

Unconditionally Stable LOD-FDTD Method for 3-D Maxwell's Equations

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Abstract—This paper presents an unconditionally stable locally one-dimensional finite-difference time-domain (LOD-FDTD) method for three-dimensional (3-D) Maxwell's equations. The method does not exhibit the second-order non-commutativity error and its second-order temporal accuracy is ascertained via numerical justification. The method also involves simpler updating procedures and facilitates exploitation of parallel and/or reduced output processing. This leads to its higher computation efficiency than the alternating direction implicit (ADI) and split-step (SS2) FDTD methods.

Index Terms—Locally one-dimensional FDTD, alternating direction implicit FDTD, split-step FDTD, unconditionally stable FDTD methods, computational electromagnetics.

I. INTRODUCTION

There has been considerable interest in the development of unconditionally stable finite-difference time-domain (FDTD) methods that are not constrained by Courant-Friedrich-Levy (CFL) condition [1]. One such method is based on the celebrated alternating direction implicit (ADI) technique [2], [3]. Recently, many researchers have proposed other unconditionally stable (nondissipative) methods such as split-step (SS) [4], [5] and locally one-dimensional (LOD) FDTD methods [6], [7]. It can be found that the LOD-FDTD methods presented in [6], [7] for two dimensions correspond to the split-step approach of [4] denoted by SS1. As is commonly anticipated by the researchers, such LOD-FDTD exhibits an extra non-commutativity error term (not present in ADI), thus making the method accurate to first order only in time. To achieve second-order temporal accuracy, Strang splitting scheme may be employed as in the split-step approach of [4] denoted by SS2. However, when field data is to be output at every time step, this scheme involves more arithmetic operations for requiring three updating procedures (see also [5]) that do not facilitate separate or parallel implementation.

In this paper, an alternative unconditionally stable LOD-FDTD method is presented which does not exhibit the aforementioned second-order non-commutativity error. For generality, we shall consider the LOD-FDTD method for full three-dimensional (3-D) Maxwell's equations. Numerical justification will be provided to ascertain its second-order temporal accuracy. Furthermore, the present 3-D LOD-FDTD method involves essentially two simple updating procedures and facilitates exploitation of parallel and/or reduced output processing. This leads to its higher computation efficiency than the ADI and SS2 FDTD methods.

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II. 3-D LOD-FDTD METHOD

In this section, the updating equations will be presented explicitly for the 3-D LOD-FDTD method. Like the previous LOD-FDTD methods [6], [7], there are two updating procedures in the main iterations. However, one major distinction is that the present two procedures signify the advancement of time steps from $n + \frac{1}{4}$ to $n + \frac{3}{4}$ and from $n + \frac{3}{4}$ to $n + 1\frac{1}{4}$, rather than from n to $n + \frac{1}{2}$ and from $n + \frac{1}{2}$ to $n + 1$ as in the previous methods. Such distinction has important implications on the temporal accuracy especially when one is dealing with field data at integer time steps (as is often the case). Based on the LOD principle of rational approximation, the two procedures can be formulated as follows (the equations for other field components can be written down simply by permuting the indices):

A. First procedure from $n + \frac{1}{4}$ to $n + \frac{3}{4}$

(i) Implicit updating for $E_x|^{n+\frac{3}{4}}$:

$$\begin{aligned} & -\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}E_x|_{i+\frac{1}{2},j-1,k}^{n+\frac{3}{4}} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta y^2}\right)E_x|_{i+\frac{1}{2},j,k}^{n+\frac{3}{4}} \\ & -\frac{\Delta t^2}{4\mu\epsilon\Delta y^2}E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{3}{4}} = \left(1 - \frac{\Delta t^2}{2\mu\epsilon\Delta y^2}\right)E_x|_{i+\frac{1}{2},j,k}^{n+\frac{1}{4}} \\ & + \frac{\Delta t^2}{4\mu\epsilon\Delta y^2}(E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{1}{4}} + E_x|_{i+\frac{1}{2},j-1,k}^{n+\frac{1}{4}}) \\ & + \frac{\Delta t}{\epsilon\Delta y}(H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{4}} - H_z|_{i+\frac{1}{2},j-\frac{1}{2},k}^{n+\frac{1}{4}}). \quad (1) \end{aligned}$$

(ii) Explicit updating for $H_z|^{n+\frac{3}{4}}$:

$$\begin{aligned} H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{3}{4}} & = H_z|_{i+\frac{1}{2},j+\frac{1}{2},k}^{n+\frac{1}{4}} \\ & + \frac{\Delta t}{2\mu\Delta y}(E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{1}{4}} - E_x|_{i+\frac{1}{2},j,k}^{n+\frac{1}{4}} \\ & + E_x|_{i+\frac{1}{2},j+1,k}^{n+\frac{3}{4}} - E_x|_{i+\frac{1}{2},j,k}^{n+\frac{3}{4}}). \quad (2) \end{aligned}$$

B. Second procedure from $n + \frac{3}{4}$ to $n + 1\frac{1}{4}$

(i) Implicit updating for $E_x|^{n+1\frac{1}{4}}$:

$$\begin{aligned} & -\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}E_x|_{i+\frac{1}{2},j,k-1}^{n+1\frac{1}{4}} + \left(1 + \frac{\Delta t^2}{2\mu\epsilon\Delta z^2}\right)E_x|_{i+\frac{1}{2},j,k}^{n+1\frac{1}{4}} \\ & -\frac{\Delta t^2}{4\mu\epsilon\Delta z^2}E_x|_{i+\frac{1}{2},j,k+1}^{n+1\frac{1}{4}} = \left(1 - \frac{\Delta t^2}{2\mu\epsilon\Delta z^2}\right)E_x|_{i+\frac{1}{2},j,k}^{n+\frac{3}{4}} \\ & + \frac{\Delta t^2}{4\mu\epsilon\Delta z^2}(E_x|_{i+\frac{1}{2},j,k+1}^{n+\frac{3}{4}} + E_x|_{i+\frac{1}{2},j,k-1}^{n+\frac{3}{4}}) \\ & - \frac{\Delta t}{\epsilon\Delta z}(H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{3}{4}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+\frac{3}{4}}). \quad (3) \end{aligned}$$

(ii) Explicit updating for $H_y|^{n+1\frac{1}{4}}$:

$$H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+1\frac{1}{4}} = H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+\frac{3}{4}} - \frac{\Delta t}{2\mu\Delta z} (E_x|_{i+\frac{1}{2},j,k+1}^{n+\frac{3}{4}} - E_x|_{i+\frac{1}{2},j,k}^{n+\frac{3}{4}} + E_x|_{i+\frac{1}{2},j,k+1}^{n+1\frac{1}{4}} - E_x|_{i+\frac{1}{2},j,k}^{n+1\frac{1}{4}}). \quad (4)$$

With field updating at quarter (and three-quarter) time steps, it will often be necessary to relate the fields to those at integer time steps. This calls for additional processing for the input and output field data as follows (the coefficient $\frac{1}{16} = \frac{1}{4^2}$ corresponds to the quarter-step updating):

C. Input processing at $n = 0$

(i) Input processing (implicit) for $E_x|^{1/4}$:

$$-\frac{\Delta t^2}{16\mu\epsilon\Delta z^2} E_x|_{i+\frac{1}{2},j,k-1}^{1/4} + \left(1 + \frac{\Delta t^2}{8\mu\epsilon\Delta z^2}\right) E_x|_{i+\frac{1}{2},j,k}^{1/4} - \frac{\Delta t^2}{16\mu\epsilon\Delta z^2} E_x|_{i+\frac{1}{2},j,k+1}^{1/4} = \left(1 - \frac{\Delta t^2}{8\mu\epsilon\Delta z^2}\right) E_x|_{i+\frac{1}{2},j,k}^0 + \frac{\Delta t^2}{16\mu\epsilon\Delta z^2} (E_x|_{i+\frac{1}{2},j,k+1}^0 + E_x|_{i+\frac{1}{2},j,k-1}^0) - \frac{\Delta t}{2\epsilon\Delta z} (H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^0 - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^0). \quad (5)$$

(ii) Input processing (explicit) for $H_y|^{1/4}$:

$$H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{1/4} = H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^0 - \frac{\Delta t}{4\mu\Delta z} (E_x|_{i+\frac{1}{2},j,k+1}^0 - E_x|_{i+\frac{1}{2},j,k}^0 + E_x|_{i+\frac{1}{2},j,k+1}^{1/4} - E_x|_{i+\frac{1}{2},j,k}^{1/4}). \quad (6)$$

D. Output processing at $n + 1$

(i) Output processing (implicit) for $E_x|^{n+1}$:

$$-\frac{\Delta t^2}{16\mu\epsilon\Delta z^2} E_x|_{i+\frac{1}{2},j,k-1}^{n+1} + \left(1 + \frac{\Delta t^2}{8\mu\epsilon\Delta z^2}\right) E_x|_{i+\frac{1}{2},j,k}^{n+1} - \frac{\Delta t^2}{16\mu\epsilon\Delta z^2} E_x|_{i+\frac{1}{2},j,k+1}^{n+1} = \left(1 - \frac{\Delta t^2}{8\mu\epsilon\Delta z^2}\right) E_x|_{i+\frac{1}{2},j,k}^{n+1\frac{1}{4}} + \frac{\Delta t^2}{16\mu\epsilon\Delta z^2} (E_x|_{i+\frac{1}{2},j,k+1}^{n+1\frac{1}{4}} + E_x|_{i+\frac{1}{2},j,k-1}^{n+1\frac{1}{4}}) + \frac{\Delta t}{2\epsilon\Delta z} (H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+1\frac{1}{4}} - H_y|_{i+\frac{1}{2},j,k-\frac{1}{2}}^{n+1\frac{1}{4}}). \quad (7)$$

(ii) Output processing (explicit) for $H_y|^{n+1}$:

$$H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+1} = H_y|_{i+\frac{1}{2},j,k+\frac{1}{2}}^{n+1\frac{1}{4}} + \frac{\Delta t}{4\mu\Delta z} (E_x|_{i+\frac{1}{2},j,k+1}^{n+1\frac{1}{4}} - E_x|_{i+\frac{1}{2},j,k}^{n+1\frac{1}{4}} + E_x|_{i+\frac{1}{2},j,k+1}^{n+1} - E_x|_{i+\frac{1}{2},j,k}^{n+1}). \quad (8)$$

Note that the input processing is to be invoked only once (unlike SS2) at $n = 0$ throughout the entire simulations (even if frequent outputs are required). Since most electromagnetic excitation problems involve null initial fields, this step may be omitted altogether. For the output processing at $n + 1$, one often does not need to update all field components (unlike SS2) except only one or two of interest at few desired observation

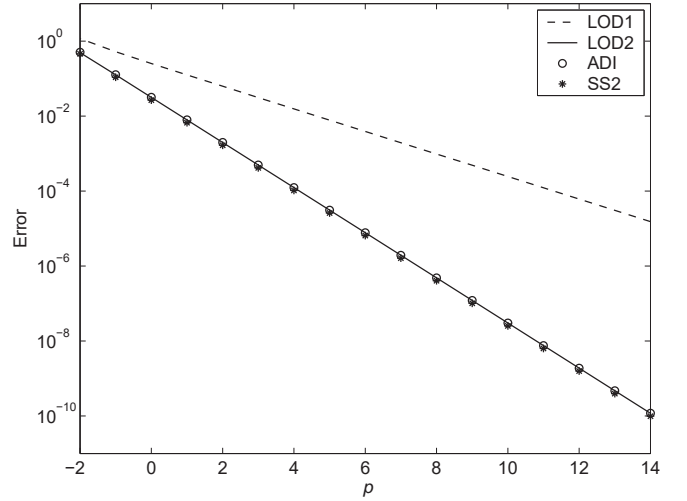


Fig. 1. Normalized norm errors versus p of geometric time subdivision ($\Delta t = \Delta t_{\text{CFL}}/2^p$) for LOD-FDTD without (LOD1) and with (LOD2) input/output processing. Also shown are the errors for ADI and SS2 FDTD.

points. Meanwhile, the previous LOD-FDTD methods [6], [7], with their field updating at (half and) integer time steps (like SS1), do not prompt for any additional processing like above. Their field data will then be prone to (uncorrected) non-commutativity error making them accurate to first order only in time.

III. NUMERICAL RESULTS AND DISCUSSIONS

In this section, numerical test is carried out to show that the present 3-D LOD-FDTD method is indeed of second-order temporal accuracy and does not exhibit the non-commutativity error. For simplicity, it is sufficient to illustrate using a small problem whereby numerically exact solution can be obtained readily for reference. Since we are interested in the temporal error alone, it is necessary to ensure that the (same) spatial discretization errors have been incorporated into the numerically exact reference. To that end, we chose an air-filled cavity meshed with $8 \times 8 \times 8$ uniform grid cells of size 2 mm each. A first-order differential matrix system is then set up for this geometry with its numerically exact solution determined from the corresponding matrix exponential.

To initialize the fields inside the cavity, they are arbitrarily set to be those of TM_{111} mode. The simulations are then carried out until $t = 4\Delta t_{\text{CFL}}$ with varying time step size $\Delta t = \Delta t_{\text{CFL}}/2^p$, where Δt_{CFL} is the Courant limit time step size and p is the integer for geometric time subdivision. A normalized norm error is defined in terms of the entire FDTD results (superscripted with n) and the numerically exact solutions (subscripted with e) as

$$\text{Error} = \frac{\|E^n - E_e\|}{\|E_e\|} + \frac{\|H^n - H_e\|}{\|H_e\|}. \quad (9)$$

Fig. 1 plots the normalized norm errors versus p for the LOD-FDTD methods without (LOD1) and with (LOD2) input/output processing. It is evident that the errors are of $O(\Delta t^2)$ and $O(\Delta t^3)$ for LOD1 and LOD2 which signify their temporal accuracies of first and second order respectively. Also

TABLE I
FLOPS COUNT FOR 3-D FDTD METHODS

3-D FDTD Method		ADI [2], [3]	LOD herein	SS2 [4]
Implicit	M/D	18	18	27
	A/S	48	24	36
Explicit	M/D	12	6	9
	A/S	24	24	36
Total	M/D	30	24	36
	A/S	72	48	72

shown in the figure are the errors associated with the ADI and SS2 FDTD methods, which are clearly second-order accurate in time as is well understood.

Having ascertained the temporal accuracy, it will be desirable to assess the computation efficiency of the 3-D LOD-FDTD method as compared to the 3-D ADI-FDTD method of [2], [3]. Table I lists the floating point operations (flops) count for both methods, taking into account the number of multiplications/divisions (M/D) and additions/subtractions (A/S) required in the right-hand sides of their main updating equations, c.f. (1)-(4) etc. For simplicity, the number of electric and magnetic field components in all directions have been taken to be the same. From the table, it is clear that the total (M/D+A/S) flops count of LOD-FDTD (72) has been reduced considerably from that of ADI-FDTD (102). Note that the flops count for the input and output processing has been excluded. This is good for the case like Fig. 1 above, where (5)-(6) are invoked only once at the beginning, while (7)-(8) are executed only at the end of many iterations. Also shown in Table I is the (right-hand side) flops count for SS2 of [4], which involves three updating procedures.

In simulations where the entire field data is to be output frequently at short periodic time intervals, the flops for output processing should be counted as well. Furthermore, in order to better evaluate the overall efficiency gain, the cost of solving tridiagonal systems should also be considered. This takes approximately $5N$ flops for a tridiagonal system of order N using precomputed bidiagonally factorized elements. Taking these flops into account, the efficiency gains achieved for LOD-FDTD over ADI-FDTD are depicted in Fig. 2 for various field data output periods m , i.e. data is output at time step $n = m, 2m, 3m, \dots$. From the figure (crossed line), we see that the LOD-FDTD method is more efficient than the ADI-FDTD method except only when $m \sim 1$. In this case, the LOD-FDTD, like the original SS2 (cf. Fig. 2 dotted line), essentially calls for three updating procedures at every time step. Hence the reduction of flops in their right-hand sides could not cover the overall cost. To alleviate such shortcoming, we may exploit the present LOD-FDTD that allows separate or parallel processing of output field data. For instance, the computations to deduce say, the first output at time step $n = m$, i.e. $m + \frac{1}{4} \rightarrow m$ [via (7)-(8)] may be performed separately and independently in parallel without disrupting the main iterations: $\frac{1}{4} \rightarrow \frac{3}{4} \rightarrow \dots \rightarrow m - \frac{3}{4} \rightarrow m - \frac{1}{4} \rightarrow m + \frac{1}{4} \rightarrow m + \frac{3}{4} \rightarrow \dots$ [via (1)-(4)]. By this way maximum efficiency gain may be achieved directly without having to approach large $m \gg 1$, see Fig. 2 (dashed line). This is in contrast to the combined SS2 approach that does not facilitate

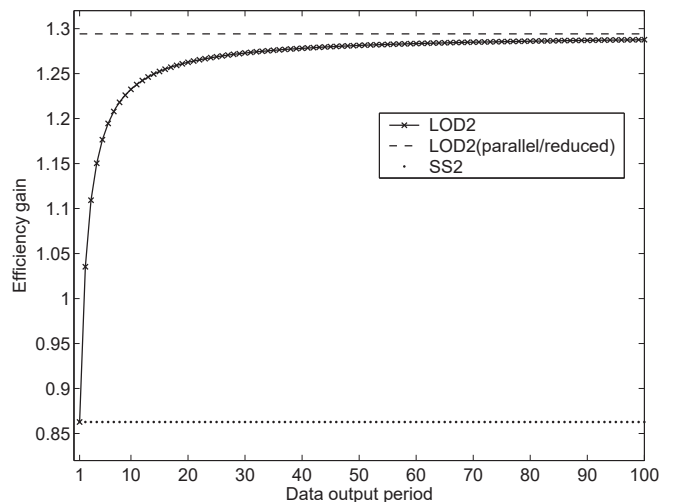


Fig. 2. Efficiency gains of LOD-FDTD over ADI-FDTD for various field data output periods. Maximum gain may be achieved directly with parallel and/or reduced output processing. Also shown is the efficiency (relative to ADI) of SS2 with three updating procedures.

such independent output processing at $n = m$ in its iterations. Moreover as mentioned above, when only a small number of field components are needed at only a few observation points, the computation cost of output processing for LOD-FDTD can be reduced further.

IV. CONCLUSION

This paper has presented an unconditionally stable 3-D LOD-FDTD method without second-order non-commutativity error. With its higher efficiency due to simpler updating procedures along with exploitation of parallel and/or reduced output processing, the present method would be very attractive for further developments and applications.

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