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The Fitting Ideal of $J_0(q)(\mathbb{F}_p)$ over the Hecke Algebra

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1. Introduction

Let $q$ be a prime. Let $X_0(q)$ be the classical modular curve and let $J = J_0(q)$ be its Jacobian variety. The curve $X_0(q)$ is endowed with well known Hecke correspondences $T_n$ for all $n \geq 1$ ([8, Chap. 7]). The modular interpretation of $X_0(q)$ and the correspondence $T_n$ allows us to define these objects over $\mathbb{Q}$. Each correspondence $T_n$ of $X_0(q)$ also induces an endomorphism of $J_0(q)$ when $J_0(q)$ is regarded as (the connected component of) the Picard variety of $X_0(q)$. We denote these Hecke operators by $T_n$ again.

Let the Hecke algebra $T$ be the subring of $\text{End}(J_0)$ generated by the Hecke operators. It is well known that $T$ is a free $\mathbb{Z}$-module of finite rank. In fact, $T = \text{End}(J_{\mathcal{C}})$ ([4, II, Prop. 9.5]).

Let $p \neq q$ be another prime and consider $J_{\mathbb{F}_p}$. Let $J(\mathbb{F}_p)$ denote the group of $\mathbb{F}_p$-rational points on $J_{\mathbb{F}_p}$ and let $\phi$ be the Frobenius endomorphism of $J_{\mathbb{F}_p}$. Then $\phi$ induces a map $J(\mathbb{F}_p) \rightarrow J(\mathbb{F}_p)$. Considering the map $1 - \phi$ induces on $J(\mathbb{F}_p)$ allows us to identify $\ker(1 - \phi)$ with $J(\mathbb{F}_p)$. In fact, we get the exact sequence of groups

$$0 \rightarrow J(\mathbb{F}_p) \rightarrow J(\bar{\mathbb{F}}_p) \rightarrow J(\bar{\mathbb{F}}_p) \rightarrow 0.$$ 

Consequently, since $1 - \phi$ is a separable endomorphism, we have

$$\text{card}(J(\mathbb{F}_p)) = \text{card}(\ker(1 - \phi)) = \text{deg}(1 - \phi) = P(1),$$

where $P(T) = \det_{\mathbb{Q}}(1 - \phi T) \in \mathbb{Q}_l[T]$ is the characteristic polynomial of $\phi$, for $l \neq p$ a prime ([5, Prop. 12.9]).

However, since

$$P(T) = N_{T \otimes \mathbb{Q}_l/\mathbb{Q}_l}(\det_{T \otimes \mathbb{Q}_l}(1 - \phi T))$$

* The author thanks Ken Ribet for an enlightening discussion.
and, from the Eichler–Shimura relation,
\[
\det_{T \otimes Q_0}(1 - \phi) = 1 + p - T_p \in T.
\]
we have
\[
P(1) = N_{T/Z}(1 + p - T_p).
\]
In other words,
\[
\text{card}(J(F_p)) = N_{T/Z}(1 + p - T_p).
\]
From the fact
\[
\text{card}(T/(1 + p - T_p)T) = N_{T/Z}(1 + p - T_p),
\]
we deduce the equality of integers
\[
\text{card}(J(F_p)) = \text{card}(T/(1 + p - T_p)T).
\]
It is easy to verify that the group structures of \( J(F_p) \) and \( T/(1 + p - T_p)T \) are not necessarily identical. For example, if \( q = 11 \), then \( J = J_0(11) \simeq X_0(11) \) is an elliptic curve described by the Weierstrass equation
\[
y^2 + y = x^3 - x^2 - 10x - 20.
\]
In this case, we have that \( T = Z \). If \( p = 31 \), the tables in [1] give \( T_p = 7 \), and hence \( T/(1 + p - T_p)T = Z/25Z \) is cyclic. However, one verifies readily that, for example, the points \((1, 12)\) and \((0, 6)\) are points of order 5 and generate distinct subgroups of \( J_0(11)(F_{31}) \). Therefore, \( J_0(11)(F_{31}) \simeq Z/5Z \times Z/5Z \) is not cyclic.

The Fitting ideal (see Section 2 for definition) is a measure of the “size” of a module. It is therefore natural to ask whether \( J(F_p) \) and \( T/(1 + p - T_p)T \) have the same Fitting ideals over \( T \). (That is, are both of the Fitting ideals equal to the ideal \((1 + p - T_p)T\)?) In this paper, we answer this equation in the affirmative. In fact, we prove the following

**Theorem 1.** If \( n \geq 1 \) is an integer and \( p \neq 2 \) is an odd prime, we have
\[
F_T(J(F_p)) = (1 - (\lambda_1^n + \lambda_2^n) + p^n)T,
\]
where \( \lambda_1, \lambda_2 \in T \otimes ZC \) satisfy \( X^2 - T_pX + p = (X - \lambda_1)(X - \lambda_2) \).

**Note.** Clearly, \( \lambda_1 + \lambda_2 = T_p \in T \) and \( \lambda_1 \lambda_2 = p \in T \). Since
\[
\hat{\lambda}_1^n + \hat{\lambda}_2^n = (\lambda_1 + \lambda_2)(\lambda_1^{n-1} + \lambda_2^{n-1}) - \lambda_1 \hat{\lambda}_2(\lambda_1^{n-2} + \lambda_2^{n-2})
\]
for $n \geq 2$, it follows by induction that

$$\lambda_1^n + \lambda_2^n \in T \quad \text{for} \quad n \geq 1.$$ 

The following is an immediate corollary of Theorem 1.

**Corollary 1.** For $p \neq 2$ an odd prime, we have the identities

(a) $F_T(J(F_{p^r})) = (1 + p - T_p) T$,

(b) $F_T(J(F_{p^r})) = (1 + p^2 - T_p^2) T$.

From Theorem 1, we can also deduce the next corollary, which may be regarded as a generalisation of the fact that, given positive integers $d$ and $n$, card($J(F_{p^d})$) divides card($J(F_{p^d})$) when $d$ divides $n$.

**Corollary 2.** Given positive integers $d$ and $n$, and that $d$ divides $n$, $J(F_{p^d})$ is a $T$-submodule of $J(F_{p^d})$.

Moreover, we have the inclusion of ideals

$$F_T(J(F_{p^d})) \subseteq F_T(J(F_{p^d})).$$

**Proof.** The first assertion is clear.

For the second statement, suppose that $n = dm$. Then

$$1 - (\lambda_1^n + \lambda_2^n) + p^n = (1 - \lambda_1^n)(1 - \lambda_2^n)$$

$$= (1 - \lambda_1^n)(1 + \lambda_1^d + \cdots + \lambda_1^{d(m-1)})(1 - \lambda_2^d)$$

$$\times (1 + \lambda_2^d + \cdots + \lambda_2^{d(m-1)})$$

$$= (1 - (\lambda_1^d + \lambda_2^d) + p^d) \left( \sum_{i=0}^{m-1} \lambda_1^d \right) \left( \sum_{j=0}^{m-1} \lambda_2^d \right)$$

$$= (1 - (\lambda_1^d + \lambda_2^d) + p^d) \left( \sum_{i,j=0}^{m-1} \lambda_1^d \lambda_2^d \right).$$

When $i = j$, we have $\lambda_1^d \lambda_2^d = (\lambda_1 \lambda_2)^d = p^d \in T$.

When $i \neq j$, say $i = \min(i, j)$, then $\lambda_1^d \lambda_2^d + \lambda_1^d \lambda_2^d = (\lambda_1 \lambda_2)^d (\lambda_1^{d(j-i)} + \lambda_2^{d(i-j)}) \in T$.

Therefore,

$$(1 - (\lambda_1^n + \lambda_2^n) + p^n) T \subseteq (1 - (\lambda_1^d + \lambda_2^d) + p^d) T,$$

i.e.,

$$F_T(J(F_{p^d})) \subseteq F_T(J(F_{p^d})).$$
2. Fitting Ideals

Let $R$ be a commutative ring with 1. In this section, we define the notion of the Fitting ideal (or, in the terminology of [7], the 0th or initial Fitting invariant) of an $R$-module over $R$ and describe some of its properties. For an $R$-homomorphism $\alpha : R^r \to R^s$, given by an $s \times r$ matrix $Ma(\alpha)$ with entries in $R$, we determine the following

**Definition 1.** The Fitting ideal of $\alpha$ over $R$, denoted by $F_R(\alpha)$, is

$$F_R(\alpha) = \begin{cases} 0 & \text{if } r < s. \\ \text{the } R\text{-ideal generated by the } s \times s \text{ minors of } Ma(\alpha) & \text{if } r \geq s. \end{cases}$$

If $M$ is an $R$-module of finite representation, we have the exact sequence

$$R^r \overset{\alpha}{\to} R^s \to M \to 0 \quad (2)$$

Since the Fitting ideal $F_R(\alpha)$, for any $\alpha : R^r \to R^s$, is dependent only on the $R$-isomorphism class of coker $\alpha$ ([7, Chap. III, Theorem 1]), we can define the Fitting ideal of $M$ over $R$, denoted by $F_R(M)$, to be $F_R(\alpha)$. In particular, if we have $r = s$, then $F_R(M)$ is simply the $R$-ideal generated by the determinant $\det_R(\alpha)$ of $\alpha$. If $M = 0$, then we have the trivial exact sequence

$$R \overset{\text{id}}{\to} R \to M \to 0,$$

and it follows that $F_R(0) = R$. If $M$ is a free $R$-module, then $F_R(M) = 0$.

If there is a surjective map $M \to N$ of $R$-modules, then $F_R(M) \subseteq F_R(N)$.

However, if $N$ is an $R$-submodule of $M$, it is not necessarily true that $F_R(M) \subseteq F_R(N)$. An example is found in [2, VII.4, Exercise 10(g)]. Corollary 2 of Theorem 1 gives examples of $M$ and $N \subseteq M$ for which $F_R(M) \subseteq F_R(N)$.

There is an intimate relationship between $F_R(M)$ and the annihilator $\text{Ann}_R(M)$ of $M$. If $M$ is of the finite presentation $(2)$, then

$$(\text{Ann}_R(M))^\perp \subseteq F_R(M) \subseteq \text{Ann}_R(M).$$

In particular, if $M$ can be generated by a single element, we have $F_R(M) = \text{Ann}_R(M)$.

If the module $M$ is finite, there is also a close connection between $F_R(M)$ and the size of $M$. For example, if

$$R = \mathbb{Z} \quad \text{and} \quad M = \mathbb{Z}/m_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s \mathbb{Z},$$

then we get the exact sequence

$$\mathbb{Z}^s \overset{\text{nat}}{\to} \mathbb{Z}^s \to M \to 0,$$
where

\[ Ma(x) = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_s \end{pmatrix}. \]

Hence,

\[ F_Z(M) = m_1 \cdots m_s \mathbb{Z} \quad \text{and} \quad \text{card}(M) = m_1 \cdots m_s. \]

In fact, more generally, if \( m_1, \ldots, m_s \) are ideals of \( R \), and

\[ M = R/m_1 \oplus \cdots \oplus R/m_s, \]

then

\[ F_R(M) = m_1 \cdots m_s. \]

Finally, we state without proof a proposition that is of use later.

**Proposition 1 (7, Chap. III, Theorem 3).** Let \( R \) and \( M \) be as in above. Let \( S \) be a multiplicatively closed subset of \( R \) not containing 0. Then we have

\[ F_{R'}(M) \otimes_{R_S} R_S = F_{R_S}(M_S), \]

where the right-hand side means the Fitting ideal of the \( R_S \)-module \( M_S \).

3. Proof of Theorem 1

Let \( V \) be an abelian variety over \( \bar{F}_p \) provided with an action of \( T \). Then, for any maximal ideal \( m \) of \( T \) of residue characteristic \( l \), we have

\[ V[m] \overset{\text{def}}{=} \bigcap_{x \in m} (\text{kernel of } x \text{ in } V) = \bigcap_{x \in m} (\text{kernel of } x \text{ in } V[l]). \]

Let

\[ V_m = \lim_{n} V[m^n] \]

and let \( Ta(V_m) \) be the Tate module \( \text{Hom}_{\mathbb{Z}_l}(\mathbb{Q}_l/\mathbb{Z}_l, V_m) \). Then

\[ Ta(V_m) = \text{Hom}_{\mathbb{Z}_l}(\mathbb{Q}_l/\mathbb{Z}_l, V_m) \]

\[ \simeq \lim_{n} \text{Hom}(\mathbb{Z}/l^n, V_m) \]

\[ \simeq \lim_{n} V_m[l^n]. \]
Let

\[ T_m = \lim_n \frac{T}{m^n}. \]

**Proposition 2.** Let \( V \) and \( W \) be abelian varieties provided with \( T \)-actions and let \( f \) be a separable isogeny \( f: V \to W \). Then, for every prime \( l \), we have the following exact sequence of \( T \otimes \mathbb{Z}_l \)-modules

\[ 0 \to \text{Ta}(V_l) \xrightarrow{\text{Ta}(f)_l} \text{Ta}(W_l) \xrightarrow{(\ker f)_l} 0, \]

where \( \text{Ta}(V_l) = \lim_n V(l^n) = \text{Hom}_{\mathbb{Z}_l}(Q_l, \mathbb{Q}_l, V_l) \) and similarly for \( \text{Ta}(W_l) \), and \( (\ker f)_l \) denotes the \( l \)-primary part of \( \ker f \).

**Proof.** For every \( n \geq 0 \), consider the commutative diagram of \( T \)-modules

\[ 0 \to V[l^n] \xrightarrow{f_n} V \xrightarrow{f} V \to 0, \]

where \( f_n \) is the restriction of \( f \) to \( V[l^n] \).

Since \( f \) is surjective (It is an isogeny!) we obtain, by the Snake Lemma, the exact sequence

\[ 0 \to (\ker f)[l^n] \xrightarrow{\ker f} \ker f \to \text{ker } f_n \to 0. \quad (3) \]

Therefore,

\[ \text{coker } f_n = \ker f/l^n \ker f. \quad (4) \]

For each integer \( m \) such that \( m \geq n \geq 0 \), we deduce, from (3) and (4), a commutative diagram

\[ 0 \to (\ker f)[l^m] \to V[l^m] \xrightarrow{f_m} W[l^m] \to \ker f/l^m \ker f \to 0. \quad (5) \]

The inverse system \((V[l^n]/(\ker f)[l^n], l^m \cdot -)\) satisfies the Mittag-Leffler condition (cf. [3, p.191]). Upon taking inverse limits, (5) gives us the exact sequence of \( T \otimes \mathbb{Z}_l \)-modules:

\[ 0 \to \text{Ta}(V_l) \xrightarrow{\text{Ta}(f)_l} \text{Ta}(W_l) \xrightarrow{(\ker f)_l} 0. \]
**Corollary 3.** For every prime $l$, we have the exact sequence of $T \otimes \mathbb{Z}_l$-modules

\[
0 \longrightarrow \mathfrak{Ta}(J_{l/F_l}) \xrightarrow{1 - \phi^*} \mathfrak{Ta}(J_{l/F_l}) \longrightarrow J(F_{\rho^*})_l \longrightarrow 0.
\]  

(6)

For any maximal ideal $m \subseteq T$ of residue characteristic $l$, the sequence of $T_m$-modules

\[
0 \longrightarrow \mathfrak{Ta}(J_m) \xrightarrow{1 - \phi^*} \mathfrak{Ta}(J_m) \longrightarrow J(F_{\rho^*})_m \longrightarrow 0
\]  

(7)

is also exact, where $J_m = \lim_n J[m^n]$, and $J(F_{\rho^*})_m = J(F_{\rho^*}) \otimes T_m$.

**Proof.** The exact sequence (6) follows immediately from Proposition 2 by using $V = W = J_{l/F_l}$ and $f = 1 - \phi^*$. The kernel of $1 - \phi^*$ is $J(F_{\rho^*})$.

The equality $T \otimes \mathbb{Z}_l = \prod_{m \subseteq T} T_m$, where the product is taken over all maximal ideals $m \subseteq T$ containing $l$, gives rise to the decompositions $\mathfrak{Ta}(J_{l/F_l}) = \prod_{m \subseteq T} \mathfrak{Ta}(J_m)$ and $J(F_{\rho^*})_l = \prod_{m \subseteq T} J(F_{\rho^*})_m$. The exact sequence (7) then follows immediately from these decompositions and (6).

From Proposition 1, we deduce the following

**Proposition 3.** We have the equalities

\[
\begin{align*}
(a) \quad F_T(J(F_{\rho^*})) \otimes \mathbb{Z}_l &= F_T \otimes \mathbb{Z}_l (J(F_{\rho^*})_l); \\
(b) \quad F_{T_m}(J(F_{\rho^*})_m) &= F_T(J(F_{\rho^*})) \otimes T_m = F_T \otimes \mathbb{Z}_l (J(F_{\rho^*})_l) \otimes T_m.
\end{align*}
\]

We next state a theorem that is an important ingredient in our proof of Theorem 1. We defer the proof of Theorem 2 until the next section. Before stating the theorem, we first define the notions of supersingular and ordinary maximal ideals in $T$, as well as those of the Eisenstein ideal and Eisenstein primes in $T$.

**Definition 2.** A maximal ideal $m$ of $T$ of residue characteristic $l$ is **supersingular** if the $l$th Hecke operator $T_l$ belongs to $m$. If $T_l \notin m$, we say that $m$ is **ordinary**.

**Definition 3.** The **Eisenstein ideal** $\mathfrak{I} \subseteq T$ is the ideal generated by the elements $1 + l - T_l$ (where $l \neq q$ is a prime) and by $1 - T_q$.

**Definition 4.** A prime ideal $\mathfrak{p} \subseteq T$ in the support of the Eisenstein ideal is called an **Eisenstein prime**.
Theorem 2. Let \( m \) be a maximal ideal in \( T \) of residue characteristic \( l \). Suppose that \( m \) is not an ordinary non-Eisenstein prime of residue characteristic \( 2 \). Then \( \text{Ta}(J_m) \) is free over \( T_m \) of rank
\[
\begin{cases}
2 & \text{if } l \neq p, \\
1 & \text{if } l = p, \text{ } m \text{ is ordinary}, \\
0 & \text{if } l = p, \text{ } m \text{ is supersingular}.
\end{cases}
\]

Proposition 4. For all primes \( l \), we have
\[ F_{T \otimes Z_l}(J(F_{p^l})) = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l). \]
where \( \lambda_1, \lambda_2 \) are the eigenvalues of the Frobenius \( \phi \), acting on \( \text{Ta}(J_l) \).

Proof. Case I: If \( l \neq p \). Corollary 3 says that we have an exact sequence of \( T \otimes Z_l \)-modules
\[
\begin{array}{c}
0 \to \text{Ta}(J_l) \xrightarrow{\text{Ta}(1 - \phi^n)} \text{Ta}(J_l) \xrightarrow{J(F_{p^l})} 0,
\end{array}
\]
and since \( \text{Ta}(J_l) \) is a free \( T \otimes Z_l \)-module, we have
\[ F_{T \otimes Z_l}(J(F_{p^l})) = (\det \text{Ta}(1 - \phi^n))(T \otimes Z_l) = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l). \]

Case II: If \( l = p \) and \( T_p \notin m \) (supersingular case). From the above theorem, we have \( \text{Ta}(J_m) = 0 \). Therefore, by (7) in Corollary 3, we have \( J(F_{p^l})_m = 0 \).

From the definition of Fitting ideals, \( F_{T_m}(J(F_{p^l})_m) = T_m \). Since \( \lambda_1 + \lambda_2 = T_p \notin m \) and \( \lambda_1, \lambda_2 = p \in m \), it follows by induction that \( \lambda_1^n + \lambda_2^n = (\lambda_1, \lambda_2) \)
\[
(\lambda_1^{n-1} + \lambda_2^{n-1}) - \lambda_1, \lambda_2(\lambda_1^{n-2} + \lambda_2^{n-2}) \in m. \]
This implies that \( 1 - (\lambda_1^n + \lambda_2^n) + p^n \) is a unit in \( T_m \). Hence,
\[ F_{T_m}(J(F_{p^l})) = T_m = (1 - (\lambda_1^n + \lambda_2^n) + p^n) \]
whenever \( m \mid p \) and \( m \) is supersingular.

Case III: If \( l = p \) and \( T_p \notin m \) (ordinary case). By Theorem 2, \( \text{Ta}(J_m) = T_m \). The endomorphism \( \phi \) acts as a "unit" root \( \lambda_1 \) in \( T_m^* \) and we have
\[ 0 = \phi^2 - T_p \phi + p = (\phi - \lambda_1)(\phi - \lambda_2). \]
The fact that \( \lambda_1 \lambda_2 = p \in mT_m \) implies that \( \lambda_2 \in mT_m \) and \( 1 - \lambda_2^n \) is a unit in \( T_m \). It then follows that
\[ (1 - (\lambda_1^n + \lambda_2^n) + p^n) T_m = (1 - \lambda_1^n)(1 - \lambda_2^n) T_m = (1 - \lambda_1^n) T_m. \]
Hence,
\[ F_{T_m}(J(F_{\rho^n})) = \det(1 - \phi^n) T_m = (1 - \lambda_1^n) T_m = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T_m. \]

Putting Cases II and III together, we obtain, if \( p = l \),
\[ F_{T \otimes Z_l}(J(F_{\rho^n})) = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l) \]

since \( T \otimes Z_l = \prod_{m \mid l} T_m \).

From Propositions 3(a) and 4, we have

**Corollary 4.** If \( p \neq 2 \), then, for all primes \( l \),
\[ F_T(J(F_{\rho^n})) \otimes Z_l = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l). \]

Now we are ready to prove the main theorem, which we restate:

**Theorem 1.** If \( n \geq 1 \) is an integer and \( p \neq 2 \) is an odd prime, we have
\[ F_T(J(F_{\rho^n})) = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T, \]
where \( \lambda_1, \lambda_2 \in T \otimes \mathbb{Z} \mathbb{C} \) satisfy \( X^2 - T_pX + p = (X - \lambda_1)(X - \lambda_2). \)

**Proof.** First, we prove a lemma:

**Lemma 1.** Let \( L, M, N \) be \( \mathbb{Z} \)-lattices and, for every prime \( l \), let \( L_l \) denote \( L \otimes Z_l \) (and similarly for \( M_l \) and \( N_l \)). If \( L \subseteq N \), \( M \subseteq N \), and \( L_l = M_l \subseteq N_l \), for all primes \( l \), then \( L = M \).

**Proof of Lemma 1.** Let \( X = L + M \). Then \( L \subseteq X \subseteq N \) and \( L \otimes Z_l = X \otimes Z_l \). The sequence
\[ 0 \longrightarrow L \overset{id}{\longrightarrow} X \longrightarrow X/L \longrightarrow 0 \]
is exact. Tensoring with \( Z_l \), we get an exact sequence of \( Z_l \)-modules
\[ 0 \longrightarrow L \otimes Z_l \overset{id \otimes 1}{\longrightarrow} X \otimes Z_l \longrightarrow (X/L) \otimes Z_l \longrightarrow 0. \]

However, \( L \otimes Z_l = X \otimes Z_l \) for all \( l \) implies \( (X/L) \otimes Z_l = 0 \) for all primes \( l \). Therefore, \( X/L = 0 \); i.e., \( L = X = L + M \). Consequently, we have the inclusion \( M \subseteq L \). Symmetry of the argument implies that \( L = M \).

To complete the proof of Theorem 1, simply take \( L = F_T(J(F_{\rho^n})) \), \( M = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T \) and \( N = T \).
4. Freeness of \( \text{Ta}(J_m(\bar{F}_p)) \)

In this section, as promised earlier, we prove Theorem 2 which was stated in the last section.

From [4, II §815.1, 15.2, 16.3, and 17.9], we see that \( \text{Ta}(J_m(\bar{Q})) \) is free over \( T_m \) of rank 2 and that \( T_m \) is Gorenstein ([4, II §15]) under each of the following hypotheses:

1. \( m \) not Eisenstein, \( \text{char} \ T/m \neq 2 \) (15.2);
2. \( m \) not Eisenstein, \( \text{char} \ T/m = 2, \) \( m \) supersingular (15.2);
3. \( m \) Eisenstein (16.3, 17.9).

We prove Theorem 2 in the three cases in the statement of the theorem.

**Case I:** \( m \not| p \); i.e., \( \text{char} \ T/m = l \neq p \). In this case,

\[
J[m]_{\bar{F}_p} = \bigcap_{\alpha \in m} (\ker \text{ of } \alpha \text{ in } J[I]_{\bar{F}_p}) \subseteq \bigcap_{\alpha \in m} (\ker \text{ of } \alpha \text{ in } J[I]_{\bar{Q}}) = J[m]_{\bar{Q}}.
\]

Therefore, \( \text{Ta}(J_m(\bar{F}_p)) \simeq \text{Ta}(J_m(\bar{Q})) \), and hence is free of rank 2 over \( T_m \).

**Case II:** \( m \mid p \), \( m \) supersingular; i.e., \( l = p \), \( T_p \in m \). In this case, \( J[m](\bar{F}_p) \) admits a Jordan–Hölder filtration whose constituents are all isomorphic to the group scheme \( \alpha_p \) ([4, II §14]). In other words, the \( p \)-rank of \( J_m(\bar{F}_p) \) is 0 ([6, p.147]). It follows that \( [6, \text{ p.171}] \) \( \text{Ta}(J_m(\bar{F}_p)) \simeq 0 \), i.e., \( \text{Ta}(J_m(\bar{F}_p)) \) is free over \( T_m \) of rank 0.

**Case III:** \( m \mid p \), \( m \) ordinary; i.e., \( l = p \), \( T_p \notin m \). We proceed by using the proposition below:

**Proposition 5.** Let \( M \) be a finitely generated module over \( T_m \). Then \( M \) is free of rank 1 over \( T_m \) if and only if

1. \( M \otimes Q_p \) is free of rank 1 over \( T_m \otimes Q_p \), and
2. \( M/mM \), as a \( T/m \)-module, is free of rank 1.

**Proof of Proposition 5.** (\( \Rightarrow \)) This direction is clear.

(\( \Leftarrow \)) Nakayama’s lemma and Condition 2 imply that \( M \) is generated over \( T_m \) by an element \( x \), i.e., \( M = T_m x \). Let \( g : T_m \to M \) be the map \( g(\tau) = \tau x \), and let \( H \) be the kernel of \( g \). Then we have an exact sequence

\[
0 \longrightarrow H \longrightarrow T_m \overset{g}{\longrightarrow} M \longrightarrow 0.
\]
We can consider these as $\mathbb{Z}_p$-modules, and since $Q_p$ is $\mathbb{Z}_p$-flat, the sequence of $Q_p$-modules

$$0 \longrightarrow H \otimes Q_p \longrightarrow T_m \otimes Q_p \longrightarrow M \otimes Q_p \longrightarrow 0$$

is exact. By Condition 1, $H \otimes Q_p = 0$. Since $T \simeq Z'$, we have $T \otimes Z_p \simeq Z'_p$ and $H \subseteq T \subseteq T \otimes Z_p \simeq Z'_p$. In particular, $H$ is $Z_p$-torsion free. Therefore, $H$ can be embedded in $H \otimes Q_p$ and hence $H = 0$. In other words, $M \simeq T_m$, i.e., $M$ is free over $T_m$ of rank 1. This prove Proposition 5.}

**Proposition 6.** Let $m \subseteq T$ be an ordinary maximal ideal of residue characteristic $p$, with the additional condition that $m$ be Eisenstein when $p = 2$. Then the $Z_p$-dual of $\text{Ta}(J_m(F_p))$ is free of rank 1 over $T_m$.

**Proof.** Let $M = \text{Ta}(J_m(F_p))$ and let $M^* = \text{Hom}_{Z_p}(M, Z_p)$ be the $Z_p$-dual of $\text{Ta}(J_m(F_p))$.

By [4, II Prop. 8.5] (see also the proof of Cor. 14.11), $M^*$ and $M$ are both of rank 1 over $T_m$; i.e., $M^* \otimes Q_p$ and $M \otimes Q_p$ are both free of rank 1 over $T_m \otimes Q_p$. Hence, Condition 1 of Proposition 5 is satisfied by $M$ and $M^*$.

We now show that Condition 2 is satisfied by $M^*/mM^*$.

**Lemma 2.** We have the isomorphism of $T/m$-modules

$$M^*/mM^* \simeq \text{Hom}_{Z_p}(J(F_p)[m], Q_p/Z_p),$$

where $M^* = \text{Hom}_{Z_p}(\text{Ta}(J_m(F_p)), Z_p)$.

(For the proof, see later.)

The module on the right-hand side of (8) is free of rank 1 over $T/m$ for $p > 2$ ([4, II Cor. 14.8]). Therefore, by Proposition 5, $M^*$ is free of rank 1 over $T_m$ for $p > 2$.

Now, let $p = 2$. We are then in the case where $m$ is Eisenstein. In this case, [4, II Cor. 14.11] implies that $M^*$ is free of rank 1 over $T_m$.

It remains, therefore, to prove Lemma 2.

Let $N$ be a $Z_p$-module with $T_m$-action, of the type $F \oplus (Q_p/Z_p)^n$, where $F$ is a finite abelian group and $n$ a non-negative integer. Let $N^* = \text{Hom}_{Z_p}(N, Q_p/Z_p)$ be the Pontrjagin $p$-dual of $N$.

**Lemma 3.** There is an isomorphism of $Z_p$-modules $N \simeq N^\wedge$. The $T_m$-action on these modules is also preserved under the isomorphism.

**Proof.** Since $\text{Hom}_{Z_p}(Q_p/Z_p, Q_p/Z_p) = Z_p$ and $\text{Hom}_{Z_p}(Z_p, Q_p/Z_p) = Q_p/Z_p$, we have the isomorphism $N \simeq N^\wedge$ if $N = (Q_p/Z_p)^n$, for $n$ a
non-negative integer. For finite groups $F$, the isomorphism $F \cong F^\wedge$ is well known. Hence, we obtain the isomorphism of $\mathbb{Z}_p$-modules $N \cong N^\wedge$, where $N = F \oplus (\mathbb{Q}_p / \mathbb{Z}_p)^\wedge$.

If $t \in T_m$, $f \in N^\wedge$, and $j \in N$ (with corresponding element $j^* \in N^\wedge$), then $t$ acts on $N^\wedge$ by $(tf)(j) = f(tj)$. Consequently,

$$(tj^*)(f) = j^*(tf) = (tf)(j) = f(tj) = (tj)^*(f).$$

(9)

Therefore, the $T_m$-action is preserved.

Now we prove Lemma 2.

**Proof of Lemma 2.** Let $N$ be $J_m(\overline{F}_p)$. Then $N^\wedge = M^*$ by [4, II §7, p. 92]. From Lemma 3, we have the duality $N \cong N^\wedge$. This duality gives rise to a one-to-one correspondence between the $T_m$-submodules $P$ of $M^*$ and the $T_m$-submodules $Q$ of $J_m(\overline{F}_p)$. Indeed, given $Q \subseteq J_m(\overline{F}_p)$, we get a quotient $Q^\wedge$ of $N^\wedge = M^*$. Since $\mathbb{Q}_p / \mathbb{Z}_p$ is injective, there exists a submodule $P \subseteq M^*$ such that $M^*/P \cong Q^\wedge$. Conversely, given $P \subseteq M^*$, $Q \overset{\text{def}}{=} \text{Hom}_{\mathbb{Z}_p}(M^*/P, \mathbb{Q}_p / \mathbb{Z}_p)$ injects into $\text{Hom}_{\mathbb{Z}_p}(M^*, \mathbb{Q}_p / \mathbb{Z}_p) = N^\wedge \cong N$. These two directions are clearly inverse to each other. Now,

$$mM^* \subseteq P \quad \iff \quad m \text{ annihilates } M^*/P$$

$$\iff m \text{ annihilates } \text{Hom}_{\mathbb{Z}_p}(Q, \mathbb{Q}_p / \mathbb{Z}_p)$$

$$\iff mQ = 0$$

$$\iff Q \subseteq J(\overline{F}_p)[m].$$

It follows then that $P = mM^*$ corresponds to $Q = J_m(\overline{F}_p)[m]$ in the above correspondence, and hence

$$M^*/mM^* \cong \text{Hom}_{\mathbb{Z}_p}(J(\overline{F}_p)[m], \mathbb{Q}_p / \mathbb{Z}_p).$$

Proposition 7 below implies that $M = \text{Ta}(J_m(\overline{F}_p))$ is free of rank 1 over $T_m$ in Case III.

**PROPOSITION 7.** Let $m \subseteq T$ be a maximal ideal of residue characteristic $p$ and let $M$ be $\text{Ta}(J_m(\overline{F}_p))$. Suppose that $M^* = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ is free of rank 1 over $T_m$. Then $M$ is also free of rank 1 over $T_m$.

**Proof.** As $J_m(\overline{F}_p) \subseteq J_m(\overline{F}_p)$, and $\text{Ta}(J_m(\overline{F}_p))$ is a free $\mathbb{Z}_p$-module, we see that $M = \text{Ta}(Z_m(\overline{F}_p))$ is also a free $\mathbb{Z}_p$-module. If the $\mathbb{Z}_p$-rank of $M$ is $s$, then $M \cong \mathbb{Z}_p^s$. Since $\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^s, \mathbb{Z}_p) \cong \mathbb{Z}_p^s$, it follows that $M \cong (M^*)^*$ as $\mathbb{Z}_p$-modules. An argument parallel to the one used in (9) shows that
$M \simeq (M^*)^*$ as $T_m$-modules. Consequently, we have isomorphisms of $T_m$-modules

$$M \simeq \text{Hom}_{Z_p}(M^*, Z_p) \simeq \text{Hom}_{Z_p}(T_m, Z_p) \simeq T_m,$$

where the last isomorphism follows from the fact that $T_m$ is Gorenstein. This shows that $M$ is free of rank 1 over $T_m$. □

The proof of Theorem 2 is now complete.

REFERENCES