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# A boundary element and radial basis function method for the Cattaneo–Vernotte equation in anisotropic media with spatially varying and temperature dependent properties

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ABSTRACT

A numerical method based on boundary integral equation formulation and radial basis function approximation is presented for the numerical solution of the Cattaneo–Vernotte hyperbolic equation in anisotropic media with spatially varying and temperature dependent properties. Specific test problems are solved to verify the validity and accuracy of the proposed numerical procedure.

## 1. Introduction

The tensor equation for the well known Cattaneo–Vernotte constitutive model relating the heat flux and the temperature gradient in the conduction of heat in solids is given by[Refs. 1,2]

$$\underline{\mathbf{q}} + \tau \frac{\partial \underline{\mathbf{q}}}{\partial t} = -\underline{\underline{\boldsymbol{\kappa}}} \cdot \underline{\nabla} T \tag{1.1}$$

where $T$ is the temperature, the vector $\underline{\mathbf{q}}$ is the heat flux, the second rank tensor $\underline{\underline{\kappa}}$ is the thermal conductivity, $t$ is the time coordinate and the scalar $\tau$ is the relaxation time.

In the absence of internal heat source, the equation expressing the law of conservation of heat energy is given by

$$\underline{\nabla} \cdot \underline{\mathbf{q}} = -\rho c \frac{\partial T}{\partial t}, \tag{1.2}$$

where $\rho$ and $c$ are the volume density and the specific heat capacity respectively.

Together with (1.2), taking the divergence of both sides of (1.1) leads to the Cattaneo–Vernotte non-classical hyperbolic heat equation

$$\underline{\nabla} \cdot (\underline{\underline{\boldsymbol{\kappa}}} \cdot \underline{\nabla} T) = \rho c \frac{\partial T}{\partial t} + \tau \frac{\partial}{\partial t}(\rho c \frac{\partial T}{\partial t}), \tag{1.3}$$

where the relaxation time $\tau$ is a constant.

In the current paper, a numerical method based on boundary integral equation formulation and radial basis function approximation is presented for the numerical solution of an initial–boundary value problem governed by the Cattaneo–Vernotte hyperbolic heat equation in anisotropic media with spatially varying and temperature dependent properties. In recent years, the Catteneo–Vernotte equation is widely used in thermal analyses of modern materials in many engineering applications (see, for example, Refs. 3–5). For material properties that vary continuously from point to point in space and with temperature, the equation is non-linear and has spatially varying coefficients. In general, such an equation is not easily amenable to analytic solutions. Thus, the task of developing a numerical method for the problem under consideration here is a research topic worthy of attention.

Existing works on numerical methods for solving the Cattaneo–Vernotte hyperbolic heat equation, mostly for the case of thermally isotropic media with constant thermal conductivity, include finite difference schemes[Refs. 6–8], finite element methods[Refs. 9,10] and finite volume methods[Refs. 11,12]. The boundary integral equation and radial basis function approach proposed here should offer an interesting and viable alternative numerical method for solving the hyperbolic heat equation. Numerical methods based on boundary integral equations do not require the solution domain to be discretized into elements. Only the boundary of the solution domain has to be discretized. Hence, such methods are relatively easy to implement, especially for two-dimensional problems[Ref. 13]. Furthermore, the boundary conditions can be easily incorporated into the discretized boundary integral equations.

## 2. Initial boundary value problem

Referring to a Cartesian coordinate frame denoted by $Ox_1x_2x_3$, consider a thermally anisotropic solid with geometry that does not change along the $x_3$-axis. On the $Ox_1x_2$ plane, the body occupies the two-dimensional region $R$ bounded by a simple closed curve $C$.

The temperature $T$ in the solid does not vary with $x_3$, that is, $T$ is a function of the Cartesian coordinates $x_1$ and $x_2$ only and the time

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coordinate $t$. The thermal conductivity $\underline{\underline{\kappa}}$, the volume density $\rho$ and the specific heat capacity $c$ vary smoothly with $x_1$, $x_2$ and $T$.

The thermal behavior of the body is governed by (1.3) which may be written in Cartesian coordinates as

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(\kappa_{ij}(x_1,x_2,T)\frac{\partial T}{\partial x_j}) = A(x_1,x_2,T)\frac{\partial^2 T}{\partial t^2} + B(x_1,x_2,T,\frac{\partial T}{\partial t})\frac{\partial T}{\partial t}, \tag{2.1}$$

where the coefficients $\kappa_{ij}$ are the Cartesian components of the thermal conductivity $\underline{\underline{\kappa}}$ satisfying the symmetry relation $\kappa_{ij} = \kappa_{ji}$ and the strict inequality $\kappa_{12}^2 - \kappa_{11}\kappa_{22} < 0$, and the coefficients $A(x_1,x_2,T)$ and $B(x_1,x_2,T)$ are given by

$$\begin{aligned} A(x_1,x_2,T) &= \tau\rho(x_1,x_2,T)c(x_1,x_2,T), \\ B(x_1,x_2,T,\frac{\partial T}{\partial t}) &= \rho(x_1,x_2,T)c(x_1,x_2,T) \\ &\quad + \tau\frac{\partial}{\partial T}(\rho(x_1,x_2,T)c(x_1,x_2,T))\frac{\partial T}{\partial t}. \end{aligned} \tag{2.2}$$

The problem of interest here is to solve the nonlinear hyperbolic heat equation (2.1) for $t > 0$ in the region $R$ subject to the initial conditions

$$\left.\begin{aligned} T(x_1,x_2,0) &= f(x_1,x_2) \\ \left.\frac{\partial T}{\partial t}\right|_{t=0} &= g(x_1,x_2) \end{aligned}\right\} \text{ for } (x_1,x_2)\in R, \tag{2.3}$$

and the boundary conditions

$$\begin{aligned} T(x_1,x_2,t) &= r(x_1,x_2,t) \text{ for } (x_1,x_2)\in C_1, \\ \sum_{i=1}^{2}\sum_{j=1}^{2}\kappa_{ij}(x_1,x_2,T)n_i(x_1,x_2)\frac{\partial T}{\partial x_j} &= s(x_1,x_2,t) \text{ for } (x_1,x_2)\in C_2, \end{aligned} \tag{2.4}$$

where $f(x_1,x_2)$, $g(x_1,x_2)$, $r(x_1,x_2,t)$ and $s(x_1,x_2,t)$ are suitably prescribed functions, $C_1$ and $C_2$ are non-intersecting curves such that $C_1 \cup C_2 = C$ and $n_i(x_1,x_2)$ is the $x_i$ component of the unit outward normal vector to $C$ at the point $(x_1,x_2)$.

## 3. Time-stepping predictor–corrector scheme

For a time-stepping scheme with three time levels, consider (2.1) for $n\Delta t < t < (n+1)\Delta t$ for $n = 0, 1, 2, \cdots$, where $\Delta t$ is a small positive time-step.

Defining

$$\left.\begin{aligned} T^{(n)}(x_1,x_2) &= T(x_1,x_2,n\Delta t) \\ T^{(n+\frac{1}{2})}(x_1,x_2) &= T(x_1,x_2,(n+\frac{1}{2})\Delta t) \end{aligned}\right\} \text{ for } n = 0,1,2,\ldots, \tag{3.1}$$

we approximate (2.1) at $t = (n+\frac{1}{2})\Delta t$ by

$$\begin{aligned} &\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(\kappa_{ij}(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))\frac{\partial}{\partial x_j}(T^{(n+\frac{1}{2})}(x_1,x_2))) \\ &= 4A(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))\frac{T^{(n+1)}(x_1,x_2) - 2T^{(n+\frac{1}{2})}(x_1,x_2) + T^{(n)}(x_1,x_2)}{(\Delta t)^2} \\ &\quad + B(x_1,x_2,\overline{T}^{(n)}(x_1,x_2),\overline{Z}^{(n)}(x_1,x_2))\frac{T^{(n+1)}(x_1,x_2) - T^{(n)}(x_1,x_2)}{\Delta t}. \end{aligned} \tag{3.2}$$

where $\overline{T}^{(n)}(x_1,x_2)$ and $\overline{Z}^{(n)}(x_1,x_2)$ are respectively given (known) estimation of $T$ and $\frac{\partial T}{\partial t}$ at time $t = (n+\frac{1}{2})\Delta t$.

Using the Lagrangian polynomial interpolation, we make the approximation

$$\begin{aligned} T(x_1,x_2,t) \simeq \frac{2}{(\Delta t)^2}((t-(n+\frac{1}{2})\Delta t)(t-(n+1)\Delta t)T^{(n)}(x_1,x_2) \\ - 2(t-n\Delta t)(t-(n+1)\Delta t)T^{(n+\frac{1}{2})}(x_1,x_2) \\ + (t-n\Delta t)(t-(n+\frac{1}{2})\Delta t)T^{(n+1)}(x_1,x_2)) \\ \text{for } n\Delta t < t < (n+1)\Delta t. \end{aligned} \tag{3.3}$$

From (3.3), we obtain the approximate formulae

$$\begin{aligned} \left.\frac{\partial T}{\partial t}\right|_{t=n\Delta t} &\simeq \frac{1}{\Delta t}(-3T^{(n)}(x_1,x_2) + 4T^{(n+\frac{1}{2})}(x_1,x_2) - T^{(n+1)}(x_1,x_2)), \\ \left.\frac{\partial T}{\partial t}\right|_{t=(n+1)\Delta t} &\simeq \frac{1}{\Delta t}(T^{(n)}(x_1,x_2) - 4T^{(n+\frac{1}{2})}(x_1,x_2) + 3T^{(n+1)}(x_1,x_2)). \end{aligned} \tag{3.4}$$

Assume that $T^{(n)}(x_1,x_2)$ is known. The time-stepping predictor–corrector scheme for finding $T^{(n+\frac{1}{2})}(x_1,x_2)$ and $T^{(n+1)}(x_1,x_2)$ for $n = 0$, 1, 2, $\cdots$ comprises the steps below.

1. Start with $n = 0$. Define $\overline{T}^{(0)}(x_1,x_2) = f^{(0)}(x_1,x_2)$ and $\overline{Z}^{(0)}(x_1,x_2) = g^{(0)}(x_1,x_2)$, where $f^{(0)}(x_1,x_2)$ and $g^{(0)}(x_1,x_2)$ are respectively the given functions $f(x_1,x_2)$ and $g(x_1,x_2)$ in the initial conditions in (2.3).
2. Solve (3.2) together with $T^{(n)}(x_1,x_2) = f^{(n)}(x_1,x_2)$ and
$$\frac{1}{\Delta t}(-3T^{(n)}(x_1,x_2) + 4T^{(n+\frac{1}{2})}(x_1,x_2) - T^{(n+1)}(x_1,x_2)) = g^{(n)}(x_1,x_2),$$
subject to the boundary conditions in (2.4), for the unknown functions $T^{(n+\frac{1}{2})}(x_1,x_2)$ and $T^{(n+1)}(x_1,x_2)$ by using the numerical method detailed in Section 4 below. Note that $g^{(n)}(x_1,x_2)$ above is an approximation of $\partial T/\partial t$ at $t = n\Delta t$ (refer to the first formula in (3.4)). The coefficients in (3.2) given by $\kappa_{ij}(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))$, $A(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))$ and $B(x_1,x_2,\overline{T}^{(n)}(x_1,x_2),\overline{Z}^{(n)}(x_1,x_2))$, are evaluated by using the latest definitions of $\overline{T}^{(n)}(x_1,x_2)$ and $\overline{Z}^{(n)}(x_1,x_2)$.
3. Check if $T^{(n+\frac{1}{2})}(x_1,x_2)$ and $T^{(n+1)}(x_1,x_2)$ converge to a prescribed number of significant figures at some chosen points in the solution domain. If the required convergence is not achieved, redefine the functions $\overline{T}^{(n)}(x_1,x_2)$ and $\overline{Z}^{(n)}(x_1,x_2)$ by
$$\begin{aligned} \overline{T}^{(n)}(x_1,x_2) &= T^{(n+\frac{1}{2})}(x_1,x_2), \\ \overline{Z}^{(n)}(x_1,x_2) &= \frac{T^{(n+1)}(x_1,x_2) - T^{(n)}(x_1,x_2)}{\Delta t}, \end{aligned}$$
and return to Step 2 above. Otherwise, if the required convergence is achieved, either stop the time-stepping scheme (if the solution of the initial boundary value problem is not required at a higher time level) or define $f^{(n+1)}(x_1,x_2) = T^{(n+1)}(x_1,x_2)$ and
$$g^{(n+1)}(x_1,x_2) = \frac{1}{\Delta t}(f^{(n)}(x_1,x_2) - 4T^{(n+\frac{1}{2})}(x_1,x_2) + 3T^{(n+1)}(x_1,x_2)),$$
and go to Step 4 below (if the solution is required at a higher time level). Note that $g^{(n+1)}(x_1,x_2)$ is an approximation of $\partial T/\partial t$ at $t = (n+1)\Delta t$ (refer to the second formula in (3.4)).
4. Increase the value of the integer $n$ by 1. Define $\overline{T}^{(n)}(x_1,x_2) = f^{(n)}(x_1,x_2)$ and $\overline{Z}^{(n)}(x_1,x_2) = g^{(n)}(x_1,x_2)$. Go to Step 2 above.

## 4. Boundary element and radial basis function method

### 4.1. Reformulation of partial differential equation

Assuming that $T^{(n)}(x_1,x_2)$, $\overline{T}^{(n)}(x_1,x_2)$ and $\overline{Z}^{(n)}(x_1,x_2)$ are known for a fixed integer $n$, we outline here a numerical method for solving (3.2) to determine the unknown functions $T^{(n+\frac{1}{2})}(x_1,x_2)$ and $T^{(n+1)}(x_1,x_2)$.

Omitting the fixed integer $n$ for convenience, we rewrite (3.2) as

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(k_{ij}(x_1,x_2)\frac{\partial T_1}{\partial x_j}) + c_0(x_1,x_2) + c_1(x_1,x_2)T_1 + c_2(x_1,x_2)T_2 = 0, \tag{4.1}$$

where $k_{ij}(x_1,x_2) = \kappa_{ij}(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))$, $T_1(x_1,x_2)$ and $T_2(x_1,x_2)$ denote $T^{(n+\frac{1}{2})}(x_1,x_2)$ and $T^{(n+1)}(x_1,x_2)$ respectively, and $c_0(x_1,x_2)$, $c_1(x_1,x_2)$ and $c_2(x_1,x_2)$ are defined by

$$c_0(x_1,x_2) = -(\frac{1}{(\Delta t)^2}4A(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))$$

$$-\frac{1}{\Delta t}B(x_1,x_2,\overline{T}^{(n)}(x_1,x_2),\overline{Z}^{(n)}(x_1,x_2)))T^{(n)}(x_1,x_2),$$
$$c_1(x_1,x_2)=\frac{8}{(\Delta t)^2}A(x_1,x_2,\overline{T}^{(n)}(x_1,x_2)),$$
$$c_2(x_1,x_2)=-\frac{4}{(\Delta t)^2}A(x_1,x_2,\overline{T}^{(n)}(x_1,x_2))$$
$$-\frac{1}{\Delta t}B(x_1,x_2,\overline{T}^{(n)}(x_1,x_2),\overline{Z}^{(n)}(x_1,x_2)). \quad (4.2)$$

The equation given in Step 2 in the time-stepping predictor–corrector scheme described in Section 3 can be written as

$$4T_1(x_1,x_2)-T_2(x_1,x_2)+c_3(x_1,x_2)=0, \quad (4.3)$$

where $c_3(x_1,x_2)$ is a known function given by

$$c_3(x_1,x_2)=-g^{(n)}(x_1,x_2)\Delta t-3T^{(n)}(x_1,x_2). \quad (4.4)$$

From (4.1) and (4.3), we write

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(k_{ij}(x_1,x_2)\frac{\partial u}{\partial x_j})+d_0(x_1,x_2)+d_1(x_1,x_2)u=0, \quad (4.5)$$

where $u(x_1,x_2)=T_1(x_1,x_2)$ and

$$d_0(x_1,x_2)=c_0(x_1,x_2)+c_2(x_1,x_2)c_3(x_1,x_2),$$
$$d_1(x_1,x_2)=c_1(x_1,x_2)+4c_2(x_1,x_2). \quad (4.6)$$

Following closely the analysis in Ref. 14, we let the solution of (4.5) be given by

$$u(x_1,x_2)=v(x_1,x_2)+w(x_1,x_2), \quad (4.7)$$

where the function $v$ is chosen to be related to $u$ by

$$\sum_{i=1}^{2}\sum_{j=1}^{2}(k_{ij}^{(0)}\frac{\partial^2 v}{\partial x_i\partial x_j}+\frac{\partial}{\partial x_i}(k_{ij}^{(1)}\frac{\partial u}{\partial x_j}))=-d_0(x_1,x_2)-d_1(x_1,x_2)u, \quad (4.8)$$

and $w$ satisfies the constant coefficient linear elliptic partial differential equation

$$\sum_{i=1}^{2}\sum_{j=1}^{2}k_{ij}^{(0)}\frac{\partial^2 w}{\partial x_i\partial x_j}=0, \quad (4.9)$$

where $k_{ij}^{(0)}$ are constant coefficients obtained by averaging $k_{ij}$ uniformly over the solution domain $R$ and $k_{ij}^{(1)}=k_{ij}-k_{ij}^{(0)}$ are, in general, functions that vary smoothly with $x_1$ and $x_2$ in the solution domain.

### 4.2. Radial basis function approximation

To discretize (4.8) into linear algebraic equations by using a meshless technique, we choose $P$ well spaced out collocation points in $R\cup C$. We denote the chosen collocation points by $(\zeta_1^{(1)},\zeta_2^{(1)})$, $(\zeta_1^{(2)},\zeta_2^{(2)})$, $\cdots$, $(\zeta_1^{(P-1)},\zeta_2^{(P-1)})$ and $(\zeta_1^{(P)},\zeta_2^{(P)})$, where $\zeta_i^{(j)}$ is the $x_i$ coordinate of the $j$th collocation point.

We make the approximations

$$\sum_{j=1}^{2}(k_{ij}^{(0)}\frac{\partial v}{\partial x_j}+k_{ij}^{(1)}(x_1,x_2)\frac{\partial u}{\partial x_j})\simeq\sum_{r=1}^{P}a_i^{(r)}\varrho^{(r)}(x_1,x_2), \quad (4.10)$$

and

$$u(x_1,x_2)\simeq\sum_{r=1}^{P}b^{(r)}\varrho^{(r)}(x_1,x_2),$$
$$v(x_1,x_2)\simeq\sum_{r=1}^{P}c^{(r)}\varrho^{(r)}(x_1,x_2), \quad (4.11)$$

where $a_i^{(r)}$, $b^{(r)}$ and $c^{(r)}$ are constants and $\varrho^{(r)}(x_1,x_2)$ is a radial basis function centered about $(\zeta_1^{(r)},\zeta_2^{(r)})$.

If $\varrho^{(r)}(x_1,x_2)$ is partially differentiable once with respect to $x_1$ or $x_2$, we may use (4.10) and (4.11), as detailed in Ref. 14, to discretize (4.8) into linear algebraic equations

$$d_1^{(n)}u^{(n)}+\sum_{m=1}^{P}(\mu^{(nm)}v^{(m)}+\omega^{(nm)}u^{(m)})=-d_0^{(n)} \text{ for } n=1,2,\ldots,P, \quad (4.12)$$

where $d_0^{(n)}$, $d_1^{(n)}$, $u^{(n)}$ and $v^{(n)}$ are respectively the values of $d_0(x_1,x_2)$, $d_1(x_1,x_2)$, $u(x_1,x_2)$ and $v(x_1,x_2)$ at $(x_1,x_2)=(\zeta_1^{(n)},\zeta_2^{(n)})$ and $\mu^{(nm)}$ and $\omega^{(nm)}$ are constants defined by

$$\mu^{(nm)}=\sum_{r=1}^{P}\sum_{i=1}^{2}\gamma_i^{(rm)}\frac{\partial}{\partial x_i}(\varrho^{(r)}(x_1,x_2))\Big|_{(x_1,x_2)=(\zeta_1^{(n)},\zeta_2^{(n)})},$$
$$\omega^{(nm)}=\sum_{r=1}^{P}\sum_{i=1}^{2}\beta_i^{(rm)}\frac{\partial}{\partial x_i}(\varrho^{(r)}(x_1,x_2))\Big|_{(x_1,x_2)=(\zeta_1^{(n)},\zeta_2^{(n)})}. \quad (4.13)$$

together with

$$\gamma_i^{(rm)}=\sum_{n=1}^{P}\varphi^{(rn)}\theta_i^{(nm)},$$
$$\beta_i^{(rm)}=\sum_{n=1}^{P}\varphi^{(rn)}\phi_i^{(nm)}, \quad (4.14)$$

and

$$\theta_i^{(nm)}=\sum_{j=1}^{2}k_{ij}^{(0)}\sum_{r=1}^{P}\varphi^{(rm)}\frac{\partial}{\partial x_j}(\varrho^{(r)}(x_1,x_2))\Big|_{(x_1,x_2)=(\zeta_1^{(n)},\zeta_2^{(n)})},$$
$$\phi_i^{(nm)}=\sum_{j=1}^{2}k_{ij}^{(1)}(\zeta_1^{(n)},\zeta_2^{(n)})\sum_{r=1}^{P}\varphi^{(rm)}\frac{\partial}{\partial x_j}(\varrho^{(r)}(x_1,x_2))\Big|_{(x_1,x_2)=(\zeta_1^{(n)},\zeta_2^{(n)})},$$
$$\sum_{m=1}^{P}\varphi^{(rm)}\varrho^{(s)}(\zeta_1^{(m)},\zeta_2^{(m)})=\begin{cases}1 & \text{if} \quad r=s,\\ 0 & \text{if} \quad r\neq s.\end{cases} \quad (4.15)$$

The unknowns in the linear algebraic equations in (4.12) are $u^{(n)}$ and $v^{(n)}$ for $n=1,2,\cdots,P$.

### 4.3. Boundary integral approximation

The use of (4.7) and the boundary integral equation of (4.9) as derived in Ref. 15 yields

$$\lambda(\zeta_1,\zeta_2)(u(\zeta_1,\zeta_2)-v(\zeta_1,\zeta_2))$$
$$=\int_C((u(x_1,x_2)-v(x_1,x_2))\Gamma(x_1,x_2,\zeta_1,\zeta_2)$$
$$-(p(x_1,x_2)-q(x_1,x_2))\Phi(x_1,x_2,\zeta_1,\zeta_2))ds(x_1,x_2), \quad (4.16)$$

where $\lambda(\zeta_1,\zeta_2)$ is such that $\lambda(\zeta_1,\zeta_2)=1$ if $(\zeta_1,\zeta_2)$ lies in the interior of the solution domain $R$ bounded by the curve $C$ and $\lambda(\zeta_1,\zeta_2)=1/2$ if $(\zeta_1,\zeta_2)$ lies on a smooth part of the curve $C$, $p(x_1,x_2)$ and $q(x_1,x_2)$ are defined by

$$p(x_1,x_2)=\sum_{i=1}^{2}\sum_{j=1}^{2}n_i(x_1,x_2)k_{ij}^{(0)}\frac{\partial}{\partial x_j}(u(x_1,x_2)),$$
$$q(x_1,x_2)=\sum_{i=1}^{2}\sum_{j=1}^{2}n_i(x_1,x_2)k_{ij}^{(0)}\frac{\partial}{\partial x_j}(v(x_1,x_2)), \quad (4.17)$$

and $\Phi(x_1,x_2,\zeta_1,\zeta_2)$ and $\Gamma(x_1,x_2,\zeta_1,\zeta_2)$ are given by

$$\Phi(x_1,x_2,\zeta_1,\zeta_2)=\frac{1}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}\operatorname{Re}\{\ln(x_1-\zeta_1+\tau_0[x_2-\zeta_2])\},$$
$$\Gamma(x_1,x_2,\zeta_1,\zeta_2)=\frac{1}{2\pi\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}\operatorname{Re}\left\{\frac{L(x_1,x_2)}{(x_1-\zeta_1+\tau_0[x_2-\zeta_2])}\right\},$$
$$L(x_1,x_2)=(k_{11}^{(0)}+\tau_0k_{12}^{(0)})n_1(x_1,x_2)+(k_{21}^{(0)}+\tau_0k_{22}^{(0)})n_2(x_1,x_2),$$
$$\tau_0=\frac{-k_{12}^{(0)}+i\sqrt{k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2}}{k_{22}^{(0)}}\quad(i=\sqrt{-1}). \quad (4.18)$$

Note that $k_{11}^{(0)}k_{22}^{(0)}-(k_{12}^{(0)})^2>0$.

Following the procedure in Ref. 15, we discretize the boundary integral equation (4.16) into linear algebraic equations as follows.

The boundary $C$ by using $M$ straight line elements denoted by $E^{(1)}$, $E^{(2)}$, $\cdots$, $E^{(M-1)}$ and $E^{(M)}$, that is, we make the approximation

$C \simeq E^{(1)} \cup E^{(2)} \cup \cdots \cup E^{(M-1)} \cup E^{(M)}$. We take the collocation point $(\zeta_1^{(m)}, \zeta_2^{(m)})$ for the meshless technique in Section 4.2 to be the midpoint of $E^{(m)}$ for $m = 1, 2, \ldots, M$, that is, we take the first $M$ collocation points to be the midpoints of the $M$ elements. The other collocation points lie in the interior of the solution domain. The number of interior collocation points is $N$. Thus, the total number of collocation points is given by $P = M + N$.

If we assume that $u(x_1, x_2)$, $v(x_1, x_2)$, $p(x_1, x_2)$ and $q(x_1, x_2)$ are respectively constants $u^{(m)}$, $v^{(m)}$, $p^{(m)}$ and $q^{(m)}$ over the element $E^{(m)}$, we can approximate (4.16) and collocate it at $(x_1, x_2) = (\zeta_1^{(n)}, \zeta_2^{(n)})$ for $n = 1, 2, \ldots, P$ to obtain the linear algebraic equations

$$\begin{aligned}
&\lambda(\zeta_1^{(n)}, \zeta_2^{(n)})(u^{(n)} - v^{(n)}) \\
&= \sum_{m=1}^{M} (u^{(m)} - v^{(m)}) \int_{E^{(m)}} \Gamma(x_1, x_2, \zeta_1^{(n)}, \zeta_2^{(n)}) ds(x_1, x_2) \\
&- \sum_{m=1}^{M} (p^{(m)} - q^{(m)}) \int_{E^{(m)}} \Phi(x_1, x_2, \zeta_1^{(n)}, \zeta_2^{(n)}) ds(x_1, x_2) \\
&\qquad \text{for } n = 1, 2, \ldots, P. \qquad (4.19)
\end{aligned}$$

Analytical formulae for evaluating the line integrals over $E^{(m)}$ are given in Ref. 14.

The unknowns in the linear algebraic equations in (4.19) are given by $u^{(n)}$ and $v^{(n)}$ for $n = 1, 2, \cdots, P$, and $p^{(m)}$ and $q^{(m)}$ for $m = 1, 2, \cdots, M$.

### 4.4. Numerical solution

As shown in Ref. 14, the unknown $q^{(m)}$ on the element $E^{(m)}$ is related to the unknowns $v^{(q)}$ $(q = 1, 2, \ldots, P)$ by

$$q^{(m)} - \sum_{q=1}^{P} v^{(q)} z^{(mq)} = 0 \text{ for } m = 1, 2, \ldots, M, \qquad (4.20)$$

where

$$z^{(mq)} = \sum_{i=1}^{2} n_i^{(m)} \sum_{j=1}^{2} k_{ij}^{(0)} \sum_{r=1}^{P} \varphi^{(rq)} \frac{\partial}{\partial x_j} (\varrho^{(r)}(x_1, x_2)) \Big|_{(x_1,x_2)=(\zeta_1^{(m)},\zeta_2^{(m)})}. \qquad (4.21)$$

From (2.4), the boundary conditions for $u(x_1, x_2)$ are given by

$$\begin{aligned}
&u(x_1, x_2) = r_1(x_1, x_2) \text{ for } (x_1, x_2) \in C_1, \\
&\sum_{i=1}^{2} n_i(x_1, x_2) \sum_{j=1}^{2} k_{ij}(x_1, x_2) \frac{\partial u}{\partial x_j} = s_1(x_1, x_2) \text{ for } (x_1, x_2) \in C_2, \qquad (4.22)
\end{aligned}$$

where $r_1(x_1, x_2) = r(x_1, x_2, (n + \frac{1}{2})\Delta t)$ and $s_1(x_1, x_2) = s(x_1, x_2, (n + \frac{1}{2})\Delta t)$.

The boundary conditions in (4.22) give

$$u^{(m)} = r_1(\zeta_1^{(m)}, \zeta_2^{(m)}) \text{ if } u \text{ is known on } E^{(m)}, \qquad (4.23)$$

or

$$\begin{aligned}
&p^{(m)} + \sum_{q=1}^{P} u^{(q)} y^{(mq)} = s_1(\zeta_1^{(m)}, \zeta_2^{(m)}) \\
&\text{if } \sum_{i=1}^{2} n_i(x_1, x_2) \sum_{j=1}^{2} k_{ij}(x_1, x_2) \frac{\partial u}{\partial x_j} \text{ is known on } E^{(m)}, \qquad (4.24)
\end{aligned}$$

where the constant coefficients $y^{(mq)}$ defined by

$$y^{(mq)} = \sum_{i=1}^{2} n_i^{(m)} \sum_{j=1}^{2} k_{ij}^{(1)}(\zeta_1^{(m)}, \zeta_2^{(m)}) \sum_{r=1}^{P} \varphi^{(rq)} \frac{\partial}{\partial x_j} (\varrho^{(r)}(x_1, x_2)) \Big|_{(x_1,x_2)=(\zeta_1^{(m)},\zeta_2^{(m)})}, \qquad (4.25)$$

with $n_i^{(m)}$ being the $x_i$ component of the unit vector that is normal to $E^{(m)}$ and that points out of the solution domain.

We solve (4.12) and (4.19) together with (2.4) and either (4.23) or (4.24) to determine $u^{(n)}$, that is, the value of $u(x_1, x_2)$ at the collocation point $(\zeta_1^{(n)}, \zeta_2^{(n)})$.

## 5. Specific test problems

The numerical procedure based on the time-stepping predictor–corrector scheme in Section 3 and the boundary element and radial basis function method in Section 4 is applied to solve some specific test problems. For the radial basis approximation in Section 4.2, we use the radial basis function proposed in Ref. 16, that is,

$$\varrho^{(r)}(x_1, x_2) = 1 + (x_1 - \zeta_1^{(r)})^2 + (x_2 - \zeta_2^{(r)})^2 + ((x_1 - \zeta_1^{(r)})^2 + (x_2 - \zeta_2^{(r)})^2)^{3/2}. \qquad (5.1)$$

**Test problem 1**. For time $t > 0$, solve

$$\begin{aligned}
&2\frac{\partial}{\partial x_1}((x_1^2 - 2x_1x_2 + 2)^2 T \frac{\partial T}{\partial x_1}) + \frac{\partial}{\partial x_1}((x_1^2 - 2x_1x_2 + 2)^2 T \frac{\partial T}{\partial x_2}) \\
&+ \frac{\partial}{\partial x_2}((x_1^2 - 2x_1x_2 + 2)^2 T \frac{\partial T}{\partial x_1}) + \frac{\partial}{\partial x_2}((x_1^2 - 2x_1x_2 + 2)^2 T \frac{\partial T}{\partial x_2}) \\
&= (x_1^2 - 2x_1x_2 + 2)^2 (T \frac{\partial T}{\partial t} + \frac{\partial}{\partial t}(T \frac{\partial T}{\partial t})) \\
&\text{for } 0 < x_1 < 1, \ 0 < x_2 < 1, \qquad (5.2)
\end{aligned}$$

subject to the initial conditions

$$\left.\begin{aligned}
T(x_1, x_2, 0) &= \sqrt{\frac{1 + x_1 - x_2}{x_1^2 - 2x_1x_2 + 2}} \\
\frac{\partial T}{\partial t}\Big|_{t=0} &= -\frac{1}{2}\sqrt{\frac{1 + x_1 - x_2}{x_1^2 - 2x_1x_2 + 2}}
\end{aligned}\right\} \text{for } 0 < x_1 < 1, \ 0 < x_2 < 1, \qquad (5.3)$$

and the boundary conditions

$$\begin{aligned}
&\left.\begin{aligned}
T(x_1, 0, t) &= e^{-t/2}\sqrt{\frac{1 + x_1}{x_1^2 + 2}} \\
T(x_1, 1, t) &= e^{-t/2}\sqrt{\frac{x_1}{x_1^2 - 2x_1 + 2}}
\end{aligned}\right\} \text{ for } 0 < x_1 < 1, \\
&\left.\begin{aligned}
T(0, x_2, t) &= e^{-t/2}\sqrt{\frac{1 - x_2}{2}} \\
T(1, x_2, t) &= e^{-t/2}\sqrt{\frac{2 - x_2}{3 - 2x_2}}
\end{aligned}\right\} \text{ for } 0 < x_2 < 1. \qquad (5.4)
\end{aligned}$$

The initial boundary value problem defined by (5.2), (5.3) and (5.4) can be recovered from (2.1), (2.3) and (2.4) by letting $\tau = 1$, $\kappa_{11} = 2(x_1^2 - 2x_1x_2 + 2)^2 T$, $\kappa_{12} = \kappa_{21} = \kappa_{22} = (x_1^2 - 2x_1x_2 + 2)^2 T$ and $\rho c = (x_1^2 - 2x_1x_2 + 2)^2 T$.

We may verify by direct substitution that the analytic solution of the initial boundary value problem is given by

$$T(x_1, x_2, t) = e^{-t/2}\sqrt{\frac{1 + x_1 - x_2}{x_1^2 - 2x_1x_2 + 2}}. \qquad (5.5)$$

For the numerical solution of (5.2) subject to (5.3) and (5.4), we divide each side of the square domain into $M_0$ equal length boundary elements, so that total number of boundary elements is given by $M = 4M_0$. The $N$ interior collocation points are taken to be given by $(i/(N_0 + 1), j/(N_0 + 1))$ for $i = 1, 2, \cdots, N_0$ and $j = 1, 2, \cdots, N_0$, that is, $N = N_0^2$.

In Table 1, numerical values of $T(1/2, 1/2, t)$ obtained using $(M_0, N_0) = (5, 4)$ and $(M_0, N_0) = (10, 6)$ with $\Delta t = 1/5$, are compared with the analytic solution in (5.5) at selected time instants $t$. Table 2 gives a comparison of the numerical and analytic values of $T(x_1, x_2, 1/2)$ at selected points in the interior of the square domain. In both tables, there is an obvious reduction in the average absolute error (A.A.E.) of the numerical values when the calculation is refined by increasing the number of boundary elements and interior collocation points.

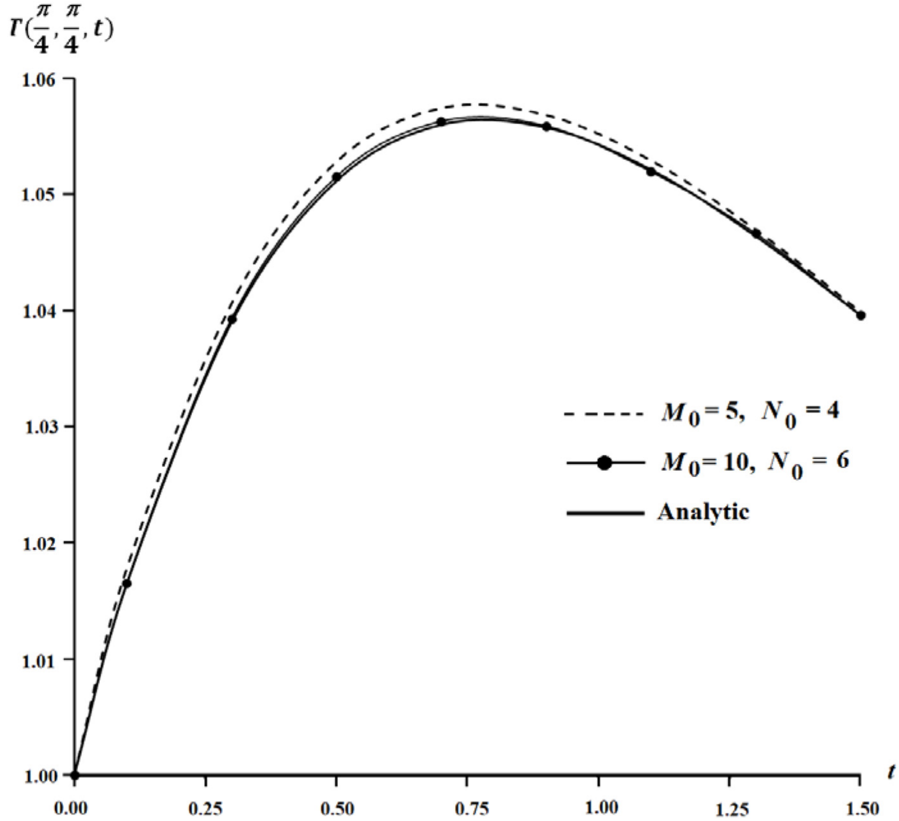


**Fig. 1.** Plots of numerical and analytic values of $T(\pi/4, \pi/4, t)$ for $0 \le t < \pi/2$.

**Table 1**
Comparison of numerical and analytic values of $T(1/2, 1/2, t)$ at selected time instants $t$.

| Time $t$ | $M_0 = 5$ $N_0 = 4$ | $M_0 = 10$ $N_0 = 6$ | Analytic |
|---|---|---|---|
| 0.10 | 0.72021 | 0.71894 | 0.71906 |
| 0.30 | 0.65301 | 0.65051 | 0.65063 |
| 0.50 | 0.59291 | 0.58880 | 0.58872 |
| 0.70 | 0.53701 | 0.53333 | 0.53269 |
| 0.90 | 0.48486 | 0.48133 | 0.48200 |
| A.A.E. | $3.0 \times 10^{-3}$ | $3.3 \times 10^{-4}$ | – |

**Table 2**
Comparison of numerical and analytic values of $T(x_1, x_2, 1/2)$ at selected points $(x_1, x_2)$.

| Point $(x_1, x_2)$ | $M_0 = 5$ $N_0 = 4$ | $M_0 = 10$ $N_0 = 6$ | Analytic |
|---|---|---|---|
| (0.25, 0.25) | 0.56226 | 0.55776 | 0.55951 |
| (0.25, 0.50) | 0.49676 | 0.50067 | 0.50098 |
| (0.25, 0.75) | 0.40819 | 0.42626 | 0.42393 |
| (0.50, 0.25) | 0.62521 | 0.61439 | 0.61570 |
| (0.50, 0.50) | 0.59291 | 0.58880 | 0.58872 |
| (0.50, 0.75) | 0.53256 | 0.55054 | 0.55069 |
| (0.75, 0.25) | 0.64950 | 0.64356 | 0.64491 |
| (0.75, 0.50) | 0.65598 | 0.64724 | 0.64676 |
| (0.75, 0.75) | 0.64099 | 0.64607 | 0.64956 |
| A.A.E. | $8.5 \times 10^{-3}$ | $1.3 \times 10^{-3}$ | – |

**Test problem 2**. For time $t > 0$, solve

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial}{\partial x_i}(\delta_{ij}T^2\frac{\partial T}{\partial x_j}) = T^2\frac{\partial T}{\partial t} + \frac{1}{2}\frac{\partial}{\partial t}(T^2\frac{\partial T}{\partial t})$$
$$\text{for } 0 < x_1 < \pi/2,\ 0 < x_2 < \pi/2, \quad (5.6)$$

subject to the initial conditions

$$\left.\begin{array}{rcl} T(x_1, x_2, 0) & = & 1 \\ \left.\dfrac{\partial T}{\partial t}\right|_{t=0} & = & \dfrac{1}{3}x_2\cos(x_1) \end{array}\right\} \text{ for } 0 < x_1 < \pi/2,\ 0 < x_2 < \pi/2, \quad (5.7)$$

and the boundary conditions

$$\left.\begin{array}{rcl} T(x_1, 0, t) & = & 1 \\ T^2\left.\dfrac{\partial T}{\partial x_2}\right|_{x_2=\pi/2} & = & \dfrac{1}{3}\cos(x_1)\sin(t)e^{-t} \end{array}\right\} \text{ for } 0 < x_1 < \pi/2,$$
$$\left.\begin{array}{rcl} T^2\left.\dfrac{\partial T}{\partial x_1}\right|_{x_1=0} & = & 0 \\ T^2\left.\dfrac{\partial T}{\partial x_1}\right|_{x_1=\pi/2} & = & -\dfrac{1}{3}x_2\sin(t)e^{-t} \end{array}\right\} \text{ for } 0 < x_2 < \pi/2. \quad (5.8)$$

The initial boundary value problem defined by (5.6), (5.7) and (5.8) can be recovered from (2.1), (2.3) and (2.4) by letting $\tau = 1/2$, $\kappa_{ij} = \delta_{ij}T^2$ and $\rho c = T^2$. Note that $\delta_{ij}$ is the Kronecker delta.

The analytic solution of the initial boundary value problem is given by

$$T(x_1, x_2, t) = (1 + x_2\cos(x_1)\sin(t)e^{-t})^{1/3}. \quad (5.9)$$

Fig. 1 gives plots of numerical and the analytic values of $T(\pi/4, \pi/4, t)$ for $0 \le t < \pi/2$. The numerical values are obtained using $(M_0, N_0) = (5, 4)$ and $(M_0, N_0) = (10, 6)$ with $\Delta t = 1/5$. The numerical plots are close to the analytic solution, even for the relatively coarse calculation given by $(M_0, N_0) = (5, 4)$. The plot of the numerical values obtained using $(M_0, N_0) = (10, 6)$ is almost indistinguishable from that of the analytic solution, showing an improvement in the numerical values when the calculation is refined by increasing the number of boundary elements and interior collocation points. Note that the deviation of both numerical plots from the analytic solution is slightly more pronounced near $t = \pi/4 \simeq 0.785\,40$, where $T(\pi/4, \pi/4, t)$ peaks with time.

## 6. Summary and final remarks

We have successfully derived and implemented on the computer a numerical method based on boundary integral equation and radial basis function approximation for solving a two-dimensional non-classical heat conduction problem governed by the Cattaneo–Vernotte hyperbolic heat equation in an anisotropic medium with properties that vary continuously from point to point in space and with temperature. The validity and accuracy of the boundary element procedure is verified by applying it to solve some specific test problems with known solutions. We find that the numerical solutions obtained agree well with the known solutions and they show convergence when the calculation is refined.

In explicit time-stepping finite difference methods for hyperbolic partial differential equations, the time step and finite difference grid spacing are required to satisfy the Courant–Friedrichs–Lewy condition for stability in the numerical calculations[17]. The condition is, however, not applicable to implicit time-stepping finite difference schemes which are generally stable irrespective of the time step and the grid spacing used.

It may be highly challenging to analyze theoretically the stability and error of the numerical method here for the nonlinear Cattaneo–Vernotte hyperbolic heat equation. Nevertheless, with the boundary integral equation discretized at the second time level of the three time level time-stepping scheme, the numerical approach used here cannot be regarded as explicit and hence the numerical method may be expected to be stable. Practical and useful insights of the numerical method may also be gained through numerical experiments, such as by applying it to solve specific test problems with known analytic solutions. For example, from the numerical experiments on the two specific test problems in Section 5, we observe that the iterative solution converges at a faster rate at collocation points nearer to the center of the solution domain, and that the accuracy of the solution at each time step may be reduced if we proceed to a higher time level before the iterative solution achieves sufficient convergence at all the collocation points.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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