, the graph of the bound of Theorem 4.10 consists of two line segments of different slopes and it is not infinitely differentiable. The graph of the bound of Theorem 4.5 is infinitely differentiable. Nevertheless, for many examples (see, for example, Fig. 4), we observe that the graph of Theorem 4.5 also has a similar bent near the intersection of the line segments of the graph of Theorem 4.10. We also observe that, for each of these two lower bounds, there are some parameters at which one lower bound outperforms the other.

The paper is organized as follows. In Sect. 2 we give some preliminaries and also we derive some new bounds. In Sect. 3, we obtain a Gilbert–Varshamov type construction and we construct infinite families of examples of optimal linear error-block codes of some types over $\mathbb{F}_2$. We study the asymptotic of linear error-block codes and we give examples with graphs of the asymptotic bounds in Sect. 4. Throughout the paper, unless stated otherwise, we assume that n is a positive integer with the arbitrary partition π given in (1.1).

2 Preliminaries and some new bounds

In this section we obtain some simple preliminary results that we use later and we define optimal linear error-block codes. Moreover we derive some new bounds on the parameters of linear error-block codes. These bounds improve on some earlier known bounds, such as the Singleton bound, in certain cases.

First we construct some simple codes.

- Simple Code $\mathcal{C}_1$: Let $\mathcal{C}_1 = \{(c, c, \ldots, c) \in \mathbb{F}_q^n : c \in \mathbb{F}_q\}$ be the $\mathbb{F}_q$-linear subspace of dimension 1. It is clear that the π-minimum distance of $\mathcal{C}_1$ is s and hence $\mathcal{C}_1$ is an $[n, 1, s]$ linear error-block code of type π over $\mathbb{F}_q$. It is easy to observe that $\mathcal{C}_1$ is an MDS (maximum distance separable) code in the sense of [3,Definition 2.2] if and only if $m_t = 1$.
- Simple Code $\mathcal{C}_2$: Let $\mathcal{C}_2 = \mathbb{F}_q^n$ be the whole $\mathbb{F}_q$-linear space. For $\boldsymbol{c} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s) \in \mathbb{F}_q^n$ with $\boldsymbol{c}_i \in V_i$ for each $1 \le i \le s$, if $\boldsymbol{c} \ne \boldsymbol{0}$ then there exists $1 \le i \le s$ such that $\boldsymbol{c}_i \ne 0$. It is easy to observe that $\mathcal{C}_2$ is an $[n, n, 1]$ linear error-block code of type π over $\mathbb{F}_q$. Note that $\mathcal{C}_2$ is an MDS code.

There is a restriction on the existence of $[n, k, d]$ linear error-block codes of type π over $\mathbb{F}_q$ with $d \ge 2$.

Lemma 2.1 *Assume that* $1 \le k \le n$. *If* $n - k < n_1$, *then each* k*-dimensional subspace* C *of* $\mathbb{F}_q^n$ *is an* $[n, k, 1]$ *linear error-block code of type* π *over* $\mathbb{F}_q$. *Otherwise if* $n - k \ge n_1$, *then there exists an* $[n, k, d]$ *linear error-block code of type* π *over* $\mathbb{F}_q$ *with* $d \ge 2$.

Proof Assume first that $n - k < n_1$ and let C be a k-dimensional subspace of $\mathbb{F}_q^n$. Let

$$H = [H_1 \; H_2 \cdots H_s]$$

be a parity check matrix of C, where H_i is an $(n - k) \times n_i$ matrix over $\mathbb{F}_q$. As $n - k < n_1$, the columns of H_1 are linearly dependent and there exists $\boldsymbol{c}_1 \in V_1 \setminus \{\boldsymbol{0}\}$ such that $H_1 \boldsymbol{c}_1^{\perp} = \boldsymbol{0}$. Then $\boldsymbol{c} = (\boldsymbol{c}_1, \boldsymbol{0}, \ldots, \boldsymbol{0}) \in C$, $w_\pi(\boldsymbol{c}) = 1$ and hence C is an $[n, k, 1]$ linear error-block code of type π.

Next assume that $n - k \ge n_1$. Let $H = [I \;\; A]$ be an $(n - k) \times n$ matrix such that I is the $(n - k) \times (n - k)$ identity matrix and A is an $(n - k) \times k$ matrix chosen as follows: Let $H_1, \ldots, H_s$ denote the blocks of H corresponding to the partition π such that $H = [H_1, \;\; H_2 \cdots H_s]$. For each $1 \le i \le s$, if H_i is completely in I, then H_i is already determined by I. If H_i is completely out of I, then we choose H_i arbitrarily such that the columns of H_i are linearly independent. Note that as $n - k \ge n_1 \ge n_i$, we can choose an $(n - k) \times n_i$ matrix H_i of rank n_i. If a part of H_i is in I and another part of H is not in I, as the part of H_i in I has linearly independent columns and $n - k \ge n_i$, we can choose the part of H_i not in I such that H_i is a matrix of rank n_i. Therefore we have chosen A and H. We note that the rows of H are linearly independent as I is a part of H. Moreover for each $1 \le i \le s$, the columns of H_i are linearly independent. For any $\boldsymbol{c}_i \in V_i \setminus \{\boldsymbol{0}\}$, as the columns of H_i are linearly independent, we have $H_i \boldsymbol{c}_i^{\perp} \ne \boldsymbol{0}$. Using also the fact that the rows of H are linearly independent, we obtain that the linear code C with the parity check matrix H is an $[n, k, d]$ linear error-block code of type π with $d \ge 2$. □

Let $B(n, d, \pi)_q$ be the maximum $\mathbb{F}_q$-dimension k of an $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$. Clearly, by the Simple Code $\mathcal{C}_2$, $B(n, 1, \pi)_q = n$ for all π. When $\pi = [1]^n$, i.e., in the case of classical linear error-correcting codes, we write $B(n, d, \pi)_q$ simply as $B(n, d)_q$. Recall that we assume the partition π is $\pi = [n_1][n_2] \cdots [n_s]$, where $n_1 \ge n_2 \ge \cdots \ge n_s$. If $d > s$, then by convention we denote $B(n, d, \pi)_q = 0$.

Lemma 2.2 *Let* C *be a* q*-ary* $[n, k, d]$ *linear error-block code of type* π. *Assume that* $d > 1$. *Puncturing the code at the* i*-th block of length* n_i, *i.e., removing all the coordinates belonging to the* i*-th block from all codewords of* C, *we obtain a* q*-ary* $[n', k, d']$ *linear error-block* *code of type* $\pi_i' = [n_1]\cdots[n_{i-1}][n_{i+1}]\cdots[n_s]$*, where* $n' = n - n_i$ *and* $d - 1 \le d' \le d$*. In particular,* $B(n, d, \pi)_q \le B(n', d', \pi_i')_q$*.*

Proof It is clear that the punctured code C_i' is still a linear error-block code, and that its length is $n - n_i$ and it is of type π_i'. Moreover, since $d > 1$, no two codewords in C can become identical after puncturing, hence the dimension of C_i' is still k. Since the minimum π_i'-distance is equal to the minimum π_i'-weight, the distance d' of C_i' must satisfy $d - 1 \le d' \le d$. □

Corollary 2.3 *When* $\pi = [m][1]^l$ *(*$m > 1$*), we have that, for* $d > 1$*,*

$$B(n, d, \pi)_2 \le \begin{cases} l & \text{if } d = 2, \\ l - 1 & \text{if } d = 3, \\ 1 & \text{if } d = l + 1, \\ l - d + 1 & \text{if } 3 < d < l + 1, \\ 0 & \text{if } d > l + 1. \end{cases}$$

Proof Taking $i = 1$, note that $\pi_1' = [1]^l$, so the punctured code is in fact a classical linear error-correcting code of length l. By Lemma 2.2,

$$B(n, d, \pi)_2 \le \max\{B(l, d)_2, B(l, d - 1)_2\}. \tag{2.1}$$

When $d = 2$, the RHS of (2.1) is l, since the existence of the $[l, l, 1]_2$ code implies that $B(l, 1)_2 = l$. When $d = 3$, the RHS of (2.1) is $l - 1$, since the existence of the $[l, l - 1, 2]_2$ code implies that $B(l, 2)_2 = l - 1$. Similarly, the existence of the $[l, 1, l]_2$ code shows that the RHS of (2.1) is 1 when $d = l + 1$. For other values of d, since binary MDS codes do not exist, the RHS of (2.1) is bounded above by $l - d + 1$. □

Remark 2.4 When the exact values of $B(l, d)_2$ and $B(l, d - 1)_2$ are known, the upper bound in Corollary 2.3 can be improved.

Example 2.5 For an integer $m \ge 2$, $n = m + 4$, $d = 2$ and $\pi = [m][1]^4$, Corollary 2.3 shows that $B(m + 4, 4, \pi)_2 \le 1$. Hence there cannot exist an $[m + 4, 2, 4]$ linear error-block code of type $[m][1]^4$ over $\mathbb{F}_2$ for an integer $m \ge 2$. When $m = 2$, we note that, in contrary, there exists a classical $[6, 2, 4]$ linear error-correcting code over $\mathbb{F}_2$.

Now consider $\pi = [n_1][n_2][1]^l$ ($n_2 > 1$), and the block of length n_1 is punctured. Then, for $d > 1$, Lemma 2.2 shows that

$$B(n, d, \pi)_2 \le \max\{B(n_2 + l, d - 1, \pi')_2, B(n_2 + l, d, \pi')_2\}$$

where $\pi' = [n_2][1]^l$. By Corollary 2.3, we have

$$B(n_2 + l, d', \pi')_2 \le \begin{cases} l & \text{if } d' = 2, \\ l - 1 & \text{if } d' = 3, \\ 1 & \text{if } d' = l + 1, \\ l - d' + 1 & \text{if } 3 < d' < l + 1, \\ 0 & \text{if } d' > l + 1. \end{cases}$$

Hence, we obtain:

Corollary 2.6 *When* $\pi = [n_1][n_2][1]^l$ *(*$n_2 > 1$*), we have that, for* $d > 1$*,*

$$B(n, d, \pi)_2 \le \begin{cases} n_2 + l & \text{if } d = 2, \\ l & \text{if } d = 3, \\ l - 1 & \text{if } d = 4, \\ 1 & \text{if } d = l + 2, \\ l - d + 2 & \text{if } 4 < d < l + 2, \\ 0 & \text{if } d > l + 2. \end{cases}$$

Remark 2.7 Using the methods above, it is possible to extend Corollaries 2.3 and 2.6 to more general types of π.

Definition 2.8 An $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$ is called *optimal* if there exists no $[n, k, d_1]$ linear error-block code of type π over $\mathbb{F}_q$ with $d_1 \geq d + 1$.

For the sake of completeness we also define the notion k-optimality as follows: An $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$ is called *k-optimal* if there exists no $[n, k_1, d]$ linear error-block code of type π over $\mathbb{F}_q$ with $k_1 \geq k + 1$. We will mainly consider optimality in the sense of Definition 2.8.

For an integer $m \geq 1$ let $\pi \oplus [m]$ denote the partition of $n + m$ corresponding to $n + m = n_1 + \cdots + n_s + m$, where m is to be rearranged compared to $n_1, \ldots, n_s$. In the next lemma, we prove some propagation rules for linear error-block codes. In its proof we use the puncturing argument of Lemma 2.2. The following lemma is also useful for proving the continuity of the real valued function $\alpha_{q,m,a}(\delta)$ defined in Sect. 4. We note that the proofs of the corresponding spoiling lemmas in [9,Sect. 1.1.4] do not hold in our case and we need some modified arguments for linear error-block codes.

Lemma 2.9 *Assume that there exists an $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$. We have the following:*

(i) *If $k \geq 2$, then there exists an $[n, k - 1, d]$ linear error-block code of type π over $\mathbb{F}_q$.*
(ii) *If $k \geq 1$ and $d \geq 2$, then there exists an $[n, k, d - 1]$ linear error-block code of type π over $\mathbb{F}_q$.*
(iii) *Assume also that n_i is a part in the partition π of n and $n > n_i$. Let $\overline{\pi}$ be the partition of $n - n_i$ satisfying $\pi = \overline{\pi} \oplus [n_i]$. If $k \geq 1$ and $d \geq 2$, then there exists an $[n - n_i, k, d - 1]$ linear error-block code of type $\overline{\pi}$ over $\mathbb{F}_q$.*
(iv) *Assume also that n_i is a part in the partition π of n, $n > n_i$ and $k \geq n_i + 1$. Let $\overline{\pi}$ be the partition of $n - n_i$ satisfying $\pi = \overline{\pi} \oplus [n_i]$. Then there exists an $[n - n_i, k - n_i, d]$ linear error-block code of type $\overline{\pi}$ over $\mathbb{F}_q$.*

Proof Let C be an $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$. We first prove item (i). Let $\boldsymbol{c}$ be a codeword of C of π-weight d. Any $\mathbb{F}_q$-linear subspace C' of C of dimension $k - 1$ containing $\boldsymbol{c}$ is an $[n, k - 1, d]$ linear error-block code of type π over $\mathbb{F}_q$.

Next we prove item (ii). Let i be an integer with $1 \leq i \leq s$ such that there exists a codeword $\boldsymbol{c} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s) \in C$ with $w_\pi(\boldsymbol{c}) = d$ and $\boldsymbol{c}_i \neq \mathbf{0}$. Replacing the i-th block of C by $\mathbf{0}$ we obtain a linear error-block code C' of type π over $\mathbb{F}_q$. It is clear that the minimum π-distance of C' is $d - 1$. As $d \geq 2$, the dimension of C' over $\mathbb{F}_q$ is k.

Then we prove item (iii). By Lemma 2.2 we have an $[n - n_i, k, d']$ linear error-block code of type $\overline{\pi}$ over $\mathbb{F}_q$, where $d - 1 \leq d' \leq d$. If $d' = d$, then using item (ii), which is proved above, we obtain an $[n - n_i, k, d - 1]$ linear error-block code of type $\overline{\pi}$ over $\mathbb{F}_q$.

Finally we prove item (iv). Let G be a $k \times n$ generator matrix for the $[n, k, d]$ linear error-block code C of type π over $\mathbb{F}_q$. Let A be the $k \times n_i$ submatrix of G corresponding to the block n_i. The rank of A is at most n_i. Therefore there exists an $\mathbb{F}_q$-linear subspace $C' \subseteq C$ of dimension $k - n_i$ such that for each codeword $\boldsymbol{c}' = (\boldsymbol{c}_1', \ldots, \boldsymbol{c}_s') \in C'$, the i-th block $\boldsymbol{c}_i'$ is zero. Hence puncturing C' at the i-th block we obtain an $[n - n_i, k - n_i, d']$ linear error-block code of type $\overline{\pi}$ over $\mathbb{F}_q$ such that $d' \geq d$. Using item (ii) we complete the proof. □

3 A Gilbert–Varshamov type construction and some optimal linear error-block codes over $\mathbb{F}_2$

In this section we give a Gilbert–Varshamov type construction in Theorem 3.1. We also obtain an infinite family of optimal linear error-block codes of some types over $\mathbb{F}_2$ in the sense of Definition 2.8. These examples show that our constructions give codes with better parameters than the ones in [3] in some cases.

Theorem 3.1 *For given $n \geq 1$, $1 \leq k \leq n-1$, $d \geq 2$, assume that there exists an $[n, k, d_0]$ linear error-block code of type π over $\mathbb{F}_q$ with $d_0 \geq d$. Let $m \geq 1$ be a given integer. Assume also that the inequality*

$$\sum_{u_1} \cdots \sum_{u_t} \binom{l_1}{u_1} \cdots \binom{l_t}{u_t} \left(q^{m_1}-1\right)^{u_1} \cdots \left(q^{m_t}-1\right)^{u_t} q^{m-1} < q^{n-k} \tag{3.1}$$

holds, where the sum is over the integers $0 \leq u_1 \leq l_1, \ldots, 0 \leq u_t \leq l_t$ such that $u_1 + \cdots + u_t \leq d-2$. Then there exists an $[n+m, k+m, d]$ linear error-block code of type $\pi \oplus [m]$ over $\mathbb{F}_q$.

Proof Let $H = [H_1\ H_2 \cdots H_s]$ be a parity check matrix corresponding to an $[n, k, d_0]$ linear error-block code of type π with $d_0 \geq d$. First we show that there exists an $(n-k) \times 1$ column A_1 over $\mathbb{F}_q$ such that A_1 cannot be written in the form

$$H_1 \boldsymbol{c}_1^{\perp} + \cdots + H_s \boldsymbol{c}_s^{\perp}, \tag{3.2}$$

where $\boldsymbol{c} = (\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s) \in \mathbb{F}_q^n$ satisfies $w_\pi(\boldsymbol{c}) \leq d-2$. Indeed, the number of all such columns is at most

$$\sum_{u_1} \cdots \sum_{u_t} \binom{l_1}{u_1} \cdots \binom{l_t}{u_t} \left(q^{m_1}-1\right)^{u_1} \cdots \left(q^{m_t}-1\right)^{u_t}, \tag{3.3}$$

where the sum is over the integers $0 \leq u_1 \leq l_1, \ldots, 0 \leq u_t \leq l_t$ such that $u_1 + \cdots + u_t \leq d-2$. This sum is less than or equal to (if $m = 1$) the left hand side of the inequality (3.1). Hence the existence of A_1 follows.

Let A_1 be chosen as above. We note that for each $a_1 \in \mathbb{F}_q \backslash \{0\}$, it implies that $A_1 a_1$ cannot be written in the form of (3.2) as well. Indeed, otherwise multiplying the corresponding coefficient vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s$ by $1/a_1$ we would obtain A_1 in the form of (3.2), which is a contradiction.

Assume that $m \geq 2$. Next we show that there exists an $(n-k) \times 1$ column A_2 over $\mathbb{F}_q$ such that A_2 cannot be written in the form

$$H_1 \boldsymbol{c}_1^{\perp} + \cdots H_s \boldsymbol{c}_s^{\perp} - A_1 a_1, \tag{3.4}$$

where $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s$ are as in the form (3.2) and $a_1 \in \mathbb{F}_q$. The number of all such columns is at most the number in (3.3) times q. This is again less than or equal to (if $m = 2$) the left hand side of the inequality (3.1). Hence the existence of A_2 follows.

Let A_1 and then A_2 be chosen as above. We note that for each $(a_1, a_2) \in \mathbb{F}_q^2 \backslash \{\mathbf{0}\}$, it implies that $[A_1\ A_2](a_1\ a_2)^{\perp}$ cannot be written in the form of (3.2) as well. Indeed first assume that $a_2 = 0$. Then $a_1 \neq 0$ and we have already proved that $[A_1\ A_2](a_1\ a_2)^{\perp} = A_1 a_1$ cannot be written in the form of (3.2). Next assume that $a_2 \neq 0$. If $[A_1\ A_2](a_1\ a_2)^{\perp}$ were written in the form of (3.2) with the coefficient vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_s$, then using the coefficient vectors $\boldsymbol{c}_1/a_1, \ldots, \boldsymbol{c}_s/a_1$ and a_1/a_2 we would obtain A_2 in the form of (3.4), which is a contradiction.

Continuing in this way we obtain an $(n-k) \times m$ matrix $A = [A_1\ A_2 \cdots A_m]$ such that for any $\boldsymbol{a} \in \mathbb{F}_q^m \setminus \{\mathbf{0}\}$, the $(n-k) \times m$ column $A\boldsymbol{a}^{\perp}$ cannot be written in the form (3.2). Let H' be the $(n-k) \times (n+m)$ matrix over $\mathbb{F}_q$ given by

$$H' = [H\ A].$$

Let C' be the $(k+m)$-dimensional linear error-block code of type $\pi \oplus [m]$, which has H' as its parity check matrix, up to the rearrangement of the block A in the blocks $H_1, \ldots, H_s, A$ of H' with respect to the order of $m_1, \ldots, m_t, m$. Let $\boldsymbol{c}' = (\boldsymbol{c}, \boldsymbol{a}) \in \mathbb{F}_q^{n+m}$ with $\boldsymbol{c} \in \mathbb{F}_q^n$ and $\boldsymbol{a} \in \mathbb{F}_q^m$. In order to complete the proof it is enough to show that if $\boldsymbol{c}' \in C' \setminus \{\mathbf{0}\}$, then $\boldsymbol{a} = \mathbf{0} \Rightarrow w_\pi(\boldsymbol{c}) \geq d$, and $\boldsymbol{a} \neq \mathbf{0} \Rightarrow w_\pi(\boldsymbol{c}) \geq d-1$. Indeed if $\boldsymbol{c}' \in C' \setminus \{\mathbf{0}\}$ and $\boldsymbol{a} = \mathbf{0}$, then $\boldsymbol{c} \in C \setminus \{\mathbf{0}\}$ and hence $w_\pi(\boldsymbol{c}) \geq d$ as $d_\pi(C) \geq d$. If $\boldsymbol{c}' \in C' \setminus \{\mathbf{0}\}$ and $\boldsymbol{a} \neq \mathbf{0}$, we have $H\boldsymbol{c}^{\perp} + A\boldsymbol{a}^{\perp} = \mathbf{0}$ and hence by the choice of A we have $w_\pi(\boldsymbol{c}) \geq d-1$. Therefore C' is an $[n+m, k+m, d']$ linear error-block code of type $\pi \oplus [m]$ over $\mathbb{F}_q$ with $d' \geq d$. Using Lemma 2.9, item (ii), we decrease the minimum distance to d if $d' > d$. □

We recall that the partition π is given by $n = l_1 m_1 + \cdots + l_t m_t$ with $m_1 > \cdots > m_t \geq 1$ and positive integers $l_1, \ldots, l_t$ (see (1.1)).

Corollary 3.2 *Assume that k is a positive integer such that there exist $1 \leq i_1 < \cdots < i_{t'} \leq t$ and $1 \leq u_{i_1} \leq l_{i_1}, \ldots, 1 \leq u_{i_{t'}} \leq l_{i_{t'}}$ with $k-1 = u_{i_1} m_{i_1} + \cdots + u_{i_{t'}} m_{i_{t'}}$. For integers $m \geq 1$ and $d \geq 2$ if the inequality (3.1) holds, then there is an $[n+m, k+m, d]$ linear error-block code of type $\pi \oplus [m]$ over $\mathbb{F}_q$.*

Proof Let π' be the partition of $n-k+1$ such that $\pi = \pi' \oplus [m_{i_1}]^{u_{i_1}} \oplus \cdots \oplus [m_{i_{t'}}]^{u_{t'}}$. Using the Simple Code $\mathcal{C}_1$ and Lemma 2.9, item (i), we obtain an $[n-k+1, 1, d]$ linear error-block code of type π' over $\mathbb{F}_q$. The partition of $k-1$ defines an induction to reach n and then $n+m$ from $n-k+1$ using Theorem 3.1. In the process of such an induction the corresponding inequality (3.1) holds at each step. We complete the proof using Theorem 3.1. □

In the rest of this section we obtain an infinite family of optimal (cf. Definition 2.8) linear error-block codes of some types over $\mathbb{F}_2$ systematically. We demonstrate that Theorem 3.1 gives better results than [3,Theorems 5.3 and 5.4] in some cases. We also show that Lemma 2.1 and Corollaries 2.3, 2.6 can be effectively used to show the optimality of some codes. We assume that $m \geq 2$ is an integer and $\pi = [m][1]^l$, i.e., $t = 2$, $n = m + l$, and $l \geq 1$. If $k = 1$ or $k = n$, then the Simple Codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are MDS and optimal. Hence we will consider the cases $2 \leq k \leq n-1$.

- $\pi = [m][1]$:
  Let $2 \leq k \leq m$ be an integer. Using a subcode of the Simple Code $\mathcal{C}_2$, we get an $[m+1, k, d]$ linear error-block code C of type $[m][1]$ over $\mathbb{F}_2$ with $d \geq 1$. Using Lemma 2.1, as $n-k \leq m-1 < n_1 = m$, we obtain that $d = 1$ and C is an optimal code. C is not MDS.
- $\pi = [m][1]^2$:
  Assume first that $m = 2$. Using [3,Theorem 5.4] we obtain a $[4, 2, 2]$ linear error-block code $A_{4,2}$ of type $[2][1]^2$ over $\mathbb{F}_2$, which is MDS and optimal.

  For $m \geq 3$, let $A_{m+2,2}$ be the linear error-block code of type $[m][1]^2$ obtained from $A_{4,2}$ by inserting 0 in the extra coordinates in the first block of $A_{m+2,2}$. Then $A_{m+2,2}$ is an $[m+2, 2, 2]$ linear error-block code of type $[m][1]^2$ over $\mathbb{F}_2$, which is again MDS and optimal.

  Let $3 \leq k \leq m+1$ be an integer. Using Lemma 2.1, we obtain an optimal $[m+2, k, 1]$ linear error-block code C of type $[m][1]^2$ as a subcode of the Simple Code $\mathcal{C}_2$. Note that C is not MDS.

- $\pi = [m][1]^3$:
  Assume first that $m = 2$. Using [3,Theorem 5.4] we obtain [5, 2, 3] and [5, 3, 2] linear error-block codes $B_{5,2}$ and $B_{5,3}$, respectively, of type $[2][1]^3$ over $\mathbb{F}_2$, which are MDS and optimal.

  For $m \geq 3$, like in the above, using the linear error-block codes $B_{5,2}$ and $B_{5,3}$, and inserting 0 in the extra coordinates we obtain linear error-block codes $B_{m+3,2}$ and $B_{m+3,3}$, respectively, of type $[m][1]^3$ over $\mathbb{F}_2$. These linear error-block codes are also MDS and optimal.

  Let $4 \leq k \leq m+2$ be an integer. Using Lemma 2.1, we obtain an optimal $[m+3, k, 1]$ linear error-block code C of type $[m][1]^3$ as a subcode of the Simple Code $\mathcal{C}_2$. Note that C is not MDS.
- $\pi = [m][1]^4$:
  Assume first that $m = 2$. As the number of degree one places of the rational function field over $\mathbb{F}_2$ is 3 and $3 < 4$, we cannot use [3,Theorem 5.4]. However there exists an elliptic curve E over $\mathbb{F}_2$ with 4 degree one places [8,Table 4.2.1]. Using [8,Corollary 1.6.9] we obtain that E has 2 degree two places. Hence using E and [3,Theorem 5.3] we obtain a [6, 2, 3] linear error-block code $C_{6,2}$ of type $[2][1]^4$ over $\mathbb{F}_2$. Note that $C_{6,2}$ is not MDS. Using Example 2.5 (see also Lemma 2.2), we obtain that $C_{6,2}$ is optimal.

  Using E and [3,Theorem 5.3] we obtain a [6, 3, 2] linear error-block code of type $[2][1]^4$ over $\mathbb{F}_2$. However Theorem 3.1 gives a better linear error-block code over $\mathbb{F}_2$ as follows: Let $\pi' = [2][1]^3$ and $B_{5,2}$ be the [5, 2, 3] linear error-block code of type π' over $\mathbb{F}_2$, which is constructed above. We have $\pi = \pi' \oplus [1]$. We apply Theorem 3.1 using π', $B_{5,2}$ and $m = 1$. The left hand side of (3.1) consists of $(u_1, u_2) \in \{(0,0), (1,0), (0,1)\}$ and the corresponding sum is 7. The right hand side of (3.1) is 8. Using also the Singleton bound (cf. [3,Theorem 2.1]), we obtain a [6, 3, 3] linear error-block code $C_{6,3}$ of type $[2][1]^4$ over $\mathbb{F}_2$. Note that $C_{6,3}$ is MDS and optimal.

  As above using E and [3,Theorem 5.3] we obtain only a [6, 4, 1] linear error-block code of type $[2][1]^4$ over $\mathbb{F}_2$. Using $\pi' = [2][1]^3$, the [5, 3, 2] linear error-block code $B_{5,3}$ of type π' over $\mathbb{F}_2$ constructed above and Theorem 3.1 with $m = 1$, we obtain a [6, 4, 2] linear error-block code $C_{6,4}$ of type $[2][1]^4$ over $\mathbb{F}_2$. $C_{6,4}$ is MDS and optimal. Note that by Lemma 2.2 we further obtain $B(6, 2, [2][1]^4)_2 \leq 4$, which shows that $C_{6,4}$ is also k-optimal.

  For $m \geq 3$, like above, the existence of the codes $C_{6,2}$, $C_{6,3}$ and $C_{6,4}$ implies the existence of $[m+4, 2, 3]$, $[m+4, 3, 3]$ and $[m+4, 4, 2]$ linear error-block codes $C_{m+4,2}$, $C_{m+4,3}$ and $C_{m+4,4}$, respectively, of type $[m][1]^4$ over $\mathbb{F}_2$. The codes $C_{m+4,3}$ and $C_{m+4,4}$ are MDS and optimal. $C_{m+4,2}$ is optimal by Example 2.5.

  Let $5 \leq k \leq m+3$ be an integer. Using Lemma 2.1, we obtain an optimal $[m+4, k, 1]$ linear error-block code C of type $[m][1]^4$ as a subcode of the Simple Code $\mathcal{C}_2$. C is not MDS.

In many cases, the linear error-block codes obtained above can also be constructed using elementary techniques. As a simple example, a generating matrix of a [4, 2, 2] linear error-block code of type $[2][1]^2$ over $\mathbb{F}_2$ (cf. the code $A_{4,2}$ above) is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

where the first two columns correspond to the block [2] in $[2][1]^2$.

Table 1 lists the optimal linear error-block codes over $\mathbb{F}_2$ obtained in this section.

Table 1 Some binary optimal $[n, k, d]$ linear error-block codes of type π, where $m \geq 2$ is an arbitrary integer

π	$[n, k, d]$	π	$[n, k, d]$
$[m][1]$	$[m+1, 1, 2]$	$[m][1]^2$	$[m+2, 1, 3]$
	$[m+1, 2, 1]$		$[m+2, 2, 2]$
	$\vdots$		$[m+2, 3, 1]$
			$\vdots$
	$[m+1, m+1, 1]$		$[m+2, m+2, 1]$
$[m][1]^3$	$[m+3, 1, 4]$	$[m][1]^4$	$[m+4, 1, 5]$
	$[m+3, 2, 3]$		$[m+4, 2, 3]$
	$[m+3, 3, 2]$		$[m+4, 3, 3]$
	$[m+3, 4, 1]$		$[m+4, 4, 2]$
	$\vdots$		$[m+4, 5, 1]$
	$[m+3, m+3, 1]$		$\vdots$
			$[m+4, m+4, 1]$

Remark 3.3 We note that [3,Theorem 5.4] gives constructions of linear error-block codes with good parameters in quite complicated and general cases. Indeed, let $n_1 \geq n_2 \geq \cdots \geq n_s \geq 1$ be positive integers with certain upper bounds coming from the numbers of prime divisors with certain degrees in the rational function field $\mathbb{F}_q(x)$ (cf. [3,Theorem 5.4] for the explicit statement of these upper bounds). Let $n = n_1 + n_2 + \cdots + n_s$. For example, when $s \geq 2$ and $1 \leq k \leq n$, [3,Theorem 5.4] guarantees the existence of an $[n, k, d]$ linear error-block code C of type $[n_1][n_2]\cdots[n_s]$ over $\mathbb{F}_q$, where $1 \leq d \leq s$ is the integer determined by

$$n_s + n_{s-1} + \cdots + n_{d+1} < k \leq n_s + n_{s-1} + \cdots + n_d.$$

Moreover C is MDS when $k = n_s + n_{s-1} + \cdots + n_d$. Furthermore, when $s \geq 3$, [3,Theorem 5.4] also guarantees the existence of $[n + r, k + r, 3]$ linear error-block codes of type $[n_1][n_2]\cdots[n_s][1]^r$ over $\mathbb{F}_q$ for certain values of k and r.

4 Asymptotic

In this section we study the asymptotic of linear error-block codes over $\mathbb{F}_q$ as the length $n \to \infty$. We fix an integer $m > 1$ and for simplicity we consider the asymptotic of linear error-block codes with partitions of possible different block sizes only m and 1. That is we will consider partitions $\{\pi(i)\}_{i=1}^{\infty}$ such that

$$\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}, \tag{4.1}$$

$n(i) = ms_m(i)+s_1(i)$ and $\lim_{i\to\infty} n(i) = \infty$. In the general case of arbitrary partitions, there would be a more complicated asymptotic problem of linear error-block codes over $\mathbb{F}_q$. Note that for such sequences of partitions as in (4.1), if $\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a$, then $\lim_{i\to\infty} \frac{s_m(i)}{n(i)} = \frac{1-a}{m}$. Therefore if $\{\pi(i)\}_{i=1}^{\infty}$ is such a sequence of partitions with $\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a$ and $C(i)$ is an $[n(i), k(i), d(i)]$ linear error-block code of type $\pi(i)$ over $\mathbb{F}_q$ for each $i \geq 1$, then we have $\limsup_{i\to\infty} \frac{d(i)}{n(i)} \leq a + \frac{1-a}{m}$.

We begin with the analog of the real valued function $\alpha_q(\delta)$ (see, for example, [8,Definition 6.2.1]) of the classical linear error-correcting codes.

Definition 4.1 Let $m > 1$ be an integer and $0 \le a \le 1$ be a real number. For a real number $0 \le \delta \le a + \frac{1-a}{m}$, let $S_{q,m,a}(\delta)$ be the set of real numbers R such that there exist a sequence of partitions $\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}$ of integers $n(i) = ms_m(i) + s_1(i)$ and a sequence of $[n(i), k(i), d(i)]$ linear error-block codes of type $\pi(i)$ over $\mathbb{F}_q$ satisfying $\lim_{i\to\infty} n(i) = \infty$, $\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a$, $\lim_{i\to\infty} \frac{d(i)}{n(i)} = \delta$ and $\lim_{i\to\infty} \frac{k(i)}{n(i)} = R$. Let $\alpha_{q,m,a}(\delta) \in \mathbb{R}$ be the supremum of the elements of $S_{q,m,a}(\delta)$.

Note for each partition $\pi = [m]^{s_m}[1]^{s_1}$ and $1 \le d \le s_m + s_1$, there exists an $[n, 1, d]$ linear error-block code of type π over $\mathbb{F}_q$. Hence in Definition 4.1, for each $0 \le \delta \le a + \frac{1-a}{m}$, $0 \in S_{q,m,a}(\delta)$ and $S_{q,m,a}(\delta) \neq \emptyset$.

Remark 4.2 We note that $\alpha_{q,m,1}(\delta) = \alpha_q(\delta)$ for $0 \le \delta \le 1$ and $\alpha_{q,m,0}(\delta) = \alpha_{q^m}(m\delta)$ for $1 \le \delta \le \frac{1}{m}$, where $\alpha_q(\delta)$ and $\alpha_{q^m}(\delta)$ are the analogous real valued functions of classical linear error-correcting codes over $\mathbb{F}_q$ and $\mathbb{F}_{q^m}$, respectively.

For $0 < \epsilon < a$, let $V_{q,m,a}(\epsilon)$ be the set of rational tuples $(\frac{d}{n}, \frac{k}{n})$ such that there exists an $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$ with $\pi = [m]^{s_m}[1]^{s_1}$ and $n = ms_m + s_1$ such that $\left|\frac{s_1}{n} - a\right| \le \epsilon$. Let $U_{q,m,a}(\epsilon) \subseteq \mathbb{R}^2$ be the set of real tuples (δ, R) such that there exists a sequence of partitions $\{\pi(i)\}_{i=1}^{\infty}$ with $\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}$ and $n(i) = ms_m(i) + s_1(i)$, and furthermore there exists a sequence $\{[n(i), k(i), d(i)]\}_{i=1}^{\infty}$ of linear error-block codes of type $\pi(i)$ over $\mathbb{F}_q$, where $\lim_{i\to\infty} n(i) = \infty$, $\lim_{i\to\infty} \frac{d(i)}{n(i)} = \delta$, $\lim_{i\to\infty} \frac{k(i)}{n(i)} = R$, and $\left|\frac{s_1(i)}{n(i)} - a\right| \le \epsilon$ for each $i \ge 1$. Note that for a real number $0 < \epsilon < a$ and positive integer N, the number of linear error-block codes of type $[m]^{s_m}[1]^{s_1}$ over $\mathbb{F}_q$ of bounded length $ms_m + s_1 \le N$ satisfying $\left|\frac{s_1}{n} - a\right| \le \epsilon$ is finite. Therefore, given any limit point (δ, R) of the set $V_{q,m,a}(\epsilon)$, the lengths of the linear error-block codes, corresponding to any infinite sequence of points in $V_{q,m,a}(\epsilon)$ approaching (δ, R), must tend to infinity. Hence, any limit point of $V_{q,m,a}(\epsilon)$ is in the set $U_{q,m,a}(\epsilon)$.

We also note that $U_{q,m,a}(\epsilon)$ is a subset of the rectangular domain in $\mathbb{R}^2$ consisting of the points (x, y) with $0 \le x \le a + \frac{1-a}{m}$ and $0 \le y \le 1$.

Let $U_{q,m,a}$ be the set of real tuples (δ, R) such that there exist a sequence of partitions $\{\pi(i)\}_{i=1}^{\infty}$ with $\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}$ and $n(i) = ms_m(i) + s_1(i)$, and a sequence $\{[n(i), k(i), d(i)]\}_{i=1}^{\infty}$ of linear error-block codes of type $\pi(i)$ over $\mathbb{F}_q$, where $\lim_{i\to\infty} n(i) = \infty$, $\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a$, $\lim_{i\to\infty} \frac{d(i)}{n(i)} = \delta$ and $\lim_{i\to\infty} \frac{k(i)}{n(i)} = R$. Note that for $a > 0$, $(\delta, R) \in U_{q,m,a}$ if and only if $(\delta, R) \in U_{q,m,a}(\epsilon)$ for each sufficiently small $\epsilon > 0$. Moreover if $(\delta, R) \in U_{q,m,a}$, then $R \le \alpha_{q,m,a}(\delta)$ by Definition 4.1.

Theorem 4.3 *Let $m > 1$ be an integer and $0 \le a \le 1$ be a real number. The real valued function $\alpha_{q,m,a}(\delta)$ is decreasing and continuous over its domain $0 \le \delta \le a + \frac{1-a}{m}$.*

Proof If $a = 0$ or $a = 1$, the continuity of the function $\alpha_{q,m,a}(\delta)$ follows from the continuity of the functions $\alpha_q(\delta)$ and $\alpha_{q^m}(\delta)$ of classical linear error-correcting codes and Remark 4.2.

Assume that $0 < a < 1$. For a sequence $\{\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}\}_{i=1}^{\infty}$ of partitions of integers $n(i) = ms_m(i) + s_1(i)$ with $\lim_{i\to\infty} n(i) = \infty$ we have that

$$\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a \Longleftrightarrow \lim_{i\to\infty} \frac{s_1(i)}{s_m(i)} = \frac{ma}{1-a}. \tag{4.2}$$

We follow similar arguments as in the proof of [9,Theorem 1.3.1]. For $0 < \delta_0 < a + \frac{1-a}{m}$, let P be the point $(\delta_0, \alpha_{q,m,a}(\delta_0))$ of the (δ, R)-plane. Let ℓ_1 be the line joining P to the point

$(a + \frac{1-a}{m}, 0)$ and let P_1 be the intersection of ℓ_1 with the R-axis. Let ℓ_2 be the line joining P to the point $(0, 1)$ and let P_2 be the intersection of ℓ_2 with the δ-axis. It is enough to prove that the line segments $[P_1, P]$ and $[P, P_2]$ are in the set $U_{q,m,a}$ defined prior to the statement of the theorem.

For a sufficiently small $\epsilon > 0$, let $N(\epsilon)$ be a sufficiently large integer to be decided later and let C be an $[n, k, d]$ linear error-block code of type π over $\mathbb{F}_q$ with $\pi = [m]^{s_m}[1]^{s_1}$, $n = ms_m + s_1 \geq N(\epsilon)$ such that $d \geq 2$, $\left|\frac{s_1}{s_m} - \frac{ma}{1-a}\right| \leq \epsilon$, $\left|\frac{d}{n} - \delta_0\right| \leq \epsilon$, and $\left|\frac{k}{n} - \alpha_{q,m,a}(\delta_0)\right| \leq \epsilon$. Such a code exists as $(\delta_0, \alpha_{q,m,a}(\delta_0)) \in U_{q,m,a}$. Let M be the set of positive integer tuples (r_1, r_m) such that

$$r_1 \leq s_1,\ r_m \leq s_m, \quad r_1 + mr_m < n, \quad \left|\frac{r_1}{r_m} - \frac{ma}{1-a}\right| \leq \epsilon.$$

Let D and K be subsets of M defined as

$$D = \{(r_1, r_m) \in M : (s_1 - r_1) + (s_m - r_m) \leq d - 1\}$$

and

$$K = \{(r_1, r_m) \in M : (s_1 - r_1) + m(s_m - r_m) \leq k - 1\}.$$

For $(r_1, r_m) \in D$ let $t_m = s_m - r_m$, $t_1 = s_1 - r_1$ and let $\bar{\pi} = [m]^{r_m}[1]^{r_1}$ be the partition of the positive integer $mr_m + r_1$. For each $(r_1, r_m) \in D$, using the $[n, k, d]$ linear error code C of type π chosen above and Lemma 2.9, item (iii), recursively with t_m times blocks of size m and t_1 times blocks of size 1, we obtain an $[mr_m + r_1, k, d - t_m - t_1]$ linear error-block code over $\mathbb{F}_q$ of type $\bar{\pi}$. There exists $\epsilon' > 0$ depending on ϵ and $N(\epsilon)$ such that for each $(r_1, r_m) \in D$ we have

$$\left(\frac{d - t_m - t_1}{mr_m + r_1}, \frac{k}{mr_m + r_1}\right) = \left(\frac{d + r_m + r_1 - s_m - s_1}{mr_m + r_1}, \frac{k}{mr_m + r_1}\right) \in V_{q,m,a}(\epsilon').$$

Moreover using (4.2) we note that if $\epsilon \to 0$ and $N(\epsilon) \to \infty$, then $\epsilon' \to 0$. Using $N(\epsilon) \to \infty$ we obtain a dense set of points with rational coordinates in the line segment $[P_1, P]$ as a subset of $V_{q,m,a}(\epsilon')$. Hence the line segment $[P_1, P]$ is in the set $U_{q,m,a}(\epsilon')$ for all sufficiently small ϵ', and hence the line segment $[P_1, P]$ is in $U_{q,m,a}$. Similarly using Lemma 2.9, item (iv), and the set K instead of Lemma 2.9, item (iii), and the set D, we obtain that the line segment $[P, P_2]$ is in $U_{q,m,a}$.

Let A be the triangular region in the (δ, R)-plane with the vertices $(0, 1)$, P_1 and P. Similarly let B be the triangular region with the vertices $(a + \frac{1-a}{m}, 0)$, P_2 and P. We note that if $0 \leq \delta' \leq \delta_0$, then the point $P' = (\delta', \alpha_{q,m,a}(\delta'))$ is in the region A. Indeed P' cannot lie below the region A since we already proved the existence of points in the line segment $[P_1, P]$ in the set $U_{q,m,a}$. Also P' cannot lie above the region A. Otherwise let ℓ_2' be the line joining P' to the point $(0, 1)$ and let P_2' be the intersection of ℓ_2' with the δ-axis. Then the line segment $[P', P_2']$ is in the set $U_{q,m,a}$. However P is under the line segment ℓ_2', which is a contradiction to the definition of the function $\alpha_{q,m,a}(\delta)$. Similarly we prove that if $\delta_0 \leq \delta' \leq a + \frac{1-a}{m}$, then the point $P' = (\delta', \alpha_{q,m,a}(\delta'))$ is in the region B. Note that the R-values of the elements of the region A are greater than or equal to the R-value of the point P. Similarly the R-values of the elements of the region B are smaller than or equal to the R-value of the point P. Moreover P is the only intersection of the regions A and B. These imply that the function $\alpha_{q,m,a}(\delta)$ is decreasing and continuous over the open interval $(0, a + \frac{1-a}{m})$. It is clear that $\lim_{\delta \to 0^+} \alpha_{q,m,a}(\delta) = 1$, $\lim_{\delta \to \left(a+\frac{1-a}{m}\right)^-} \alpha_{q,m,a}(\delta) = 0$ and hence $\alpha_{q,m,a}(\delta)$ is decreasing and continuous over the closed interval $[0, a + \frac{1-a}{m}]$. □

Definition 4.4 Let $E_q(x)$ be the real valued function defined on the closed interval [0, 1] as follows: For $0 < x < 1$, let $E_q(x) = -x \log_q x - (1-x) \log_q(1-x)$. For $x \in \{0, 1\}$, let $E_q(x) = 0$.

For $x \in [0, 1]$ we have

$$\lim_{n\to\infty} \frac{\log_q \binom{n}{\lfloor xn \rfloor}}{n} = E_q(x). \tag{4.3}$$

The result in (4.3) is well known. For example it immediately follows from [6,Chap. 10, Lemma 7]. For $x \in (0, 1)$, the derivative of $E_q(x)$ is given by

$$\frac{\mathrm{d}E_q(x)}{\mathrm{d}x} = \log_q \frac{1-x}{x}. \tag{4.4}$$

In the next theorem, we obtain an asymptotic Gilbert–Varshamov type lower bound on $\alpha_{q,m,a}(\delta)$ using the finite Gilbert–Varshamov type bound of Theorem 3.1.

Theorem 4.5 *Let $m > 1$ be an integer and $0 < a < 1$ be a real number. For $0 \le \delta \le \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}$, there exists a uniquely determined real number μ satisfying*

$$\max\left\{0, \delta - \frac{a(q-1)}{q}\right\} \le \mu \le \min\left\{\delta, \frac{(1-a)(q^m-1)}{mq^m}\right\} \tag{4.5}$$

and

$$(\delta - \mu)(1 - a - m\mu)(q^m - 1) = m\mu(a - \delta + \mu)(q - 1)(\delta - \mu). \tag{4.6}$$

Using this μ we have

$$\alpha_{q,m,a}(\delta) \ge 1 - \frac{1-a}{m} E_q\left(\frac{m\mu}{1-a}\right) - aE_q\left(\frac{\delta-\mu}{a}\right) - \mu \log_q(q^m-1) - (\delta-\mu)\log_q(q-1).$$

Proof Let $\{\pi(i)\}_{i=1}^{\infty}$ be a sequence of partitions $\pi(i) = [m]^{s_m(i)}[1]^{s_1(i)}$ of integers $n(i) = ms_m(i) + s_1(i)$ such that $\lim_{i\to\infty} n(i) = \infty$ and $\lim_{i\to\infty} \frac{s_1(i)}{n(i)} = a$. For $0 \le \delta \le \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}$, let $\{d(i)\}_{i=1}^{\infty}$ be a sequence of positive integers such that $\lim_{i\to\infty} \frac{d(i)}{n(i)} = \delta$. Let $f(m, a, \delta)$ be the real number

$$\lim_{i\to\infty} \frac{\log_q \left\{\sum_{u_1(i)} \sum_{u_2(i)} \binom{\lfloor \frac{1-a}{m} n(i) \rfloor}{u_1(i)} \binom{\lfloor an(i) \rfloor}{u_2(i)} (q^m-1)^{u_1(i)} (q-1)^{u_2(i)} q^{m-1}\right\}}{n(i)},$$

where for each $i \ge 1$, the sum is over the set of integers $0 \le u_1(i), u_2(i)$ satisfying $u_1(i) \le \lfloor \frac{1-a}{m} n(i) \rfloor$, $u_2(i) \le \lfloor an(i) \rfloor$ and $u_1(i) + u_2(i) \le \lfloor \delta n(i) \rfloor$. It is easy to observe that $f(m, a, \delta)$ depends only on a, m, δ and is independent from the choice of the sequences.

For the sequence $\{\pi(i)\}_{i=1}^{\infty}$ of partitions chosen above and each $0 < \epsilon < 1 - f(m, a, \delta)$, using Corollary 3.2 we obtain a sequence $\{[n(i), k(i), d(i)]\}_{i=1}^{\infty}$ of linear error-block codes of type $\pi(i)$ over $\mathbb{F}_q$ such that

$$\lim_{i\to\infty} \frac{d(i)}{n(i)} = \delta, \quad \text{and} \quad \lim_{i\to\infty} \frac{k(i)}{n(i)} \ge 1 - f(m, a, \delta) - \epsilon.$$

As $\alpha_{q,m,a}(\delta)$ is continuous (see Theorem 4.3), we obtain that

$$\alpha_{q,m,a}(\delta) \geq R = 1 - f(m, a, \delta). \tag{4.7}$$

It remains to determine $f(m, a, \delta)$. First note that

$$f(m, a, \delta) = \max_{x,y} \lim_{n\to\infty} \left\{ \frac{\log_q \binom{\lfloor \frac{1-a}{m} \rfloor n}{\lfloor xn \rfloor}}{n} + \frac{\log_q \binom{\lfloor an \rfloor}{\lfloor yn \rfloor}}{n} + x \log_q(q^m - 1) + y \log_q(q-1) \right\},$$

where the maximum is over the domain D of real number tuples (x, y) such that $0 \leq x \leq \frac{1-a}{m}$, $0 \leq y \leq a$ and $x + y \leq \delta$. Using (4.3) we obtain that

$$\lim_{n\to\infty} \frac{\log_q \binom{\lfloor \frac{1-a}{m} \rfloor n}{\lfloor xn \rfloor}}{n} = \frac{1-a}{m} E_q\left(\frac{m}{1-a} x\right), \quad \text{and} \quad \lim_{n\to\infty} \frac{\log_q \binom{\lfloor an \rfloor}{\lfloor yn \rfloor}}{n} = a E_q\left(\frac{y}{a}\right).$$

Let $G_q(x, y)$ be a real valued function defined for $0 \leq x, y \in \mathbb{R}$ as

$$G_q(x, y) = \frac{1-a}{m} E_q\left(\frac{m}{1-a} x\right) + a E_q\left(\frac{y}{a}\right) + x \log_q\left(q^m - 1\right) + y \log_q(q-1).$$

For the partial derivatives of $G_q(x, y)$ we have

$$\frac{\partial G_q}{\partial x} = \log_q \frac{(1-a-mx)(q^m-1)}{mx}, \quad \frac{\partial G_q}{\partial y} = \log_q \frac{(a-y)(q-1)}{y}.$$

Therefore $\frac{\partial G_q}{\partial x} \geq 0$ for $0 < x \leq \frac{(1-a)(q^m-1)}{mq^m}$, and $\frac{\partial G_q}{\partial y} \geq 0$ for $0 < y \leq \frac{a(q-1)}{q}$.

Let P_1 be the point on the line $x + y = \delta$ with $y = \frac{a(q-1)}{q}$ in the (x, y)-plane. Also let P_2 be the point on the line $x + y = \delta$ with $x = \frac{(1-a)(q^m-1)}{mq^m}$ in the (x, y)-plane. Let P_1' and P' be the points on the (x, y)-plane defined as

$$P_1' = \begin{cases} P_1 & \text{if } \frac{a(q-1)}{q} \leq \delta, \\ (0, \delta) & \text{otherwise,} \end{cases} \quad P_2' = \begin{cases} P_2 & \text{if } \frac{(1-a)(q^m-1)}{mq^m} \leq \delta, \\ (\delta, 0) & \text{otherwise.} \end{cases}$$

Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points P_1' and P_2', respectively. It follows from the assumptions that $x_1 \leq x_2$, and $x_1 = x_2$ if and only if $\delta \in \left\{0, \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}\right\}$.

If $\delta = 0$, then $\mu = 0$ is the unique real number satisfying (4.5) and (4.6). Moreover the domain D defined above is just $D = \{(0, 0)\}$ and hence $f(m, a, 0) = G_q(0, 0)$. Using (4.7) we complete the proof for the case $\delta = 0$.

If $\delta = \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}$, then $\mu = \frac{(1-a)(q^m-1)}{mq^m}$ is the unique real number satisfying (4.5) and (4.6). In this case the point $\left(\frac{(1-a)(q^m-1)}{mq^m}, \frac{a(q-1)}{q}\right)$ is a point in the domain D and using the partial derivatives of the function $G_q(x, y)$ we obtain that

$$f\left(m, a, \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}\right) = G_q\left(\frac{(1-a)(q^m-1)}{mq^m}, \frac{a(q-1)}{q}\right).$$

Then the proof of this case follows from (4.7).

Finally assume that $x_1 < x_2$. Note that the normal vector to the line $x+y = \delta$ in the (x, y)-plane is parallel to the vector $(1, 1)$. Using the signs of the partial derivatives of $G_q(x, y)$

inside the domain D we obtain that $G_q(x, y)$ assumes its maximum over the closed domain D at the unique point $P = (\mu, \delta - \mu)$ of the line segment $[P_1', P_2']$ at which

$$\frac{\partial G_q}{\partial x}(\mu, \delta - \mu) = \frac{\partial G_q}{\partial y}(\mu, \delta - \mu). \tag{4.8}$$

Note that (4.8) means that the gradient of $G_q(x, y)$ is parallel to the normal of the line $x + y = \delta$ at the point P. The equalities (4.8) and (4.6) are equivalent in this case. Hence we complete the proof similarly using (4.7). □

Remark 4.6 Recall that the q-ary (resp. q^m-ary) entropy function $H_q(\delta)$ (resp. $H_{q^m}(\delta)$) is given by $H_q(\delta) = E_q(\delta) + \delta \log_q(q-1)$ (resp. $H_{q^m}(\delta) = E_{q^m}(\delta) + \delta \log_{q^m}(q^m - 1)$). As $a \to 1^-$, for $0 \le \delta \le \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}$ the uniquely determined real number μ in Theorem 4.5 approaches 0 and the lower bound on $\alpha_{q,m,a}(\delta)$ in Theorem 4.5 approaches $1 - H_q(\delta)$. Similarly as $a \to 0^+$, for $0 \le \delta \le \frac{a(q-1)}{q} + \frac{(1-a)(q^m-1)}{mq^m}$ the uniquely determined real number μ in Theorem 4.5 approaches δ and the lower bound $\alpha_{q,m,a}(\delta)$ in Theorem 4.5 approaches $1 - H_{q^m}(m\delta)$. Using Remark 4.2 and the asymptotic Gilbert–Varshamov bound of classical linear error-correcting codes (see, for example [9,Theorem 1.3.15] or [8,Corollary 6.2.5]), we also obtain these lower bounds when $a = 1$ and $a = 0$, respectively.

Now we give a concatenation type construction. This allows us to construct linear error-block codes over $\mathbb{F}_q$ from classical linear error-correcting codes over $\mathbb{F}_{q^m}$. Moreover using algebraic geometry codes over $\mathbb{F}_{q^m}$ we will obtain another lower bound on $\alpha_{q,m,a}(\delta)$ in Theorem 4.10. This is better than the bound of Theorem 4.5 for some parameters.

Let $m > 1$ be an integer and $\{\omega_1, \ldots, \omega_m\}$ be a fixed basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. For each $a \in \mathbb{F}_{q^m}$, let $\varphi(a) = (a_1, \ldots, a_m) \in \mathbb{F}_q^m$ where $a = a_1\omega_1 + \cdots + a_m\omega_m$. Let A be an $[n, m, d]$ classical linear error-correcting code over $\mathbb{F}_q$. Let G_A be a fixed $m \times n$ generator matrix over $\mathbb{F}_q$ of the code A. Let $N \ge 1$ and $0 \le s_m \le N$ be integers and π' be the partition of $N' := s_m m + (N - s_m)n$ defined as

$$\pi' = [m]^{s_m}[1]^{(N-s_m)n}. \tag{4.9}$$

Recall that V_1 and V_2 are the $\mathbb{F}_q$-linear spaces $\mathbb{F}_q^m$ and $\mathbb{F}_q$, respectively (cf. Sect. 1). Let Φ be the $\mathbb{F}_q$-linear injective map from $\mathbb{F}_{q^m}^N$ into $V_1^{s_m} \oplus V_2^{(N-s_m)n}$ defined as

$$\Phi(a_1, \ldots, a_N) = \left(\varphi(a_1), \ldots, \varphi\left(a_{s_m}\right), \varphi\left(a_{s_m+1}\right)G_A, \ldots, \varphi(a_N)G_A\right), \tag{4.10}$$

where $\varphi(a_i)G_A$ is the codeword of $A \subseteq \mathbb{F}_q^n$ with the information symbol $\varphi(a_i) \in \mathbb{F}_q^m$ for $s_m + 1 \le i \le N$. Note that the map Φ depends on φ, s_m and G_A.

Theorem 4.7 *Let $m > 1$ be an integer and C be an $[N, K, D]$ classical linear error-correcting code over $\mathbb{F}_{q^m}$. Let $0 \le s_m \le N$ be an integer and A be an $[n, m, d]$ classical linear error-correcting code over $\mathbb{F}_q$. For the partition π' of $s_m m + (N - s_m)n$ defined in (4.9), let C' be the q-ary linear error-block code of type π' defined as*

$$C' = \{\Phi(\boldsymbol{c}) : \boldsymbol{c} \in C\},$$

where the map Φ is defined in (4.10). Then C' is an $[N', Km, D']$ linear error-block code of type π' over $\mathbb{F}_q$, where $N' = s_m m + (N - s_m)n$ and

$$D' \ge \max\{D, Dd - (d-1)s_m\}.$$

Proof Let $\boldsymbol{c}' \in C'$ be a nonzero codeword of C'. There exists a uniquely determined nonzero codeword $\boldsymbol{c} \in C$ such that $\boldsymbol{c}' = \Phi(\boldsymbol{c})$. Let $\boldsymbol{c} = (c_1, \ldots, c_{s_m}, b_1, \ldots, b_{N-s_m}) \in \mathbb{F}_{q^m}^N$. Let α and β be the nonnegative integers defined as

$$\alpha = |\{i : c_i \neq 0,\ 1 \leq i \leq s_m\}|, \quad \beta = |\{i : b_i \neq 0,\ 1 \leq i \leq N - s_m\}|.$$

Then we have

$$\alpha + \beta \geq D \quad \text{and} \quad \sum_{i=1}^{N-s_m} w_H\left(\varphi(b_i) G_A\right) \geq d\beta,$$

where for $\boldsymbol{b} \in \mathbb{F}_q^n$, $w_H(\boldsymbol{b})$ is the Hamming weight of the vector $\boldsymbol{b}$. Hence for the π'-weight $w_{\pi'}(\boldsymbol{c}')$ of the codeword $\boldsymbol{c}'$ we have

$$w_{\pi'}(\boldsymbol{c}') \geq \alpha + d\beta \geq \alpha + \beta \geq D.$$

Moreover $\beta \geq D - \alpha$ and hence

$$w_{\pi'}(\boldsymbol{c}') \geq \min\{\alpha + d(D - \alpha) : 0 \leq \alpha \leq s_m\} = Dd - (d-1)s_m.$$

This completes the proof. □

Remark 4.8 If $s_m = 0$, then Theorem 4.7 corresponds to [3,Theorem 3.1]. If $s_m = N$, then Theorem 4.7 corresponds to the concatenation construction of classical linear error-correcting codes.

Remark 4.9 We use Theorem 4.7 mainly for obtaining an algebraic geometry lower bound on $\alpha_{q,m,a}(\delta)$ in Theorem 4.10 below. For a partition π of finite length, in certain cases, it would be possible to construct linear error-block codes of type π over $\mathbb{F}_q$ with better parameters using [3,Theorem 5.3] than the ones constructed using Theorem 4.7.

In the next theorem, we give an algebraic geometry type lower bound on $\alpha_{q,m,a}(\delta)$.

Theorem 4.10 *Let $m > 1$ be an integer and $0 < a < 1$ be a real number. Assume that there exists a sequence $\{F_i/\mathbb{F}_{q^m}\}_{i=1}^{\infty}$ of global function fields with the common full constant field $\mathbb{F}_{q^m}$ such that $g_i \to \infty$ as $i \to \infty$ and $\lim_{i\to\infty} \frac{n_i}{g_i} = \gamma > 0$, where n_i and g_i are the number of rational places and genus of F_i. Assume also that there exists an $[n, m, d]$ linear classical linear error-correcting codes over $\mathbb{F}_q$. Then for $0 \leq \delta \leq a + \frac{1-a}{m}$ we have*

$$\alpha_{q,m,a}(\delta) \geq \begin{cases} \left(1 - \frac{1}{\gamma}\right) \frac{ma + n(1-a)}{n} - m\delta & \text{if } \delta \leq \frac{1-a}{m}, \\ \left(1 - \frac{1}{\gamma}\right) \frac{ma + n(1-a)}{n} - \frac{m\delta}{d} - \frac{(d-1)(1-a)}{d} & \text{otherwise}. \end{cases}$$

Proof Using the sequence of global function fields and the corresponding algebraic geometry codes over $\mathbb{F}_{q^m}$ (see, for example, [9,Chapter 3.4]) for sufficiently large N_1 we can assume the existence of an $[N_1, K_1, D_1]$ linear code C over $\mathbb{F}_{q^m}$ such that for $R_1 := \frac{K_1}{N_1}$ and $\delta_1 := \frac{D_1}{N_1}$ we have

$$\delta_1 = 1 - \frac{1}{\gamma} - R_1. \tag{4.11}$$

In fact the equality (4.11) can be assumed in the limit as $N_1 \to \infty$. However for the sake of simpler notation, instead we will assume the slightly stronger condition of (4.11). It is easy to observe that passing to the limit $N_1 \to \infty$, our arguments hold without any extra assumption than the ones of the theorem.

For an integer $0 \leq s_m \leq N_1$, let $\alpha = \frac{s_m}{N_1}$ and $\pi' = [m]^{\alpha N_1}[1]^{(1-\alpha)N_1 n}$ be the partition of the positive integer $N_2 = N_1(\alpha m + (1-\alpha)n)$. Using Theorem 4.7 we obtain an $[N_2, K_1 m, D_2]$ linear error-block code of type π' over $\mathbb{F}_q$, where

$$D_2 = \begin{cases} D_1 & \text{if } D_1 \geq D_1 d - (d-1)\alpha N_1, \\ D_1 d - (d-1)\alpha N_1 & \text{otherwise.} \end{cases}$$

For $d \geq 2$ we have

$$D_1 \geq D_1 d - (d-1)\alpha N_1 \Longleftrightarrow \delta_1 \leq \alpha.$$

Let $K_2 = mK_1$, $R_2' = \frac{K_2}{N_1}$ and $\delta_2' = \frac{D_2}{N_1}$. Hence for $d \geq 1$ we get

$$\delta_2' = \begin{cases} \delta_1 & \text{if } \delta_1 \leq \alpha, \\ \delta_1 d - (d-1)\alpha & \text{otherwise.} \end{cases}$$

Then we get

$$\delta_1 = \begin{cases} \delta_2' & \text{if } \delta_2' \leq \alpha, \\ \dfrac{\delta_2'}{d} + \dfrac{(d-1)\alpha}{d} & \text{otherwise.} \end{cases} \tag{4.12}$$

Note that $\pi' = [m]^{s_m}[1]^{s_1}$ with $s_1 = aN_2$, where

$$a = \frac{n(1-\alpha)}{m\alpha + n(1-\alpha)}, \quad \text{and} \quad \alpha = \frac{n(1-a)}{ma + n(1-a)}. \tag{4.13}$$

Moreover we have

$$\frac{N_1}{N_2} = \frac{1}{m\alpha + n(1-\alpha)} = \frac{ma + n(1-a)}{mn}. \tag{4.14}$$

Let $\delta_2 = \frac{D_2}{N_2}$. Using (4.12), (4.13) and (4.14) we get

$$\delta_1 = \begin{cases} \delta_2 \left(\dfrac{mn}{ma + n(1-a)} \right) & \text{if } \delta_2 \leq \dfrac{1-a}{m}, \\ \frac{\delta_2}{d} \left(\dfrac{mn}{ma + n(1-a)} \right) + \left(\dfrac{d-1}{d} \right) \left(\dfrac{n(1-a)}{ma + n(1-a)} \right) & \text{otherwise.} \end{cases}$$

Then using (4.11) we obtain that

$$R_2' = \begin{cases} m\left(1 - \dfrac{1}{\gamma}\right) - \dfrac{m^2 n}{ma + n(1-a)}\delta_2 & \text{if } \delta_2 \leq \dfrac{1-a}{m}, \\ m\left(1 - \dfrac{1}{\gamma}\right) - \dfrac{m^2 n}{ma + n(1-a)} \left(\dfrac{\delta_2}{d} \right) - \left(\dfrac{d-1}{d} \right) \dfrac{mn(1-a)}{ma + n(1-a)} & \text{otherwise.} \end{cases} \tag{4.15}$$

Let $R_2 = \frac{K_2}{N_2} = \frac{N_1}{N_2} R_2'$. From (4.14) and (4.15) we get

$$R_2 = \begin{cases} \left(1 - \dfrac{1}{\gamma}\right) \dfrac{ma + n(1-a)}{n} - m\delta_2 & \text{if } \delta_2 \leq \dfrac{1-a}{m}, \\ \left(1 - \dfrac{1}{\gamma}\right) \dfrac{ma + n(1-a)}{n} - \dfrac{m\delta_2}{d} - \dfrac{(d-1)(1-a)}{d} & \text{otherwise.} \end{cases}$$

Fig. 1 $q = 2, m = 10, a = 0.1$

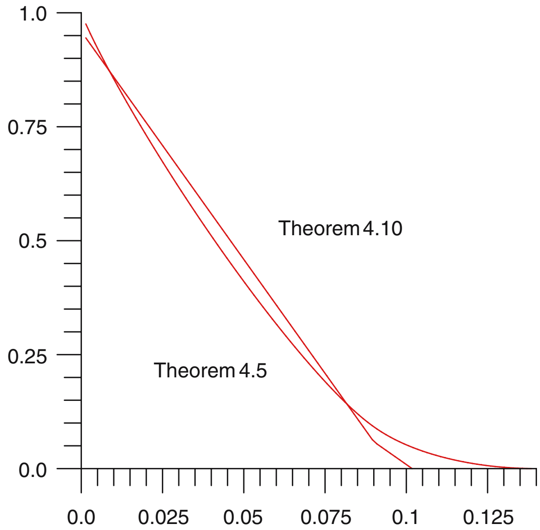
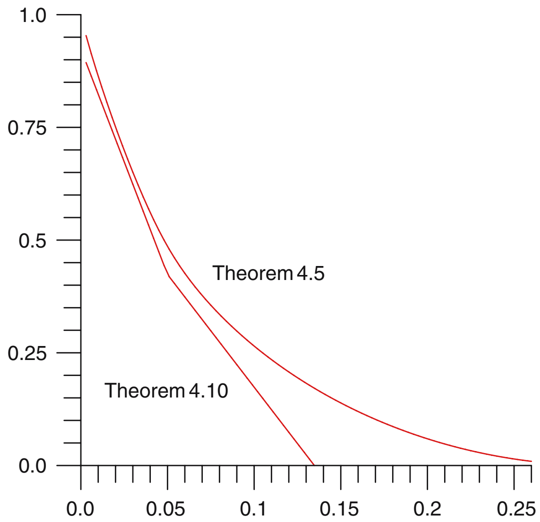


Fig. 2 $q = 2, m = 10, a = 0.5$



This completes the proof. □

Remark 4.11 If q^m is an even power of the characteristic of the field $\mathbb{F}_q$, then by the well known Tsfasman–Vlăduţ–Zink bound [10] there exists a sequence of function fields satisfying the properties in the statement of Theorem 4.10 with $\gamma = q^{m/2} - 1$. If $q^m = p^l$, where $p = \text{char } \mathbb{F}_q$, such that l is an integer divisible by 3, then by the recent bound of Bezerra et al. [2] there exists a sequence of function fields satisfying the properties in the statement of Theorem 4.10 with $\gamma = \frac{2(q^{2m/3}-1)}{q^{m/3}+2}$.

Remark 4.12 Another type of asymptotic problem would be the following. Let $1 \leq u$ be an integer, $0 \leq b < 1$ and $0 \leq \delta$ be real numbers. Assume that we consider the sequences

Fig. 3 $q = 5, m = 9, a = 0.1$

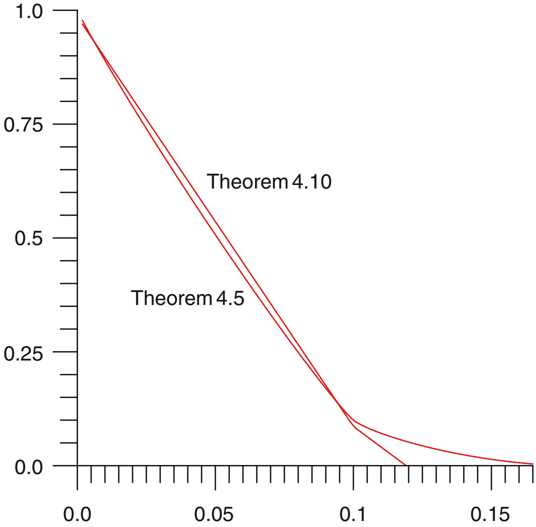
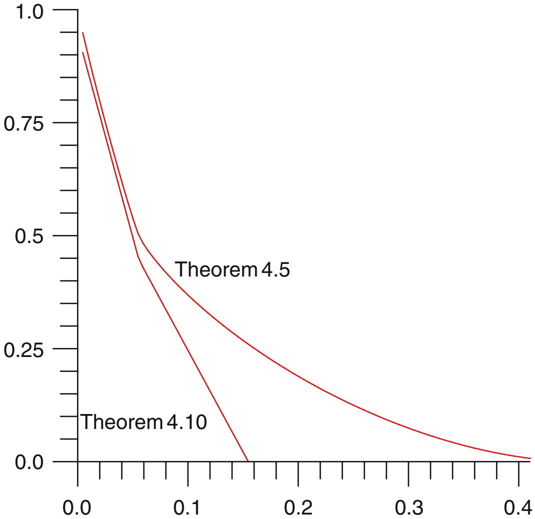


Fig. 4 $q = 5, m = 9, a = 0.5$



$\{\pi(i)\}_{i=1}^{\infty}$ of partitions such that

$$\pi(i) = [m(i)]^u[1]^{v(i)},$$

where $\lim_{i\to\infty} um(i) + v(i) = \infty$, $\lim_{i\to\infty} \frac{m(i)}{n(i)} = b$ and $\delta \le \lim_{i\to\infty} \frac{v(i)}{n(i)}$. Here $n(i) = um(i) + v(i)$ for $i \ge 1$. For such a sequence $\{\pi(i)\}_{i=1}^{\infty}$ of partitions, assume that there exists an $[n(i), k(i), d(i)]$ linear error-block code $C(i)$ of type $\pi(i)$ over $\mathbb{F}_q$ for each $i \ge 1$ and $\lim_{i\to\infty} \frac{d(i)}{n(i)} = \delta$. A corresponding asymptotic problem would be to estimate the supremum of $\frac{k(i)}{n(i)}$ among all such sequences $\{C(i)\}_{i=1}^{\infty}$ of linear error-block codes. We note that $\lim_{i\to\infty} \frac{u}{n(i)} = 0$ and the asymptotic problem is different than the one considered in Theorem 4.10. In this asymptotic problem, using [3,Theorem 5.3] would be better than using Theorem 4.7 (see also Remark 4.9).

In the following examples we compare Theorems 4.5 and 4.10.

Example 4.13 For $q = 2$ and $m = 10$, using Theorem 4.5 and Theorem 4.10 with $\gamma = q^{m/2} - 1$ (see Remark 4.11) and an [11, 10, 2] classical linear error-correcting binary code, we obtain Figs. 1 and 2 for $a = 0.1$ and $a = 0.5$, respectively.

Example 4.14 For $q = 5$ and $m = 9$, using Theorem 4.5 and Theorem 4.10 with $\gamma = \frac{2(q^{2m/3}-1)}{q^{m/3}+2}$ and an [10, 9, 2] classical linear error-correcting 5-ary code, we obtain Figs. 3 and 4 for $a = 0.1$ and $a = 0.5$, respectively.

In all the figures, the x-axis and the y-axis correspond to the parameters δ and R of Definition 4.1, respectively.

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