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The Fitting Ideal of $J_0(q)(\mathbb{F}_p^n)$ over the Hecke Algebra

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1. INTRODUCTION

Let $q$ be a prime. Let $X_0(q)$ be the classical modular curve and let $J = J_0(q)$ be its Jacobian variety. The curve $X_0(q)$ is endowed with well known Hecke correspondences $T_n$ for all $n \geq 1$ ([8, Chap. 7]). The modular interpretation of $X_0(q)$ and the correspondence $T_n$ allows us to define these objects over $\mathbb{Q}$. Each correspondence $T_n$ of $X_0(q)$ also induces an endomorphism of $J_0(q)$ when $J_0(q)$ is regarded as (the connected component of) the Picard variety of $X_0(q)$. We denote these Hecke operators by $T_n$ again.

Let the Hecke algebra $T$ be the subring of $\text{End}(J) \otimes \mathbb{Q}$ generated by the Hecke operators. It is well known that $T$ is a free $\mathbb{Z}$-module of finite rank. In fact, $T = \text{End}(J_{\mathbb{Q}})$ ([4, II, Prop. 9.5]).

Let $p \neq q$ be another prime and consider $J_{\mathbb{F}_p}$. Let $J(\mathbb{F}_p)$ denote the group of $\mathbb{F}_p$-rational points on $J_{\mathbb{F}_p}$ and let $\phi$ be the Frobenius endomorphism of $J_{\mathbb{F}_p}$. Then $\phi$ induces a map $J(\mathbb{F}_p) \rightarrow J(\mathbb{F}_p)$. Considering the map $1 - \phi$ induces on $J(\overline{\mathbb{F}}_p)$ allows us to identify $\ker(1 - \phi)$ with $J(\mathbb{F}_p)$. In fact, we get the exact sequence of groups

$$0 \rightarrow J(\mathbb{F}_p) \rightarrow J(\overline{\mathbb{F}}_p) \rightarrow J(\overline{\mathbb{F}}_p) \rightarrow 0.$$ 

Consequently, since $1 - \phi$ is a separable endomorphism, we have

$$\text{card}(J(\mathbb{F}_p)) = \text{card}(\ker(1 - \phi)) = \text{deg}(1 - \phi) = P(1),$$

where $P(T) = \det_{\mathbb{Q}_l}(1 - \phi T) \in \mathbb{Q}_l[T]$ is the characteristic polynomial of $\phi$, for $l \neq p$ a prime ([5, Prop. 12.9]).

However, since

$$P(T) = N_{T \otimes \mathbb{Q}_l / \mathbb{Q}_l}(\det_{T \otimes \mathbb{Q}_l}(1 - \phi T))$$

* The author thanks Ken Ribet for an enlightening discussion.
and, from the Eichler–Shimura relation,

$$\det_{T \otimes \mathbb{Q}_p}(1 - \phi) = 1 + p - T_p \in T,$$

we have

$$P(1) = N_{T/Z}(1 + p - T_p).$$

In other words,

$$\text{card}(J(F_p)) = N_{T/Z}(1 + p - T_p).$$

From the fact

$$\text{card}(T/(1 + p - T_p) T) = N_{T/Z}(1 + p - T_p),$$

we deduce the equality of integers

$$\text{card}(J(F_p)) = \text{card}(T/(1 + p - T_p) T).$$

It is easy to verify that the group structures of $J(F_p)$ and $T/(1 + p - T_p) T$ are not necessarily identical. For example, if $q = 11$, then $J = J_0(11) \simeq X_0(11)$ is an elliptic curve described by the Weierstrass equation

$$y^2 + y = x^3 - x^2 - 10x - 20. \quad (1)$$

In this case, we have that $T = Z$. If $p = 31$, the tables in [1] give $T_p = 7$, and hence $T/(1 + p - T_p) T = Z/25Z$ is cyclic. However, one verifies readily that, for example, the points $(1, 12)$ and $(0, 6)$ are points of order 5 and generate distinct subgroups of $J_0(11)(F_{31})$. Therefore, $J_0(11)(F_{31}) \simeq Z/5Z \times Z/5Z$ is not cyclic.

The Fitting ideal (see Section 2 for definition) is a measure of the “size” of a module. It is therefore natural to ask whether $J(F_p)$ and $T/(1 + p - T_p) T$ have the same Fitting ideals over $T$. (That is, are both of the Fitting ideals equal to the ideal $(1 + p - T_p) T$?) In this paper, we answer this equation in the affirmative. In fact, we prove the following

**Theorem 1.** If $n \geq 1$ is an integer and $p \neq 2$ is an odd prime, we have

$$F_T(J(F_p)) = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T,$$

where $\lambda_1, \lambda_2 \in T \otimes \mathbb{Z} C$ satisfy $X^2 - T_p X + p = (X - \lambda_1)(X - \lambda_2)$.

**Note.** Clearly, $\lambda_1 + \lambda_2 = T_p \in T$ and $\lambda_1 \lambda_2 = p \in T$. Since

$$\lambda_1^n + \lambda_2^n = (\lambda_1 + \lambda_2)(\lambda_1^{n-1} + \lambda_2^{n-1}) - \lambda_1 \lambda_2(\lambda_1^{n-2} + \lambda_2^{n-2})$$
for \( n \geq 2 \), it follows by induction that

\[
\hat{\lambda}_1^n + \hat{\lambda}_2^n \in \mathbf{T} \quad \text{for} \quad n \geq 1.
\]

The following is an immediate corollary of Theorem 1.

**Corollary 1.** For \( p \neq 2 \) an odd prime, we have the identities

(a) \( F_T(J(F_{p^n})) = (1 + p - T_p) \mathbf{T} \),

(b) \( F_T(J(F_{p^n})) = (1 + p^2 - T_{p^2}) \mathbf{T} \).

From Theorem 1, we can also deduce the next corollary, which may be regarded as a generalisation of the fact that, given positive integers \( d \) and \( n \), \( \text{card}(J(F_{p^n})) \) divides \( \text{card}(J(F_{p^n})) \) when \( d \) divides \( n \).

**Corollary 2.** Given positive integers \( d \) and \( n \), and that \( d \) divides \( n \), \( J(F_{p^n}) \) is a \( \mathbf{T} \)-submodule of \( J(F_{p^n}) \).

Moreover, we have the inclusion of ideals

\[
F_T(J(F_{p^n})) \subseteq F_T(J(F_{p^n})).
\]

**Proof.** The first assertion is clear.

For the second statement, suppose that \( n = dm \). Then

\[
1 - (\hat{\lambda}_1^n + \hat{\lambda}_2^n) + p^n = (1 - \hat{\lambda}_1^n)(1 - \hat{\lambda}_2^n)
\]

\[
= (1 - \hat{\lambda}_1^n)(1 + \hat{\lambda}_1^d + \cdots + \hat{\lambda}_1^{d(m-1)})(1 - \hat{\lambda}_2^d)
\]

\[
\times (1 + \hat{\lambda}_2^d + \cdots + \hat{\lambda}_2^{d(m-1)}).
\]

\[
= (1 - (\lambda_1^d + \lambda_2^d) + p^d) \left( \sum_{i=0}^{m-1} \lambda_1^{di} \right) \left( \sum_{j=0}^{m-1} \lambda_2^{dj} \right)
\]

\[
= (1 - (\lambda_1^d + \lambda_2^d) + p^d) \left( \sum_{i,j=0}^{m-1} \lambda_1^{di} \lambda_2^{dj} \right).
\]

When \( i = j \), we have \( \lambda_1^{di} \lambda_2^{dj} = (\lambda_1 \lambda_2)^{di} = p^d \in \mathbf{T} \).

When \( i \neq j \), say \( i = \min(i, j) \), then \( \lambda_1^{di} \lambda_2^{dj} + \lambda_1^{dj} \lambda_2^{di} = (\lambda_1 \lambda_2)^{di} (\lambda_1^{d(i-j)} + \lambda_2^{d(j-i)}) \in \mathbf{T} \).

Therefore,

\[
(1 - (\lambda_1^d + \lambda_2^d) + p^d) \mathbf{T} \subseteq (1 - (\lambda_1^d + \lambda_2^d) + p^d) \mathbf{T},
\]

i.e.,

\[
F_T(J(F_{p^n})) \subseteq F_T(J(F_{p^n})).
\]
2. Fitting Ideals

Let $R$ be a commutative ring with 1. In this section, we define the notion of the Fitting ideal (or, in the terminology of [7], the 0th or initial Fitting invariant) of an $R$-module over $R$ and describe some of its properties. For an $R$-homomorphism $\alpha: R^r \to R^s$, given by an $s \times r$ matrix $Ma(\alpha)$ with entries in $R$, we determine the following

**Definition 1.** The Fitting ideal of $\alpha$ over $R$, denoted by $F_R(\alpha)$, is

$$F_R(\alpha) = \begin{cases} 0 & \text{if } r < s, \\ \text{the } R\text{-ideal generated by the } s \times s \text{ minors of } Ma(\alpha) & \text{if } r \geq s. \end{cases}$$

If $M$ is an $R$-module of finite representation, we have the exact sequence

$$R^r \xrightarrow{z} R^s \to M \to 0 \quad (2)$$

Since the Fitting ideal $F_R(\alpha)$, for any $\alpha: R^r \to R^s$, is dependent only on the $R$-isomorphism class of coker $\alpha$ ([7, Chap. III, Theorem 1]), we can define the Fitting ideal of $M$ over $R$, denoted by $F_R(M)$, to be $F_R(\alpha)$. In particular, if we have $r = s$, then $F_R(M)$ is simply the $R$-ideal generated by the determinant $\det_R(\alpha)$ of $\alpha$. If $M = 0$, then we have the trivial exact sequence

$$R \xrightarrow{id} R \to M \to 0,$$

and it follows that $F_R(0) = R$. If $M$ is a free $R$-module, then $F_R(M) = 0$.

If there is a surjective map $M \to N$ of $R$-modules, then $F_R(M) \subseteq F_R(N)$.

However, if $N$ is an $R$-submodule of $M$, it is not necessarily true that $F_R(M) \subseteq F_R(N)$. An example is found in [2, VII.4, Exercise 10(g)]. Corollary 2 of Theorem 1 gives examples of $M$ and $N \subseteq M$ for which $F_R(M) \nsubseteq F_R(N)$.

There is an intimate relationship between $F_R(M)$ and the annihilator $\text{Ann}_R(M)$ of $M$. If $M$ is of the finite presentation (2), then

$$(\text{Ann}_R(M))^c \subseteq F_R(M) \subseteq \text{Ann}_R(M).$$

In particular, if $M$ can be generated by a single element, we have $F_R(M) = \text{Ann}_R(M)$.

If the module $M$ is finite, there is also a close connection between $F_R(M)$ and the size of $M$. For example, if

$$R = \mathbb{Z} \quad \text{and} \quad M = \mathbb{Z}/m_1 \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s \mathbb{Z},$$

then we get the exact sequence

$$\mathbb{Z}^r \xrightarrow{z} \mathbb{Z}^r \to M \to 0,$$
where
\[ \mathbf{Ma}(x) = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & m_s \end{pmatrix}. \]

Hence,
\[ F_z(M) = m_1 \cdots m_s \mathbb{Z} \quad \text{and} \quad \text{card}(M) = m_1 \cdots m_s. \]

In fact, more generally, if \( m_1, \ldots, m_s \) are ideals of \( R \), and
\[ M = R/m_1 \oplus \cdots \oplus R/m_s, \]
then
\[ F_R(M) = m_1 \cdots m_s. \]

Finally, we state without proof a proposition that is of use later.

**Proposition 1 (7, Chap. III, Theorem 3).** Let \( R \) and \( M \) be as in above. Let \( S \) be a multiplicatively closed subset of \( R \) not containing 0. Then we have
\[ F_R(M) R_S = F_{R_S}(M_S), \]
where the right-hand side means the Fitting ideal of the \( R_S \)-module \( M_S \).

3. **Proof of Theorem 1**

Let \( V \) be an abelian variety over \( \mathbb{F}_p \) provided with an action of \( T \). Then, for any maximal ideal \( m \) of \( T \) of residue characteristic \( l \), we have
\[ V[m] \overset{\text{def}}{=} \bigcap_{z \in m} (\text{kernel of } z \text{ in } V) = \bigcap_{z \in m} (\text{kernel of } z \text{ in } V[l]). \]

Let
\[ V_m = \varprojlim_n V[m^n] \]
and let \( \text{Ta}(V_m) \) be the Tate module \( \text{Hom}_{\mathbb{Z}_l}(\mathbb{Q}_l/\mathbb{Z}_l, V_m) \). Then
\[ \text{Ta}(V_m) = \text{Hom}_{\mathbb{Z}_l}(\mathbb{Q}_l/\mathbb{Z}_l, V_m) \]
\[ \approx \varprojlim_n \text{Hom}(\mathbb{Z}/l^n\mathbb{Z}, V_m) \]
\[ \approx \varprojlim_n V_m[l^n]. \]
Let

\[ T_m = \lim_{n} T/m^n. \]

**Proposition 2.** Let \( V \) and \( W \) be abelian varieties provided with \( T \)-actions and let \( f \) be a separable isogeny \( f: V \to W \). Then, for every prime \( l \), we have the following exact sequence of \( T \otimes \mathbb{Z}_l \)-modules

\[
0 \longrightarrow \text{Ta}(V)_{l} \xrightarrow{\text{Ta}(f)} \text{Ta}(W)_{l} \longrightarrow (\ker f)_{l} \longrightarrow 0,
\]

where \( \text{Ta}(V) \overset{\text{def}}{=} \lim_{n} V[l^n] = \text{Hom}_{\mathbb{Z}_l}(\mathbb{Q}_l/\mathbb{Z}_l, V) \) and similarly for \( \text{Ta}(W)_{l} \), and \( (\ker f)_{l} \) denotes the \( l \)-primary part of \( \ker f \).

**Proof.** For every \( n \geq 0 \), consider the commutative diagram of \( T \)-modules

\[
\begin{array}{ccccc}
0 & \longrightarrow & V[l^n] & \xrightarrow{f_n} & V \xrightarrow{l^n} V \longrightarrow 0 \\
 & & \downarrow f & & \downarrow f \\
0 & \longrightarrow & W[l^n] & \xrightarrow{l^n} W \longrightarrow 0,
\end{array}
\]

where \( f_n \) is the restriction of \( f \) to \( V[l^n] \).

Since \( f \) is surjective (It is an isogeny!) we obtain, by the Snake Lemma, the exact sequence

\[
0 \longrightarrow (\ker f)[l^n] \longrightarrow \ker f \longrightarrow (\ker f)_{l} \longrightarrow 0. \tag{3}
\]

Therefore,

\[
\text{coker } f_n = \ker f/l^n \ker f. \tag{4}
\]

For each integer \( m \) such that \( m \geq n \geq 0 \), we deduce, from (3) and (4), a commutative diagram

\[
\begin{array}{ccccc}
0 & \longrightarrow & (\ker f)[l^n] & \xrightarrow{f_n} & V[l^n] \xrightarrow{l^n} W[l^n] \longrightarrow \ker f/l^n \ker f \longrightarrow 0 \\
 & & \downarrow & & \downarrow l^n & & \downarrow l^n & & \downarrow (5)
\end{array}
\]

\[
0 \longrightarrow (\ker f)[l^m] \longrightarrow V[l^m] \longrightarrow W[l^m] \longrightarrow \ker f/l^m \ker f \longrightarrow 0.
\]

The inverse system \((V[l^n]/(\ker f)[l^n], l^m \to l^n)\) satisfies the Mittag-Leffler condition (cf. [3, p.191]). Upon taking inverse limits, (5) gives us the exact sequence of \( T \otimes \mathbb{Z}_l \)-modules:

\[
0 \longrightarrow \text{Ta}(V)_{l} \xrightarrow{\text{Ta}(f)} \text{Ta}(W)_{l} \longrightarrow (\ker f)_{l} \longrightarrow 0.
\]
COROLLARY 3. For every prime $l$, we have the exact sequence of $\mathbf{T} \otimes \mathbf{Z}_l$-modules

\[
0 \longrightarrow \mathbf{Ta}(J_{l/F_l}) \overset{1-\phi^l}{\longrightarrow} \mathbf{Ta}(J_{l/F_l}) \longrightarrow J(F_{\rho^l})_l \longrightarrow 0. \tag{6}
\]

For any maximal ideal $m \subseteq \mathbf{T}$ of residue characteristic $l$, the sequence of $\mathbf{T}_m$-modules

\[
0 \longrightarrow \mathbf{Ta}(J_m) \overset{1-\phi^l}{\longrightarrow} \mathbf{Ta}(J_m) \longrightarrow J(F_{\rho^l})_m \longrightarrow 0 \tag{7}
\]

is also exact, where $J_m = \lim_n J[m^n]$, and $J(F_{\rho^l})_m = J(F_{\rho^l}) \otimes \mathbf{T}_m$.

**Proof.** The exact sequence (6) follows immediately from Proposition 2 by using $V = W = J_{l/F_l}$ and $f = 1 - \phi^l$. The kernel of $1 - \phi^l$ is $J(F_{\rho^l})$.

The equality $\mathbf{T} \otimes \mathbf{Z}_l = \prod_{m|l} \mathbf{T}_m$, where the product is taken over all maximal ideals $m \subseteq \mathbf{T}$ containing $l$, gives rise to the decompositions $\mathbf{Ta}(J_{l/F_l}) = \prod_{m|l} \mathbf{Ta}(J_m)$ and $J(F_{\rho^l})_l = \prod_{m|l} J(F_{\rho^l})_m$. The exact sequence (7) then follows immediately from these decompositions and (6).

From Proposition 1, we deduce the following

**PROPOSITION 3.** We have the equalities

(a) $F_\mathbf{T}(J(F_{\rho^l})) \otimes \mathbf{Z}_l = F_\mathbf{T} \otimes \mathbf{Z}_l (J(F_{\rho^l})_l)$;

(b) $F_{\mathbf{T}_m}(J(F_{\rho^l})_m) = F_\mathbf{T}(J(F_{\rho^l})) \otimes \mathbf{T}_m = F_\mathbf{T} \otimes \mathbf{Z}_l (J(F_{\rho^l})_l) \otimes \mathbf{T}_m$.

We next state a theorem that is an important ingredient in our proof of Theorem 1. We defer the proof of Theorem 2 until the next section. Before stating the theorem, we first define the notions of supersingular and ordinary maximal ideals in $\mathbf{T}$, as well as those of the Eisenstein ideal and Eisenstein primes in $\mathbf{T}$.

**DEFINITION 2.** A maximal ideal $m$ of $\mathbf{T}$ of residue characteristic $l$ is **supersingular** if the $l$th Hecke operator $T_l$ belongs to $m$. If $T_l \notin m$, we say that $m$ is **ordinary**.

**DEFINITION 3.** The **Eisenstein ideal** $\mathfrak{I} \subseteq \mathbf{T}$ is the ideal generated by the elements $1 + l - T_l$ (where $l \neq q$ is a prime) and by $1 - T_q$.

**DEFINITION 4.** A prime ideal $\mathfrak{P} \subseteq \mathbf{T}$ in the support of the Eisenstein ideal is called an **Eisenstein prime**.
THEOREM 2. Let \( m \) be a maximal ideal in \( T \) of residue characteristic \( l \). Suppose that \( m \) is not an ordinary non-Eisenstein prime of residue characteristic 2. Then \( \text{Ta}(J_m) \) is free over \( T_m \) of rank

\[
\begin{cases}
2 & \text{if } l \neq p, \\
1 & \text{if } l = p, m \text{ is ordinary,} \\
0 & \text{if } l = p, m \text{ is supersingular.}
\end{cases}
\]

PROPOSITION 4. For all primes \( l \), we have

\[ F_{T \otimes Z_l}(J(F_{p^r})){_l} = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l). \]

where \( \lambda_1, \lambda_2 \) are the eigenvalues of the Frobenius \( \phi \), acting on \( \text{Ta}(J_l) \).

Proof. Case I: If \( l \neq p \). Corollary 3 says that we have an exact sequence of \( T \otimes Z_l \)-modules

\[
0 \longrightarrow \text{Ta}(J_l) \xrightarrow{\text{Ta}(1 - \phi^n)} \text{Ta}(J_l) \xrightarrow{J(F_{p^r})){_l} \longrightarrow 0.
\]

and since \( \text{Ta}(J_l) \) is a free \( T \otimes Z_l \)-module, we have

\[ F_{T \otimes Z_l}(J(F_{p^r})){_l} = (\det \text{Ta}(1 - \phi^n))(T \otimes Z_l) = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l). \]

Case II: If \( l = p \) and \( T_p \in m \) (supersingular case). From the above theorem, we have \( \text{Ta}(J_m) = 0 \). Therefore, by (7) in Corollary 3, we have \( J(F_{p^r})_m = 0 \).

From the definition of Fitting ideals, \( F_{T_m}(J(F_{p^r})_m) = T_m \). Since \( \lambda_1 + \lambda_2 = T_p \in m \) and \( \lambda_1 \lambda_2 = p \in m \), it follows by induction that

\[
(\lambda_1^n - 1 + \lambda_2^n - 1) - \lambda_1 \lambda_2 (\lambda_1^{n-2} + \lambda_2^{n-2}) \in m.
\]

This implies that \( 1 - (\lambda_1^n + \lambda_2^n) \) is a unit in \( T_m \). Hence,

\[ F_{T_m}(J(F_{p^r})) = T_m = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T_m \]

whenever \( m \mid p \) and \( m \) is supersingular.

Case III: If \( l = p \) and \( T_p \notin m \) (ordinary case). By Theorem 2, \( \text{Ta}(J_m) \cong T_m \). The endomorphism \( \phi \) acts as a “unit” root \( \lambda_1 \) in \( T_m^* \) and we have

\[ 0 = \phi^2 - T_p \phi + p = (\phi - \lambda_1)(\phi - \lambda_2). \]

The fact that \( \lambda_1 \lambda_2 = p \in mT_m \) implies that \( \lambda_2 \in mT_m \) and \( 1 - \lambda_2^n \) is a unit in \( T_m \). It then follows that

\[ (1 - (\lambda_1^n + \lambda_2^n) + p^n) T_m = (1 - \lambda_1^n)(1 - \lambda_2^n) T_m = (1 - \lambda_1^n) T_m. \]
Hence,
\[ F_{T_m}(J(F_{p^n})) = \det (1 - \phi^n) T_m = (1 - \lambda_1^n) T_m = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T_m. \]

Putting Cases II and III together, we obtain, if \( p = l \),
\[ F_{T \otimes Z_l}(J(F_{p^n})) = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l) \]
since \( T \otimes Z_l = \prod_{m/l} T_m \).

From Propositions 3(a) and 4, we have

**Corollary 4.** If \( p \neq 2 \), then, for all primes \( l \),
\[ F_{T}(J(F_{p^n})) \otimes Z_l = (1 - (\lambda_1^n + \lambda_2^n) + p^n)(T \otimes Z_l). \]

Now we are ready to prove the main theorem, which we restate.

**Theorem 1.** If \( n \geq 1 \) is an integer and \( p \neq 2 \) is an odd prime, we have
\[ F_{T}(J(F_{p^n})) = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T, \]
where \( \lambda_1, \lambda_2 \in T \otimes Z \mathbb{C} \) satisfy \( X^2 - T_p X + p = (X - \lambda_1)(X - \lambda_2) \).

**Proof.** First, we prove a lemma:

**Lemma 1.** Let \( L, M, N \) be \( \mathbb{Z} \)-lattices and, for every prime \( l \), let \( L_l \) denote \( L \otimes Z_l \) (and similarly for \( M_l \) and \( N_l \)). If \( L \subseteq N, M \subseteq N, \) and \( L_l = M_l \subseteq N_l \) for all primes \( l \), then \( L = M \).

**Proof of Lemma 1.** Let \( X = L + M \). Then \( L \subseteq X \subseteq N \) and \( L \otimes Z_l = X \otimes Z_l \). The sequence
\[ 0 \rightarrow L \rightarrow X \rightarrow X/L \rightarrow 0 \]
is exact. Tensoring with \( Z_l \), we get an exact sequence of \( \mathbb{Z}_l \)-modules
\[ 0 \rightarrow L \otimes Z_l \rightarrow X \otimes Z_l \rightarrow (X/L) \otimes Z_l \rightarrow 0. \]
However, \( L \otimes Z_l = X \otimes Z_l \) for all \( l \) implies \( (X/L) \otimes Z_l = 0 \) for all primes \( l \). Therefore, \( X/L = 0 \); i.e., \( L = X = L + M \). Consequently, we have the inclusion \( M \subseteq L \). Symmetry of the argument implies that \( L = M \).

To complete the proof of Theorem 1, simply take \( L = F_{T}(J(F_{p^n})) \), \( M = (1 - (\lambda_1^n + \lambda_2^n) + p^n) T \) and \( N = T \).
4. **Freeness of \( \text{Ta}(J_m(\overline{F}_p)) \)**

In this section, as promised earlier, we prove Theorem 2 which was stated in the last section.

From [4, II §15.1, 15.2, 16.3, and 17.9], we see that \( \text{Ta}(J_m(\overline{Q})) \) is free over \( T_m \) of rank 2 and that \( T_m \) is Gorenstein ([4, II §15]) under each of the following hypotheses:

1. \( m \) not Eisenstein, \( \text{char } T/m \neq 2 \) (15.2);
2. \( m \) not Eisenstein, \( \text{char } T/m = 2, m \) supersingular (15.2);
3. \( m \) Eisenstein (16.3, 17.9).

We prove Theorem 2 in the three cases in the statement of the theorem.

**Case I:** \( m \nmid p; \text{ i.e., char } T/m = l \neq p. \) In this case,

\[
J[m]_{\overline{F}_p} = \bigcap_{\alpha \in m} (\ker of \alpha in J[\overline{F}_p]) = \bigcap_{\alpha \in m} (\ker of \alpha in J[l]_{\overline{F}_p})
\]

\[
\simeq \bigcap_{\alpha \in m} (\ker of \alpha in J[l]_{\overline{Q}}) = J[m]_{\overline{Q}}.
\]

Therefore, \( \text{Ta}(J_m(\overline{F}_p)) \simeq \text{Ta}(J_m(\overline{Q})) \), and hence is free of rank 2 over \( T_m \).

**Case II:** \( m \mid p, m \) supersingular; i.e., \( l = p, T_p \in m \). In this case, \( J[m](\overline{F}_p) \) admits a Jordan–Hölder filtration whose constituents are all isomorphic to the group scheme \( \alpha_p \) ([4, II §14]). In other words, the \( p \)-rank of \( J_m(\overline{F}_p) \) is 0 ([6, p.147]). It follows that ([6, p.171]) \( \text{Ta}(J_m(\overline{F}_p)) = 0 \), i.e., \( \text{Ta}(J_m(\overline{F}_p)) \) is free over \( T_m \) of rank 0.

**Case III:** \( m \mid p, m \) ordinary; i.e., \( l = p, T_p \notin m \). We proceed by using the proposition below:

**Proposition 5.** Let \( M \) be a finitely generated module over \( T_m \). Then \( M \) is free of rank 1 over \( T_m \) if and only if

1. \( M \otimes Q_p \) is free of rank 1 over \( T_m \otimes Q_p \), and
2. \( M/mM \), as a \( T/m \)-module, is free of rank 1.

**Proof of Proposition 5.** \( ( \Rightarrow ) \) This direction is clear.

\( ( \Leftarrow ) \) Nakayama’s lemma and Condition 2 imply that \( M \) is generated over \( T_m \) by an element \( x \), i.e., \( M = T_m x \). Let \( g : T_m \to M \) be the map \( g(\tau) = \tau x \), and let \( H \) be the kernel of \( g \). Then we have an exact sequence

\[
0 \longrightarrow H \longrightarrow T_m \xrightarrow{g} M \longrightarrow 0.
\]
We can consider these as $\mathbb{Z}_p$-modules, and since $Q_p$ is $\mathbb{Z}_p$-flat, the sequence of $Q_p$-modules

$$0 \longrightarrow H \otimes Q_p \longrightarrow T_m \otimes Q_p \xrightarrow{\phi \otimes 1} M \otimes Q_p \longrightarrow 0$$

is exact. By Condition 1, $H \otimes Q_p = 0$. Since $T \cong T'$, we have $T \otimes Q_p \cong T'_p$ and $H \subseteq T_m \otimes Q_p \cong T'_p$. In particular, $H$ is $T'_p$-torsion free. Therefore, $H$ can be embedded in $H \otimes Q_p$ and hence $H = 0$. In other words, $M \cong T_m$, i.e., $M$ is free over $T_m$ of rank 1. This prove Proposition 5.

**Proposition 6.** Let $m \subseteq T$ be an ordinary maximal ideal of residue characteristic $p$, with the additional condition that $m$ be Eisenstein when $p = 2$. Then the $\mathbb{Z}_p$-dual of $Ta(J_m(F_p))$ is free of rank 1 over $T_m$.

**Proof.** Let $M$ be $Ta(J_m(F_p))$ and let $M^* = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ be the $\mathbb{Z}_p$-dual of $Ta(J_m(F_p))$.

By [4, II Prop. 8.5] (see also the proof of Cor. 14.11), $M^*$ and $M$ are both of rank 1 over $T_m$; i.e., $M^* \otimes Q_p$ and $M \otimes Q_p$ are both free of rank 1 over $T_m \otimes Q_p$. Hence, Condition 1 of Proposition 5 is satisfied by $M$ and $M^*$.

We now show that Condition 2 is satisfied by $M^*/mM^*$.

**Lemma 2.** We have the isomorphism of $T/m$-modules

$$M^*/mM^* \cong \text{Hom}_{\mathbb{Z}_p}(M \langle J(F_p)[m], Q_p/\mathbb{Z}_p \rangle),$$

where $M^* = \text{Hom}_{\mathbb{Z}_p}(Ta(J_m(F_p)), \mathbb{Z}_p)$.

(For the proof, see later.)

The module on the right-hand side of (8) is free of rank 1 over $T/m$ for $p > 2$ ([4, II Cor. 14.8]). Therefore, by Proposition 5, $M^*$ is free of rank 1 over $T_m$ for $p > 2$.

Now, let $p = 2$. We are then in the case where $m$ is Eisenstein. In this case, [4, II Cor. 14.11] implies that $M^*$ is free of rank 1 over $T_m$.

It remains, therefore, to prove Lemma 2.

Let $N$ be a $\mathbb{Z}_p$-module with $T_m$-action, of the type $F \oplus (Q_p/\mathbb{Z}_p)^n$, where $F$ is a finite abelian group and $n$ a non-negative integer. Let $N^\wedge = \text{Hom}_{\mathbb{Z}_p}(N, Q_p/\mathbb{Z}_p)$ be the Pontrjagin $p$-dual of $N$.

**Lemma 3.** There is an isomorphism of $\mathbb{Z}_p$-modules $N \cong N^\wedge$. The $T_m$-action on these modules is also preserved under the isomorphism.

**Proof.** Since $\text{Hom}_{\mathbb{Z}_p}(Q_p/\mathbb{Z}_p, Q_p/\mathbb{Z}_p) = \mathbb{Z}_p$ and $\text{Hom}_{\mathbb{Z}_p}(Q_p/\mathbb{Z}_p, Q_p/\mathbb{Z}_p) = Q_p/\mathbb{Z}_p$, we have the isomorphism $N \cong N^\wedge$ if $N = (Q_p/\mathbb{Z}_p)^n$, for $n$ a
non-negative integer. For finite groups $F$, the isomorphism $F \cong F^\wedge$ is well known. Hence, we obtain the isomorphism of $\mathbb{Z}_p$-modules $N \cong N^\wedge$, where $N = F \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^f$.

If $t \in T_m$, $f \in N^\wedge$, and $j \in N$ (with corresponding element $j^* \in N^\wedge$), then $t$ acts on $N^\wedge$ by $(tf)(j) = f(tj)$. Consequently,

$$ (tj^*)(f) = j^*(tf) = (tf)(j) = f(tj) = (tj)^*(f). \tag{9} $$

Therefore, the $T_m$-action is preserved.

Now we prove Lemma 2.

Proof of Lemma 2. Let $N$ be $J_m(\bar{F}_p)$. Then $N^\wedge = M^*$ by [4, II §7, p. 92]. From Lemma 3, we have the duality $N \cong N^\wedge$. This duality gives rise to a one-to-one correspondence between the $T_m$-submodules $P$ of $M^*$ and the $T_m$-submodules $Q$ of $J_m(\bar{F}_p)$. Indeed, given $Q \subseteq J_m(\bar{F}_p)$, we get a quotient $Q^\wedge$ of $N^\wedge = M^*$. Since $\mathbb{Q}_p/\mathbb{Z}_p$ is injective, there exists a submodule $P \subseteq M^*$ such that $M^*/P \cong Q^\wedge$. Conversely, given $P \subseteq M^*$, $Q \overset{\text{def}}{=} \text{Hom}_{\mathbb{Z}_p}(M^*/P, \mathbb{Q}_p/\mathbb{Z}_p)$ injects into $\text{Hom}_{\mathbb{Z}_p}(M^*, \mathbb{Q}_p/\mathbb{Z}_p) = N^\wedge \cong N$.

These two directions are clearly inverse to each other. Now,

$$ mM^* \subseteq P \iff m \text{ annihilates } M^*/P $$
$$ \iff m \text{ annihilates } \text{Hom}_{\mathbb{Z}_p}(Q, \mathbb{Q}_p/\mathbb{Z}_p) $$
$$ \iff m \mathbb{Q} = 0 $$
$$ \iff Q \subseteq J(\bar{F}_p)[m]. $$

It follows then that $P = mM^*$ corresponds to $Q = J_m(\bar{F}_p)[m]$ in the above correspondence, and hence

$$ M^*/mM^* \cong \text{Hom}_{\mathbb{Z}_p}(J(\bar{F}_p)[m], \mathbb{Q}_p/\mathbb{Z}_p). \tag{10} $$

Proposition 7 below implies that $M = \text{Ta}(J_m(\bar{F}_p))$ is free of rank 1 over $T_m$ in Case III.

**Proposition 7.** Let $m \leq T$ be a maximal ideal of residue characteristic $p$ and let $M$ be $\text{Ta}(J_m(\bar{F}_p))$. Suppose that $M^* = \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ is free of rank 1 over $T_m$. Then $M$ is also free of rank 1 over $T_m$.

**Proof.** As $J_m(\bar{F}_p) \subseteq J_p(\bar{F}_p)$, and $\text{Ta}(J_m(\bar{F}_p))$ is a free $\mathbb{Z}_p$-module, we see that $M = \text{Ta}(J_m(\bar{F}_p))$ is also a free $\mathbb{Z}_p$-module. If the $\mathbb{Z}_p$-rank of $M$ is $s$, then $M \cong \mathbb{Z}_p^s$. Since $\text{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p^s, \mathbb{Z}_p) \cong \mathbb{Z}_p^s$, it follows that $M \cong (M^*)^*$ as $\mathbb{Z}_p$-modules. An argument parallel to the one used in (9) shows that
$M \simeq (M^*)^*$ as $T_m$-modules. Consequently, we have isomorphisms of $T_m$-modules

$$M \simeq \text{Hom}_{Z_p}(M^*, Z_p) \simeq \text{Hom}_{Z_p}(T_m, Z_p) \simeq T_m,$$

where the last isomorphism follows from the fact that $T_m$ is Gorenstein. This shows that $M$ is free of rank 1 over $T_m$. 

The proof of Theorem 2 is now complete.

REFERENCES