Robust Non-Zero-Sum Stochastic Differential Reinsurance Game ☆

Chi Seng Pun[a], Hoi Ying Wong[*,a]

[a]*Department of Statistics, The Chinese University of Hong Kong, Shatin, Hong Kong*

Abstract

This paper considers the non-zero-sum stochastic differential game problem between two ambiguity-averse insurers (AAIs) who encounter model uncertainty and seek the optimal reinsurance decision under relative performance concerns. Each AAI manages her own risks by purchasing reinsurance with the objective of maximizing the expected utility of her relative terminal surplus with respect to that of her counterparty. The two AAIs' decisions influence each other through the insurers' relative performance concerns and the correlation between their surplus processes. We establish a general framework of Nash equilibrium for the associated non-zero-sum game with model uncertainty. For the representative case of exponential utilities, we solve the equilibrium strategies explicitly. Numerical studies are conducted to draw economic interpretations.

Key words: Reinsurance, Non-Zero-Sum Stochastic Differential Game, Relative Performance Concerns, Model Uncertainty, Hamiltonian-Jacobi-Bellman-Isaacs Equation, Nash Equilibrium

1. Introduction

Investment and reinsurance (IR) are major financial activities by which insurers manage their risk and make a profit from their surplus. The associated stochastic control problem is widely studied in the insurance field. For example, Browne (1995) investigates the relation between minimizing ruin probability and maximizing the exponential utility of terminal wealth. Schmidli (2002) considers the IR optimization problem of minimizing ruin probability, and Luo et al. (2008)

☆First version: August 1, 2015; This version: March 8, 2016.

[*] Corresponding author

Email addresses: `cspun@link.cuhk.edu.hk` (Chi Seng Pun), `hywong@cuhk.edu.hk` (Hoi Ying Wong)

identify the optimal policy for the constrained problem. Bai and Guo (2008) study the IR optimization problem of maximizing exponential utility with a no-shorting constraint. Cao and Wan (2009) investigate the Hamilton-Jacobi-Bellman (HJB) equations for related problems. Liu and Ma (2009) revisit the IR problem under a general insurance model.

The IR optimization problem for an ambiguity-averse insurer (AAI) has also attracted considerable research attention. Ambiguity here refers to the Knightian (model) uncertainty originating with Knight (1921), who clarified the difference between risk and uncertainty. The Ellsberg paradox (1961) reveals the inadequacy of utility theory, and argues that human beings are ambiguity-averse. This human characteristic should also be taken into account for IR decisions. Related studies include those of Maenhout (2004), who considers a robust asset allocation framework via a regularized max-min expected utility approach in accordance with the robust decision rules in Anderson et al. (1999); Zhang and Siu (2009), who focus on the robust IR problem via a max-min expected utility approach; Yi et al. (2013), who extend the approach of Maenhout (2004) to the robust IR problem under the Heston stochastic volatility (SV) model; and Pun and Wong (2015), who explore the robust IR problem with a general class of concave utilities and multiscale SV models.

All of the aforementioned studies focus only on single-agent optimization problems. However, in a competitive economy, firms tend to compare themselves with one another, and relative performance concerns thus play a key role in decision-making. DeMarzo et al. (2008) point out that the concept of relative performance concerns is relevant to financial bubbles and excess volatility. A tractable framework for modeling the interaction among firms is proposed by Espinosa and Touzi (2015). The associated optimization problems for maximizing the utilities of relative wealths in their study constitute a non-zero-sum stochastic differential game with N heterogeneous agents whose utility functions possibly differ. They also show the existence and uniqueness of Nash equilibrium for the associated game problem. Subsequently, Bensoussan et al. (2014) extended their results to the IR game with two insurers.

This paper combines the ambiguity aversion and relative performance concerns to investigate robust reinsurance decisions.[1] We formulate the optimal decisions of two insurers benchmarking

---

[1] The extension to IR problems is rather immediate, particularly when the investment and reinsurance decisions are

each other as a stochastic differential game. Their utilities may differ and their claim processes jointly follow correlated Brownian motions with drifts, as proposed by Promislow and Young (2005) and many others. The reinsurance proportion is the control variable in the surplus process. Both proportional reinsurance and permission to acquire a new business are possible in our formulation. Under the framework of Pun and Wong (2015), we derive the Nash equilibrium of the robust non-zero-sum stochastic differential reinsurance game (RNSDRG) as the solution of a system of Hamiltonian-Jacobi-Bellman-Issaacs (HJBI) equations. We characterize the equilibrium strategies under general settings. For the representative case of exponential utilities, we provide an explicit solution for the whole problem in equilibrium, as well as the economic interpretation through our numerical studies.

Although this paper inevitably makes use of the stochastic differential game analysis, our primary interest is not the interaction between insurers through trading risks. Therefore, our consideration abstracts the possibility that insurers themselves could also be the reinsurers of their competitors. Here, we investigate the influence of relative performance concerns and ambiguity aversion on insurers' reinsurance demand. In the single-agent setting, the classical theory of Garven and Lamm-Tennant (2003) suggests some of the major factors influencing reinsurance demand, such as leverage, asset volatility and claim delay. For corresponding empirical studies based on these factors, see, for example, Garven and Lamm-Tennant (2003) and Cole and McCullough (2006).

In the context of relative performance concerns, we find that ambiguity aversion (or the fear of failure to predict a crisis) is also an important factor in reinsurance demand. Self-reinsurance demand increases with self-ambiguity aversion, of other things being fixed. When two insurers' portfolios are positively correlated, the effect of ambiguity aversion is magnified. For instance, once a competitor increases (decreases) her reinsurance demand owing to an increase (decrease) in her degree of ambiguity aversion, the underlying insurer will also increase (decrease) her demand to maintain her economic status quo. That is the consequence of competition in the face of relative performance concerns. Eventually, the total reinsurance demand of this two-person insurance

---

statistically independent, and can be found at http://ssrn.com/abstract=2744495 industry increases (decreases) to reach an equilibrium level. Our game theoretic model implies that crisis analytic technology reduces insurers' ambiguity aversion and, consequently, leads to reduced reinsurance demand. This implication echoes the uncertain future of the reinsurance industry owing to technological advancement documented in Boyle (2015).

The remainder of the paper is organized as follows. Section 2 presents the problem formulation, including the definition of the surplus processes and financial market. We apply the dynamic programming principle to the problem and characterize the Nash equilibrium in Section 3. Despite its complex structure, we derive an explicit solution for the case of exponential utilities and the Heston SV model in Section 4. We also numerically illustrate the effects of ambiguity aversion (particularly that of the other party) that and relative interest on the equilibrium strategies in this section. Section 5 concludes.

2. Problem Formulation

Consider the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by two-dimensional $\mathbb{P}$-Brownian motion $\{W^G(t)\}_{t\geq 0} = \{(W^{G_1}(t), W^{G_2}(t)\}_{t\geq 0}$.

2.1. Reference Model

Two competing insurers manage their risks through reinsurance. Therefore, the insurers' economic decision depends on their individual claim processes.

In accordance with Promislow and Young (2005), we model the claim process with the standard Cramér-Lundberg approximation (see Klugman et al. (2008) for more details). More specifically, the claim process $\{C_k(t)\}_{t\geq 0}$ of the insurer $k \in \{1, 2\}$ follows the stochastic differential equation:

$$dC_k(t) = a_k dt - b_k dW^{G_k}(t),$$

where $a_k, b_k > 0$ are the claim rate and the volatility of the claim process, respectively, and $\{W^{G_k}(t)\}_{t\geq 0}$ is the standard $\mathbb{P}$-Brownian motion, for $k = 1, 2$, such that $\mathbb{E}[dW^{G_1}(t)dW^{G_2}(t)] = \rho\, dt$. We further assume that the ratio a_k/b_k is sufficiently large ($a_k/b_k > 3$) that the probability of realizing a negative claim is small in any period of time. In the absence of a reinsurance market, the insurance premium rate is $\varsigma_k^{in} = (1+\tau_k)a_k$, with safety loading $\tau_k > 0$, implying that the surplus process $\{G_k(t)\}_{t\geq 0}$ of the insurer $k \in \{1,2\}$ evolves as

$$dG_k(t) = \varsigma_k^{in}dt - dC_k(t) = \tau_k a_k dt + b_k dW^{G_k}(t).$$

When there is a reinsurance market, the two insurers become able to manage their insurance risks through reinsurance. Let $1 - q_k(t)$ be the reinsurance fraction of the insurer $k \in \{1,2\}$ at time t. The stochastic process $\{q_k(t)\}_{t\geq 0}$ is an $\mathcal{F}_t$ progressively measurable process valued in $\mathbb{R}^+$, representing the reinsurance strategy of insurer k. When $q_k(t) > 1$, insurer k provides a reinsurance service to a reinsurance company or acquires a new business. When $q_k(t) \in [0,1]$, insurer k purchases proportional reinsurance protection from the reinsurance company, which will cover $1-q_k(t)$ of the claims and charge a reinsurance premium at rate $\varsigma_k^{re} = (1+\eta_k)(1-q_k(t))a_k$, with safety loading $\eta_k \geq \tau_k > 0$. Let $Q_k^{[0,1]} = \{q_k(t) : q_k(t) \in [0,1]\}$ and $Q_k^{[0,+\infty)} = \{q_k(t) : q_k(t) \in [0,+\infty)\}$ be the sets of admissible reinsurance strategies of insurer k for the cases in which acquiring a new business is prohibited and allowed, respectively. Our subsequent analyses hold true for both strategy sets, although their solutions are discussed separately. With reinsurance strategy $q_k(t)$, the surplus process $\{G_k^{q_k}(t)\}_{t\geq 0}$ of the insurer $k \in \{1,2\}$ becomes

$$dG_k^{q_k}(t) = \varsigma_k^{in}dt - q_k(t)dC_k(t) - \varsigma_k^{re}dt = [\lambda_k + \eta_k q_k(t)]a_k dt + b_k q_k(t) dW^{G_k}(t), \tag{1}$$

where $\lambda_k = \tau_k - \eta_k \leq 0$.

Suppose that the insurer k's surplus is put into the risk-free asset at rate r, a constant positive risk-free interest rate. Let $\{X_k^{q_k}(t)\}_{t\geq 0}$ be the wealth process of the insurer $k \in \{1,2\}$. The self-financing strategy yields the dynamics of $\{X_k^{q_k}(t)\}_{t\geq 0}$ as

$$dX_k^{q_k}(t) = dG_k^{q_k}(t) + rX_k^{q_k}dt = [a_k\lambda_k + a_k\eta_k q_k(t) + rX_k^{q_k}(t)]dt + b_k q_k(t) dW^{G_k}(t),$$

where initial wealth $X_k^{q_k}(0) = x_k > 0$, for $k = 1,2$. The variance-covariance matrix Σ_G of $\{W^G(t)\}_{t\geq 0}$ is given by

$$\Sigma_G = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}. \tag{2}$$

2.2. Ambiguity-averse Insurers under Relative Performance Concerns

We are interested in AAIs under relative performance concerns. The insurer $k \in \{1,2\}$ has utility function $U_k : \mathbb{R} \to \mathbb{R}$, which is assumed to be C^1, increasing, and strictly concave and to satisfy Inada conditions:

$$U_k'(-\infty) = +\infty,\ U_k'(+\infty) = 0.$$

Ambiguity aversion refers to distrust in reference measure $\mathbb{P}$. An AAI is concerned about the estimation of and misspecification errors in the reference model, and considers a class of alternative measures $\mathcal{Q}$ that are similar to $\mathbb{P}$. To clarify the meaning of "similar" here, we employ the concept of equivalent measures, analogous to Maenhout (2004), Yi et al. (2013) and Pun and Wong (2015). More specifically, the alternative measures are probability measures equivalent to $\mathbb{P}$: $\mathcal{Q} = \{\mathbb{Q}|\mathbb{Q} \sim \mathbb{P}\}$. By the Girsanov theorem, for each $\mathbb{Q} \in \mathcal{Q}$, there is a stochastic process $\varphi_{\mathbb{Q}}^G(t) = (\varphi_{\mathbb{Q}}^{G_1}(t), \varphi_{\mathbb{Q}}^{G_2}(t))'$ that can be regarded as model misspecification factors, such that

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \nu(t) := \exp\left(\int_0^t \varphi_{\mathbb{Q}}^G(s)'dW(s) - \frac{1}{2}\int_0^t \varphi_{\mathbb{Q}}^G(s)'\Sigma_G\varphi_{\mathbb{Q}}^G(s)ds\right).$$

If $\varphi_{\mathbb{Q}}^G(t)$ satisfies the Novikov condition,

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \varphi_{\mathbb{Q}}^G(s)'\Sigma_G\varphi_{\mathbb{Q}}^G(s)ds\right)\right] < \infty, \tag{3}$$

then process $\nu(t)$ is a positive $\mathbb{P}$-martingale and $\widetilde{W}^G(t) := (\widetilde{W}^{G_1}(t), \widetilde{W}^{G_2}(t))$ becomes a $\mathbb{Q}$-Brownian motion in $\mathbb{R}^2$ with covariance matrix Σ_G, where $d\widetilde{W}^G(t) = dW^G(t) - \Sigma_G\varphi_{\mathbb{Q}}^G(t)dt$. Hence, choosing an alternative $\mathbb{Q}$ is equivalent to determining a stochastic process $\varphi_{\mathbb{Q}}(t)$ that satisfies the Novikov condition (3). In this sense, $\varphi_{\mathbb{Q}}^G(t)$ acts as a control in the following robust problem. An equivalent definition of ambiguous measures, adopted in Yi et al. (2013) and Pun and Wong (2015), uses a stochastic process,

$$\bar{\varphi}_{\mathbb{Q}}^G(t) = (\bar{\varphi}_{\mathbb{Q}}^{G_1}(t), \bar{\varphi}_{\mathbb{Q}}^{G_2}(t))' := A'\varphi_{\mathbb{Q}}^G(t), \tag{4}$$

to represent the transformation from $\mathbb{P}$ to $\mathbb{Q}$, where $W^G(t)$ is replaced by $A\bar{W}^G(t)$ and $\Sigma_G = AA'$. In such a case, (3) becomes

$$\mathbb{E}^{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^T \bar{\varphi}_{\mathbb{Q}}^G(s)'\bar{\varphi}_{\mathbb{Q}}^G(s)ds\right)\right] < \infty,\ \nu(t) = \exp\left(\int_0^t \bar{\varphi}_{\mathbb{Q}}^G(s)'d\bar{W}(s) - \frac{1}{2}\int_0^t \bar{\varphi}_{\mathbb{Q}}^G(s)'\bar{\varphi}_{\mathbb{Q}}^G(s)ds\right).$$

We combine the concepts of ambiguity aversion in Maenhout (2004) and relative performance concerns in Espinosa and Touzi (2015) in such a way that each individual insurer aims to maximize the expected utility of her performance relative to that of her competitor at terminal time $T > 0$ under the worst-case scenario of the alternative measures. The robust optimization problem for the insurer $k \in \{1, 2\}$ thus reads:

$$\begin{aligned}
& \sup_{q_k \in Q_k} \inf_{\mathbb{Q}_k \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_k} \left[U_k((1-\kappa_k) X_k^{q_k}(T) + \kappa_k (X_k^{q_k}(T) - X_m^{q_m}(T))) + D_k^0(\mathbb{P}||\mathbb{Q}_k) \right] \\
= & \sup_{q_k \in Q_k} \inf_{\mathbb{Q}_k \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_k} \left[U_k(X_k^{q_k}(T) - \kappa_k X_m^{q_m}(T)) + D_k^0(\mathbb{P}||\mathbb{Q}_k) \right], \text{ for } m \neq k \in \{1, 2\},
\end{aligned}$$

where $\kappa_k \in [0, 1]$ reflects the level of insurer k's relative performance concerns, $Q_k = Q_k^{[0,1]}$ or $Q_k^{[0,+\infty)}$ is a set of admissible strategies q_k, and $D_k^0(\mathbb{P}||\mathbb{Q}_k) \geq 0$ is a penalty function measuring the divergence of $\mathbb{Q}_k$ from $\mathbb{P}$. The inside infimum problem corresponds to the worst-case scenario of $\mathbb{Q}_k$ with the minimal penalty. Such a maximin formulation is standard in the literature on robust optimization theory; see Maenhout (2004), Yi et al. (2013), Fouque et al. (2014), and Pun and Wong (2015).

The optimization problems of AAIs constitute an RNSDRG. A Nash equilibrium for the AAIs is a 2-tuple $(q_1^*, q_2^*) \in Q_1 \times Q_2$ that satisfies the following inequalities.

$$\begin{aligned}
& \inf_{\mathbb{Q}_1 \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_1} \left[U_1(X_1^{q_1}(T) - \kappa_1 X_2^{q_2^*}(T)) + D_1^0(\mathbb{P}||\mathbb{Q}_1) \right] \\
\leq & \inf_{\mathbb{Q}_1 \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_1} \left[U_1(X_1^{q_1^*}(T) - \kappa_1 X_2^{q_2^*}(T)) + D_1^0(\mathbb{P}||\mathbb{Q}_1) \right], \ \forall q_1 \in Q_1, \\
& \inf_{\mathbb{Q}_2 \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_2} \left[U_2(X_2^{q_2}(T) - \kappa_2 X_1^{q_1^*}(T)) + D_2^0(\mathbb{P}||\mathbb{Q}_2) \right] \\
\leq & \inf_{\mathbb{Q}_2 \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}_2} \left[U_2(X_2^{q_2^*}(T) - \kappa_2 X_1^{q_1^*}(T)) + D_2^0(\mathbb{P}||\mathbb{Q}_2) \right], \ \forall q_2 \in Q_2.
\end{aligned}$$

3. Nash Equilibrium

Our approach rests on the dynamic programming principle to find a Nash equilibrium for the RNSDRG problem. To this end, we first denote $\hat{X}_k^{q_k,q_m}(t) := X_k^{q_k}(t) - \kappa_k X_m^{q_m}(t)$ as the relative wealth (performance) process of the insurer $k \in \{1, 2\}$. The dynamics of $\{\hat{X}_k^{q_k,q_m}(t)\}_{t \geq 0}$ under the reference measure $\mathbb{P}$ is given by:

$$\begin{aligned} d\hat{X}_k^{q_k,q_m}(t) &= [(a_k\lambda_k - \kappa_k a_m\lambda_m) + a_k\eta_k q_k(t) - \kappa_k a_m\eta_m q_m(t) + r\hat{X}_k^{q_k,q_m}(t)]dt \\ &\quad + b_k q_k(t)dW^{G_k}(t) - \kappa_k b_m q_m(t)dW^{G_m}(t) \end{aligned} \tag{5}$$

with $\hat{X}_k^{q_k,q_m}(0) = x_k - \kappa_k x_m$. As the robust optimization problem is defined in an alternative measure $\mathbb{Q}_k$, the dynamics of $\{\hat{X}_k^{q_k,q_m}(t)\}_{t\geq 0}$ must also be expressed under the alternative measure $\mathbb{Q}_k$. The following notation simplifies the discussion.

$$\hat{\lambda}_k = a_k\lambda_k - \kappa_k a_m\lambda_m,\ \hat{q}_1(t) = (b_1q_1(t), -\kappa_1 b_2 q_2(t))',\ \hat{q}_2(t) = (-\kappa_2 b_1 q_1(t), b_2q_2(t))'.$$

By the Girsanov theorem, we have

$$d\hat{X}_k^{q_k,q_m}(t) = [\hat{\lambda}_k + a_k\eta_k q_k(t) - \kappa_k a_m\eta_m q_m(t) + \varphi_k^G(t)'\Sigma_G\hat{q}_k(t) + r\hat{X}_k^{q_k,q_m}(t)]dt + \hat{q}_k(t)'d\widetilde{W}^G(t),$$

where Σ_G is defined as in (2), $\widetilde{W}^G(t) = (\widetilde{W}^{G_1}(t), \widetilde{W}^{G_2}(t))'$ is the $\mathbb{Q}_k$-Brownian motion in $\mathbb{R}^2$, and $\varphi_k^G(t) = (\varphi_k^{G_1}(t), \varphi_k^{G_2}(t))'$ is a stochastic process underlying the transformation of $\mathbb{Q}_k$ from $\mathbb{P}$.

We define the value function of the insurer $k \neq m \in \{1,2\}$ as

$$V_k(t,\hat{x}_k) := \sup_{q_k\in Q_k}\inf_{\mathbb{Q}_k\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}_k}\left[U_k(\hat{X}_k^{q_k,q_m^*}(T)) + D_k^t(\mathbb{P}||\mathbb{Q}_k)\middle|\, \hat{X}_k^{q_k,q_m^*}(t) = \hat{x}_k\right], \tag{6}$$

where V_k is smooth on $[0,T]\times\mathbb{R}$ and $D_k^T(\mathbb{P}||\mathbb{Q}_k) \equiv 0$. We look for an increasing and strictly concave $V_k(t,\hat{x}_k)$ in $\hat{x}_k$, $\forall t\in[0,T]$.

Inspired by Maenhout (2004), we consider a penalty function of the form

$$D_k^t(\mathbb{P}||\mathbb{Q}_k) := \int_t^T \frac{P_k(s)}{\phi_k(s)}ds, \text{ for } k = 1,2,$$

where $\phi_k(t) \geq 0$ is the preference function related to the ambiguity aversion of the insurer k, and

$$P_k(t) := \varphi_k^G(t)'A_k\Xi_k^{-1}A_k'\varphi_k^G(t)/2,\ \Xi_k := \mathrm{diag}(\xi_k^{G_k}, \xi_k^{\bar{G}_m}), \tag{7}$$

$$A_1 = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}, A_2 = \begin{pmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{pmatrix}, \tag{8}$$

in which $\xi_k^{G_k}, \xi_k^{\bar{G}_m}$ are positive ambiguity aversion coefficients for the different diffusion processes specified below.

This choice of penalty function is justified as follows. From (4), we can write

$$P_k(t) = \bar{\varphi}_k^G(t)'\Xi_k^{-1}\bar{\varphi}_k^G(t)/2 = \frac{1}{2}\left[\left(\frac{\varphi_k^{G_k}(t)}{\xi_k^{G_k}}\right)^2 + \left(\frac{\bar{\varphi}_k^{\bar{G}_m}(t)}{\xi_k^{\bar{G}_m}}\right)^2\right]$$

for $m \neq k$, where $\bar{\varphi}_k(t) = A_k'\varphi_k(t)$ and the Radon-Nikodym derivative $\nu(t)$ is a stochastic exponential of $\int_0^t \bar{\varphi}_k^G(s)'d\bar{W}_k^G(s)$, in which $\bar{W}_k^G(t) = (W^{G_k}(t), W^{\bar{G}_m}(t))' = A_k^{-1}W_k^G(t)$, and $W^{G_m}(t) = \rho W^{G_k}(t) + \sqrt{1-\rho^2}W^{\bar{G}_m}(t)$. Note that $A_kA_k' = \Sigma_G$ for $k = 1, 2$, and $\bar{W}^G(t)$ has an identity variance-covariance matrix. Hence, $P_k(t)$ measures the relative entropy between $\mathbb{P}$ and $\mathbb{Q}_k$, adjusted by $\xi_k^{G_k}, \xi_k^{\bar{G}_m}$, which are ambiguity aversion coefficients for the insurer k's surplus process, and the additional noise process of her competitor's surplus process, respectively. Hence, the chosen penalty reflects the focal insurer's preference for the model uncertainty associated with her own claim process, and her competitor's claim process.

Once $\xi_k^M \downarrow 0$ for $M = G_k, \bar{G}_m$, insurer k is fully confident about the modeling of (the drift of) the diffusion process, $\{W^M(t)\}_{t\geq 0}$ as $\varphi_k^M(t) \equiv 0$. We consider individual ambiguity aversion coefficients instead of the aggregate ambiguity aversion coefficient in Yi et al. (2013) and Pun and Wong (2015) because insurer k usually has confidence in her own surplus process, and in this case she can send $\xi_k^{G_k}$ to zero while keeping the robustness of her competitor's surplus process. A typical setting could be $\xi_k^{G_k} = 0$, $\xi_k^{G_m} =: \xi_k$. If all ambiguity aversion coefficients converge to zero ($\xi_k \downarrow 0$), then insurer k picks the reference measure ($\mathbb{Q}_k = \mathbb{P}$), as $D_k^0(\mathbb{P}||\mathbb{Q}_k) \equiv 0$. Accordingly, our model permits one insurer to consider model uncertainty while the other does not or even for both to not consider it. Our analyses assume that $\xi_k^{G_k}, \xi_k^{\bar{G}_m}$ are positive constant, which gives more general results.

As suggested by Pun and Wong (2015), we consider the form of ϕ to be

$$\phi_k(t, \hat{x}_k) := \frac{1}{R_k(t, \hat{x}_k)\partial_{\hat{x}_k}V_k(t, \hat{x}_k)} = -\frac{\partial_{\hat{x}_k\hat{x}_k}V_k(t, \hat{x}_k)}{(\partial_{\hat{x}_k}V_k(t, \hat{x}_k))^2}, \tag{9}$$

where $R_k(t, \hat{x}_k) := -\partial_{\hat{x}_k}V_k(t, \hat{x}_k)/\partial_{\hat{x}_k\hat{x}_k}V_k(t, \hat{x}_k) \geq 0$ is the risk-tolerance function of the insurer $k \in \{1, 2\}$ and the notation $\partial_x f(x) = \frac{\partial f}{\partial x}$. As discussed in Pun and Wong (2015), this choice of preference function is economically meaningful in the sense that ϕ_k is decreasing with respect to risk-tolerance R_k and implicitly imposes homotheticity, ensuring that the robustness does not wear off as wealth rises. Moreover, it facilitates analytical tractability, as shown in later analyses.

In what follows, we first solve the optimal strategy of one insurer given the other's strategy using the dynamic programming approach. We then provide sufficient conditions for the existence of a Nash equilibrium for the RNSDRG.

3.1. Hamilton-Jacobi-Bellman Framework

To ease the notational burden, we suppress the arguments of the functions. The associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation of the value function (6) for the insurer $k \neq m \in \{1,2\}$ is given by

$$\begin{aligned}\partial_t V_k + [\hat{\lambda}_k + r\hat{x}_k]\partial_{\hat{x}_k} V_k + \sup_{q_k} \inf_{\varphi_k^G} \{ & [a_k\eta_k q_k(t) - \kappa_k a_m \eta_m q_m^*(t)]\partial_{\hat{x}_k} V_k \\ + \frac{1}{2}\hat{q}_k(t)'\Sigma_G \hat{q}_k(t)\partial_{\hat{x}_k\hat{x}_k} V_k + \varphi_k^G(t)'\Sigma_G \hat{q}_k(t)\partial_{\hat{x}_k} V_k + & \left.\frac{\varphi_k^G(t)' A_k \Xi_k^{-1} A_k' \varphi_k^G(t)}{2\phi_k}\right\} = 0,\end{aligned}$$

with terminal condition $V_k(T, \hat{x}_k) = U_k(x)$, where Σ_G is defined as in (2), A_k for $k = 1, 2$ is defined as in (8), and Ξ_k is defined as in (7).

The infimum φ_k^{G*} can be solved by minimizing the quadratic forms of φ_k^G:

$$\varphi_k^{G*} = -\frac{(A_k')^{-1}\Xi_k A_k' \hat{q}_k(t)}{R_k(t, \hat{x}_k)}, \tag{10}$$

and recall that $R_k(t, \hat{x}_k) = -\partial_{\hat{x}_k} V_k / \partial_{\hat{x}_k\hat{x}_k} V_k$, $\hat{q}_1(t) = (b_1 q_1(t), -\kappa_1 b_2 q_2^*(t))'$, and $\hat{q}_2(t) = (-\kappa_2 b_1 q_1^*(t), b_2 q_2(t))'$. Then, with the substitution of (10), the foregoing HJBI equation is reduced to an HJB equation:

$$\begin{aligned}\partial_t V_k + [\hat{\lambda}_k + r\hat{x}_k]\partial_{\hat{x}_k} V_k + \frac{\rho^2 \xi_k^{G_k} + (1-\rho^2)\xi_k^{\bar{G}_m} + 1}{2}\kappa_k^2 b_m^2 q_m^{*2}(t)\partial_{\hat{x}_k\hat{x}_k} V_k - \kappa_k a_m \eta_m q_m^*(t)\partial_{\hat{x}_k} V_k \qquad (11) \\ + \sup_{q_k}\left\{\frac{\xi_k^{G_k}+1}{2}[b_k^2 q_k^2(t) - 2\rho\kappa_k b_m b_k q_m^*(t) q_k(t)]\partial_{\hat{x}_k\hat{x}_k} V_k + a_k\eta_k q_k(t)\partial_{\hat{x}_k} V_k\right\} = 0,\end{aligned}$$

with terminal condition $V_k(T, \hat{x}_k, y) = U_k(x)$.

Maximizing the quadratic forms of q_k in (11) yields the optimal strategy $q_k^*(t)$, which takes the form

$$q_k^* = \begin{cases} \left[\min\left(\rho\kappa_k \frac{b_m}{b_k} q_m^* + \frac{a_k\eta_k}{(\xi_k^{G_k}+1)b_k^2} R_k, 1\right)\right]^+, & \text{if } Q_k = Q_k^{[0,1]}, \\ \left(\rho\kappa_k \frac{b_m}{b_k} q_m^* + \frac{a_k\eta_k}{(\xi_k^{G_k}+1)b_k^2} R_k\right)^+, & \text{if } Q_k = Q_k^{[0,+\infty)}, \end{cases} \tag{12}$$

where $x^+ = \max(x, 0)$. Substituting the optimal controls in (12) into the HJB equation (11), we obtain a highly nonlinear partial differential equation (PDE) driving V_k:

$$\begin{aligned} \partial_t V_k + [\hat{\lambda}_k + r\hat{x}_k]\partial_{\hat{x}_k} V_k + \frac{\rho^2 \xi_k^{G_k} + (1-\rho^2)\xi_k^{\bar{G}_m} + 1}{2}\kappa_k^2 b_m^2 q_m^{*2}(t)\partial_{\hat{x}_k\hat{x}_k} V_k & \\ + \frac{\xi_k^{G_k}+1}{2}[b_k^2 q_k^{*2}(t) - 2\rho\kappa_k b_m b_k q_m^*(t) q_k^*(t)]\partial_{\hat{x}_k\hat{x}_k} V_k + a_k\eta_k q_k^*(t)\partial_{\hat{x}_k} V_k &= 0, \end{aligned} \tag{13}$$

with terminal condition $V_k(T, \hat{x}_k, y) = U_k(x)$, where $q_k^*(t)$ can be deduced from (12). A Nash equilibrium is found by solving (q_1^*, q_2^*) from equations (12). Hence, the sufficient condition for the existence of a Nash equilibrium is equivalent to the solvability condition of the linear system, as stated in the following theorem.

Theorem 3.1. If $\kappa_1\kappa_2 < 1$, a Nash equilibrium strategy (q_1^*, q_2^*) solves the following coupled system

$$\left\{ \begin{array}{rcl} q_1^* &=& \left[\min\left(\rho\kappa_1\frac{b_2}{b_1}q_2^* + \frac{a_1\eta_1}{(\xi_1^{G_1}+1)b_1^2}R_1, 1\right)\right]^+, \\ q_2^* &=& \left[\min\left(\rho\kappa_2\frac{b_1}{b_2}q_1^* + \frac{a_2\eta_2}{(\xi_2^{G_2}+1)b_2^2}R_2, 1\right)\right]^+, \end{array} \right. \text{if } Q_1 = Q_1^{[0,1]}, Q_2 = Q_2^{[0,1]} \tag{14}$$

$$\left\{ \begin{array}{rcl} q_1^* &=& \left(\rho\kappa_1\frac{b_2}{b_1}q_2^* + \frac{a_1\eta_1}{(\xi_1^{G_1}+1)b_1^2}R_1\right)^+, \\ q_2^* &=& \left(\rho\kappa_2\frac{b_1}{b_2}q_1^* + \frac{a_2\eta_2}{(\xi_2^{G_2}+1)b_2^2}R_2\right)^+, \end{array} \right. \text{if } Q_1 = Q_1^{[0,+\infty)}, Q_2 = Q_2^{[0,+\infty)} \tag{15}$$

where, for $k \neq m \in \{1, 2\}$, $R_k(t, \hat{x}_k, y) = -\partial_{\hat{x}_k} V_k(t, \hat{x}_k, y)/\partial_{\hat{x}_k\hat{x}_k} V_k(t, \hat{x}_k, y)$, and V_k is driven by (13) with the substitution of q_k^* and q_m^* solved from (14) or (15).

Our consideration embraces the problems studied in Bensoussan et al. (2014). In fact, their results can be recovered by setting $\xi_k = 0$, i.e., no robustness and $Q_k = Q_k^{[0,1]}$. In addition, when $\kappa_1 = \kappa_2 = 0$, the result reduces to the single-agent analysis with ambiguity aversion in Pun and Wong (2015). Other things being fixed, it is clear from (14) and (15) that the self-reinsurance demand ($1 - q_1^*$ for insurer 1) increases with self-ambiguity aversion ($\xi_1^{G_1}$ for insurer 1). However, the effect of $\xi_2^{G_2}$ on $1 - q_1^*$ needs to be examined numerically after solving for the Nash equilibrium. Determining the Nash equilibrium strategy for a general RNSDRG is a very difficult task. Thus, we derive an explicit solution for insurers with constant absolute risk aversion (CARA).

4. Exponential Utility and the Heston Model

In this section, we deduce the explicit Nash equilibrium of the RNSDRG for CARA insurers. In other words, insurer k has a utility function of the form

$$U_k(x) = -\frac{1}{\gamma_k} e^{-\gamma_k x}, \tag{16}$$

where $\gamma_k > 0$ is the risk aversion coefficient of insurer k, $k = 1, 2$.

Theorem 4.1. If $\kappa_1 \kappa_2 < 1$, then the value function of CARA insurer k, driven by (6), is

$$V_k(t, \hat{x}_k) = -\frac{1}{\gamma_k} \exp\left(-\gamma_k e^{r(T-t)} \hat{x}_k + f_k(t)\right), \tag{17}$$

where, for $m \neq k \in \{1, 2\}$,

$$\begin{aligned} f_k(t) \;=\; & \int_t^T \left\{ -\hat{\lambda}_k \gamma_k e^{r(T-s)} + \frac{\rho^2 \xi_k^{G_k} + (1-\rho^2)\xi_k^{\bar{G}_m} + 1}{2} \kappa_k^2 b_m^2 q_m^{*2}(s) \gamma_k^2 e^{2r(T-s)} \right. \\ & \left. + \frac{\xi_k^{G_k} + 1}{2} [b_k^2 q_k^{*2}(s) - 2\rho\kappa_k b_m b_k q_m^*(s) q_k^*(s)] \gamma_k^2 e^{2r(T-s)} - a_k \eta_k q_k^*(s) \gamma_k e^{r(T-s)} \right\} ds. \end{aligned}$$

In addition, the equilibrium reinsurance strategies are presented as follows. Define

$$\tilde{q}_1(t) := \frac{e^{-r(T-t)}}{1 - \kappa_1\kappa_2\rho^2} \left(\frac{\rho\kappa_1 a_2 \eta_2}{\gamma_2(\xi_2^{G_2} + 1) b_1 b_2} + \frac{a_1 \eta_1}{\gamma_1(\xi_1^{G_1} + 1) b_1^2} \right), \tag{18}$$

$$\tilde{q}_2(t) := \frac{e^{-r(T-t)}}{1 - \kappa_1\kappa_2\rho^2} \left(\frac{\rho\kappa_2 a_1 \eta_1}{\gamma_1(\xi_1^{G_1} + 1) b_1 b_2} + \frac{a_2 \eta_2}{\gamma_2(\xi_2^{G_2} + 1) b_2^2} \right), \tag{19}$$

$$h_{1\ell_h}(t) := \rho\kappa_1 \frac{b_2}{b_1} + \frac{e^{-r(T-t)} a_1 \eta_1}{\gamma_1(\xi_1^{G_1} + 1) b_1^2}, \; h_{2\ell_v}(t) := \rho\kappa_2 \frac{b_1}{b_2} + \frac{e^{-r(T-t)} a_2 \eta_2}{\gamma_2(\xi_2^{G_2} + 1) b_2^2},$$

$$h_{1q_1}(t) := \frac{e^{-r(T-t)} a_1 \eta_1}{\gamma_1(\xi_1^{G_1} + 1) b_1^2}, \; h_{2q_2}(t) := \frac{e^{-r(T-t)} a_2 \eta_2}{\gamma_2(\xi_2^{G_2} + 1) b_2^2}.$$

The following cases are possible.

- If $Q_k = Q_k^{[0,1]}$:

  1. If $\tilde{q}_1(t) > 0,\ \tilde{q}_2(t) > 0$, and $\rho \geq 0$, then

$$(q_1^*(t), q_2^*(t)) = \left(\min\left(\tilde{q}_1(t), h_{1\ell_h}(t), 1\right), \min\left(\tilde{q}_2(t), h_{2\ell_v}(t), 1\right)\right).$$

    2. If $\tilde{q}_1(t) > 0,\ \tilde{q}_2(t) > 0$, and $\rho < 0$, then

$$(q_1^*(t), q_2^*(t)) = \left(\min\left(\max(\tilde{q}_1(t), h_{1\ell_h}(t)), 1\right), \min\left(\max(\tilde{q}_2(t), h_{2\ell_v}(t)), 1\right)\right).$$

    3. If $\tilde{q}_1(t) > 0$ and $\tilde{q}_2(t) \leq 0$, then $(q_1^*(t), q_2^*(t)) = \left(\min\left(h_{1q_1}(t), 1\right), \left[\min\left(h_{2\ell_v}(t), 1\right)\right]^+\right)$.
    4. If $\tilde{q}_1(t) \leq 0$ and $\tilde{q}_2(t) > 0$, then $(q_1^*(t), q_2^*(t)) = \left(\left[\min\left(h_{1\ell_h}(t), 1\right)\right]^+, \min\left(h_{2q_2}(t), 1\right)\right)$.

- If $Q_k = Q_k^{[0,+\infty)}$:
    1. If $\tilde{q}_1(t) > 0$ and $\tilde{q}_2(t) > 0$, then $(q_1^*(t), q_2^*(t)) = (\tilde{q}_1(t), \tilde{q}_2(t))$;
    2. If $\tilde{q}_1(t) > 0$ and $\tilde{q}_2(t) \leq 0$, then $(q_1^*(t), q_2^*(t)) = (h_{1q_1}(t), 0)$;
    3. If $\tilde{q}_1(t) \leq 0$ and $\tilde{q}_2(t) > 0$, then $(q_1^*(t), q_2^*(t)) = (0, h_{2q_2}(t))$.

Remark: The condition that $\kappa_1\kappa_2 < 1$ rules out the unreasonable situation that $\tilde{q}_1(t) \leq 0$ and $\tilde{q}_2(t) \leq 0$.

Proof. We start with the ansatz (17) of the value function of the insurer $k \in \{1, 2\}$, where $f_k(t)$ and $g_k(t)$ are real-value functions. We have

$$\begin{aligned}
\partial_t V_k(t, \hat{x}_k) &= (\partial_t f_k(t) + r\hat{x}_k\gamma_k e^{r(T-t)})V_k(t, \hat{x}_k), \\
\partial_{\hat{x}_k} V_k(t, \hat{x}_k) &= -\gamma_k e^{r(T-t)} V_k(t, \hat{x}_k), \\
\partial_{\hat{x}_k\hat{x}_k} V_k(t, \hat{x}_k) &= \gamma_k^2 e^{2r(T-t)} V_k(t, \hat{x}_k), \\
R_k(t, \hat{x}_k) &= e^{-r(T-t)}/\gamma_k. \qquad (20)
\end{aligned}$$

We now solve the coupled system of (14) and (15) with the substitution of (20). The $(\tilde{q}_1, \tilde{q}_2)$ defined in (18)-(19) is the intersection point of the following lines on the (q_1, q_2) plane.

$$\iota_1 :\ q_1 = \rho\kappa_1 \frac{b_2}{b_1} q_2 + \frac{e^{-r(T-t)} a_1 \eta_1}{\gamma_1(\xi_1^{G_1} + 1)b_1^2}, \quad \iota_1 :\ q_2 = \rho\kappa_2 \frac{b_1}{b_2} q_1 + \frac{e^{-r(T-t)} a_2 \eta_2}{\gamma_2(\xi_2^{G_2} + 1)b_2^2}.$$

Denote the q_1- and q_2-intercepts of ι_1 and ι_2 by $(h_{1q_1}, 0), (0, h_{1q_2})$ and $(h_{2q_1}, 0), (0, h_{2q_2})$, respectively. Then,

$$\begin{aligned}
h_{1q_1} &= \frac{e^{-r(T-t)} a_1 \eta_1}{\gamma_1(\xi_1^{G_1} + 1)b_1^2},\ h_{1q_2} = -\frac{e^{-r(T-t)} a_1 \eta_1}{\rho\kappa_1\gamma_1(\xi_1^{G_1} + 1)b_1 b_2}, \\
h_{2q_1} &= -\frac{e^{-r(T-t)} a_2 \eta_2}{\rho\kappa_2\gamma_2(\xi_2^{G_2} + 1)b_1 b_2},\ h_{2q_2} = \frac{e^{-r(T-t)} a_2 \eta_2}{\gamma_2(\xi_2^{G_2} + 1)b_2^2}.
\end{aligned}$$

Let ℓ_h be the horizontal line $\ell_h : q_2 = 1$, ℓ_v be the vertical line $\ell_v : q_1 = 1$, $h_{1\ell_h}$ $(h_{2\ell_h})$ be the q_1-intersect of ι_1 (ι_2) and ℓ_h, and $h_{1\ell_v}$ $(h_{2\ell_v})$ be the q_2-intersect of ι_1 (ι_2) and ℓ_v. Then,

$$\begin{aligned} h_{1\ell_h} &= \rho\kappa_1\frac{b_2}{b_1} + \frac{e^{-r(T-t)}a_1\eta_1}{\gamma_1(\xi_1^{G_1}+1)b_1^2},\ h_{2\ell_h} = \left(1 - \frac{e^{-r(T-t)}a_2\eta_2}{\gamma_2(\xi_2^{G_2}+1)b_2^2}\right) / \left(\rho\kappa_2\frac{b_1}{b_2}\right), \\ h_{1\ell_v} &= \left(1 - \frac{e^{-r(T-t)}a_1\eta_1}{\gamma_1(\xi_1^{G_1}+1)b_1^2}\right) / \left(\rho\kappa_1\frac{b_2}{b_1}\right),\ h_{2\ell_v} = \rho\kappa_2\frac{b_1}{b_2} + \frac{e^{-r(T-t)}a_2\eta_2}{\gamma_2(\xi_2^{G_2}+1)b_2^2}. \end{aligned}$$

- For $\rho = 0$,
  - if $Q_k = Q_k^{[0,+\infty)}$, then $(q_1^*(t), q^*(t)) = (h_{1q_1}, h_{2q_2})$; and
  - if $Q_k = Q_k^{[0,1]}$, then $(q_1^*(t), q^*(t)) = (\min(h_{1q_1}, 1), \min(h_{2q_2}, 1))$.

- For any $\rho \in (0,1]$, $(\tilde{q}_1, \tilde{q}_2)$ lies in the first quadrant of the (q_1, q_2) plane. It is clear that $h_{1q_2} < 0 < h_{2q_2} < h_{2\ell_v}$ and $h_{2q_1} < 0 < h_{1q_1} < h_{1\ell_h}$.
  - If $Q_k = Q_k^{[0,+\infty)}$, then $(q_1^*, q_2^*) = (\tilde{q}_1, \tilde{q}_2)$.
  - If $Q_k = Q_k^{[0,1]}$, then (q_1^*, q_2^*) is the intersection point of the following increasing lines.

$$\ell_1 : \begin{cases} q_1 = 0, & q_2 \le h_{1q_2}, \\ \iota_1, & h_{1q_2} < q_2 < h_{1\ell_v}, \\ q_1 = 1, & q_2 \ge h_{1\ell_v}, \end{cases} \qquad \ell_2 : \begin{cases} q_2 = 0, & q_1 \le h_{2q_1}, \\ \iota_2, & h_{2q_1} < q_1 < h_{2\ell_h}, \\ q_2 = 1, & q_1 \ge h_{2\ell_h}. \end{cases}$$

1. If $\tilde{q}_1 < 1$ and $\tilde{q}_2 < 1$, then we have $\tilde{q}_1 < h_{1\ell_h} < h_{2\ell_h}$ and $\tilde{q}_2 < h_{2\ell_v} < h_{1\ell_v}$, and it is straightforward to see that $(q_1^*, q_2^*) = (\tilde{q}_1, \tilde{q}_2)$.
2. If $\tilde{q}_1 < 1$ and $\tilde{q}_2 \ge 1$, then we have $h_{2\ell_h} < h_{1\ell_h} \le \tilde{q}_1 < 1$ and $h_{1\ell_v} > h_{2\ell_v} > \tilde{q}_2 \ge 1$. Hence, ℓ_2 for $q_1 < h_{2\ell_h}$ does not intersect with ℓ_1. Thus, $(q_1^*, q_2^*) = (h_{1\ell_h}, 1)$.
3. If $\tilde{q}_1 \ge 1$ and $\tilde{q}_2 < 1$, then we have $h_{1\ell_v} < h_{2\ell_v} \le \tilde{q}_2 < 1$ and $h_{2\ell_h} > h_{1\ell_h} > \tilde{q}_1 \ge 1$. Hence, ℓ_1 for $q_2 < h_{1\ell_v}$ does not intersect with ℓ_2. Thus, $(q_1^*, q_2^*) = (1, h_{2\ell_v})$.
4. If $\tilde{q}_1 \ge 1$ and $\tilde{q}_2 \ge 1$, then we have $h_{1\ell_v} < h_{2\ell_v}$ and $h_{2\ell_h} < h_{1\ell_h}$. Note that $h_{1\ell_h} < 1 \Leftrightarrow h_{1\ell_v} > 1$ and $h_{2\ell_h} < 1 \Leftrightarrow h_{2\ell_v} > 1$.
   (a) If $h_{2\ell_h} \le 1$ and $h_{1\ell_v} \le 1$, then we have $h_{1\ell_h} \ge 1 \ge h_{2\ell_h}$ and $h_{2\ell_v} \ge 1 \ge h_{1\ell_v}$. Hence, ℓ_2 for $q_1 < h_{2\ell_h}$ does not intersect with ℓ_1 for $q_2 < h_{1\ell_v}$. Thus, $(q_1^*, q_2^*) = (1, 1)$.

(b) If $h_{2\ell_h} > 1$, then we have $h_{1\ell_v} < h_{2\ell_v} < 1$. Hence, ℓ_1 for $q_2 < h_{1\ell_v}$ does not intersect with ℓ_2. Thus, $(q_1^*, q_2^*) = (1, h_{2\ell_v})$.

(c) If $h_{1\ell_v} > 1$, then we have $h_{2\ell_h} < h_{1\ell_h} < 1$. Hence, ℓ_2 for $q_1 < h_{2\ell_h}$ does not intersect with ℓ_1. Thus, $(q_1^*, q_2^*) = (h_{1\ell_h}, 1)$.

In summary, for any $\rho \in (0, 1]$ and $Q_k = Q_k^{[0,1]}$, we have

$$(q_1^*, q_2^*) = (\min(\tilde{q}_1, h_{1\ell_h}, 1), \min(\tilde{q}_2, h_{2\ell_v}, 1)).$$

- For any $\rho \in [-1, 0)$, simple calculation yields $h_{1q_1}, h_{1q_2}, h_{2q_1}, h_{2q_2}$ that are positive: $h_{1q_1} > h_{2q_1} \Leftrightarrow \tilde{q}_2 < 0$; $h_{1q_2} > h_{2q_2} \Leftrightarrow \tilde{q}_1 > 0$; $h_{1q_1} > 1 \Leftrightarrow h_{1\ell_v} > 0$; $h_{2q_2} > 1 \Leftrightarrow h_{2\ell_h} > 0$; $h_{1q_1} > h_{1\ell_h}, h_{1q_2} > h_{1\ell_v}$; and $h_{2q_1} > h_{2\ell_h}, h_{2q_2} > h_{2\ell_v}$. It is impossible for $(\tilde{q}_1, \tilde{q}_2)$ to lie in the third quadrant of the (q_1, q_2) plane; otherwise,

$$\begin{cases} \rho\frac{\kappa_1 a_2\eta_2}{\gamma_2(\xi_2^{G_2}+1)b_1b_2} + \frac{a_1\eta_1}{\gamma_1(\xi_1^{G_1}+1)b_1^2} & \leq \ 0 \\ \rho\frac{\kappa_2 a_1\eta_1}{\gamma_1(\xi_1^{G_1}+1)b_1b_2} + \frac{a_2\eta_2}{\gamma_2(\xi_2^{G_2}+1)b_2^2} & \leq \ 0 \end{cases} \Rightarrow -\rho \geq -\frac{1}{\kappa_1}\frac{a_1\eta_1\gamma_2(\xi_2^{G_2}+1)b_2}{a_2\eta_2\gamma_1(\xi_1^{G_1}+1)b_1} \geq -\frac{1}{\kappa_1\kappa_2}\frac{1}{\rho} > -\frac{1}{\rho},$$

  which contradicts $\rho \in [-1, 0)$ for $\kappa_1\kappa_2 < 1$. Consider the following cases.

  - If $Q_k = Q_k^{[0,+\infty)}$:
    1. If $\tilde{q}_1 > 0$ and $\tilde{q}_2 > 0$, then $(q_1^*, q_2^*) = (\tilde{q}_1, \tilde{q}_2)$.
    2. If $\tilde{q}_1 > 0$ and $\tilde{q}_2 \leq 0$, then we have $h_{1q_1} \geq h_{2q_1}, h_{1q_2} > h_{2q_2}$, and thus ι_1 must lie above ι_2 in the first quadrant of the (q_1, q_2) plane. It is straightforward to see that $(q_1^*, q_2^*) = (h_{1q_1}, 0)$.
    3. If $\tilde{q}_1 \leq 0$ and $\tilde{q}_2 > 0$, then we have $h_{1q_1} < h_{2q_1}$ and $h_{1q_2} \leq h_{2q_2}$, and thus ι_1 must lie below ι_2 in the first quadrant of the (q_1, q_2) plane. It is straightforward to see that $(q_1^*, q_2^*) = (0, h_{2q_2})$.
  - If $Q_k = Q_k^{[0,1]}$, then (q_1^*, q_2^*) is the intersection point of the following decreasing lines.

$$\ell_1 : \begin{cases} q_1 = 1, & q_2 \leq h_{1\ell_v}, \\ \iota_1, & h_{1\ell_v} < q_2 < h_{1q_2}, \\ q_1 = 0, & q_2 \geq h_{1q_2}, \end{cases} \qquad \ell_2 : \begin{cases} q_2 = 1, & q_1 \leq h_{2\ell_h}, \\ \iota_2, & h_{2\ell_h} < q_1 < h_{2q_1}, \\ q_2 = 0, & q_1 \geq h_{2q_1}. \end{cases}$$

1. If $\tilde{q}_1 > 0$ and $\tilde{q}_2 > 0$, then we have $h_{1q_1} < h_{2q_1}, h_{1q_2} > h_{2q_2}$.
    (a) If $\tilde{q}_1 < 1$ and $\tilde{q}_2 < 1$, then we have $h_{2\ell_h} < h_{1\ell_h} < \tilde{q}_1 < 1$ and $h_{1\ell_v} < h_{2\ell_v} < \tilde{q}_2 < 1$, and it is straightforward to see that $(q_1^*, q_2^*) = (\tilde{q}_1, \tilde{q}_2)$.
    (b) If $\tilde{q}_1 < 1$ and $\tilde{q}_2 \geq 1$, then ι_2 must lie above ι_1 for $q_1 > \tilde{q}_1$, and $\tilde{q}_1 < h_{1\ell_h} < h_{2\ell_h}$ and $h_{1\ell_v} \leq h_{2\ell_v} \leq \tilde{q}_2$.
        i. If $h_{1\ell_h} \geq 1$, then we have $h_{1\ell_v} \geq 1$. Thus, $(q_1^*, q_2^*) = (1, 1)$.
        ii. If $h_{1\ell_h} < 1$, then $(q_1^*, q_2^*) = (h_{1\ell_h}, 1)$.
    (c) If $\tilde{q}_1 \geq 1$ and $\tilde{q}_2 < 1$, then ι_1 must lie above ι_2 for $q_2 > \tilde{q}_2$, and $\tilde{q}_1 \geq h_{1\ell_h} \geq h_{2\ell_h}$ and $h_{1\ell_v} > h_{2\ell_v} > \tilde{q}_2$.
        i. If $h_{2\ell_v} \geq 1$, then we have $h_{2\ell_h} \geq 1$. Thus, $(q_1^*, q_2^*) = (1, 1)$.
        ii. If $h_{2\ell_v} < 1$, then $(q_1^*, q_2^*) = (1, h_{2\ell_v})$.
    (d) If $\tilde{q}_1 \geq 1$ and $\tilde{q}_2 \geq 1$, then we have $h_{2\ell_h} \geq h_{1\ell_h} \geq \tilde{q}_1 \geq 1$ and $h_{1\ell_v} \geq h_{2\ell_v} \geq \tilde{q}_2 \geq 1$. Thus, $(q_1^*, q_2^*) = (1, 1)$.

    To sum up, $(q_1^*, q_2^*) = (\min(\max(\tilde{q}_1, h_{1\ell_h}), 1), \min(\max(\tilde{q}_2, h_{2\ell_v}), 1))$.
2. If $\tilde{q}_1 > 0$ and $\tilde{q}_2 \leq 0$, then we have $h_{1q_1} \geq h_{2q_1}$ and $h_{1q_2} > h_{2q_2}$, and thus ι_1 must lie above ι_2 in the first quadrant of the (q_1, q_2) plane.
    (a) If $h_{1q_1} \leq 1$, then we have $h_{2q_1} \leq 1$ and $h_{2\ell_v} \leq 0$ (ℓ_2 is decreasing). Hence, ℓ_2 for $q_1 < h_{2q_2}$ does not intersect with ℓ_1. Thus, $(q_1^*, q_2^*) = (h_{1q_1}, 0)$.
    (b) If $h_{1q_1} > 1$, then we have $h_{1\ell_v} > 0$ and $h_{1\ell_v} > h_{2\ell_v}$, and ℓ_1 for $q_2 > h_{1\ell_v}$ does not intersect with ℓ_1.
        i. If $h_{2q_1} \leq 1$, then we have $h_{2\ell_v} \leq 0$ (ℓ_2 is decreasing). Thus, $(q_1^*, q_2^*) = (1, 0)$.
        ii. If $h_{2q_1} < 1$, then we have $h_{2\ell_v} > 0$ (ℓ_2 is decreasing). Thus, $(q_1^*, q_2^*) = (1, \min(h_{2\ell_v}, 1))$.

    To sum up, $(q_1^*, q_2^*) = (\min(h_{1q_1}, 1), (\min(h_{2\ell_v}, 1))^+)$.
3. If $\tilde{q}_1 \leq 0$ and $\tilde{q}_2 > 0$, similar to the second case, we have

$$(q_1^*, q_2^*) = ((\min(h_{1\ell_h}, 1))^+, \min(h_{2q_2}, 1)).$$

Finally, the PDE governing V_k (13) becomes

$$\begin{aligned}\partial_t f_k(t) - \hat{\lambda}_k \gamma_k e^{r(T-t)} + \frac{\rho^2 \xi_k^{G_k} + (1-\rho^2)\xi_k^{\bar{G}_m} + 1}{2} \kappa_k^2 b_m^2 q_m^{*2}(t) \gamma_k^2 e^{2r(T-t)} & \\ + \frac{\xi_k^{G_k} + 1}{2} [b_k^2 q_k^{*2}(t) - 2\rho\kappa_k b_m b_k q_m^*(t) q_k^*(t)] \gamma_k^2 e^{2r(T-t)} - a_k \eta_k q_k^*(t) \gamma_k e^{r(T-t)} &= 0,\end{aligned}$$

with $f_k(T) = 0$. The solutions to f_k are present in the theorem. □

Theorem 4.1 asserts that the equilibrium strategy does not depend on the current wealth level, which is consistent with the nature of a CARA agent. An interesting finding is that the insurer's ambiguity aversion in her competitor's surplus process ($\xi_k^{\bar{G}_m}$) has no effects on her reinsurance decision, having an effect only on her value function. In the context of relative performance concerns, each insurer considers only her competitor's robust reinsurance strategy, and reacts in accordance with her ambiguity aversion in her own claim process ($\xi_k^{G_k}$).

If $\kappa_k = 0$ for $k \in \{1, 2\}$, then the $q_k^*(t)$ presented in Theorem 4.1 becomes the optimal reinsurance strategy for the single-agent optimization problem, as in Yi et al. (2013) and Pun and Wong (2015). We can further forfeit robustness by setting all ξ_ks equal to zero. The effects of relative performance concerns and ambiguity aversion on an insurer's own model parameters are studied in Bensoussan et al. (2014) and Pun and Wong (2015), respectively. More specifically, the insurer's reinsurance proportion is a decreasing (increasing) function of her relative interest when the correlation between her and her competitor is positive (negative). As Pun and Wong (2015) point out, the formulation of ambiguity aversion primarily addresses robustness to the drift estimate of the (relative) wealth process rather than to diffusion. When an insurer is more ambiguity-averse, her robust reinsurance strategy tends to be more conservative. However, it is interesting to examine the effect of a competitor's ambiguity aversion on an insurer's strategy and value function under relative performance concerns. We provide an economic interpretation via numerical studies.

4.1. Numerical Studies

We use numerical studies to illustrate the effects of relative performance concerns and ambiguity aversion, and examine the sensitivity of our model parameters to the equilibrium reinsurance strategies. We use the same set of model parameters for the insurers' surplus processes as those in Bensoussan et al. (2014). The model parameters are reported in Table 1, except for those that vary in our sensitivity analyses.

Table 1: Model Parameters.

Base Parameters							
		r	T	ρ			
		0.05	4	0.2			
Insurance Companies							
Insurer 1							
$a_1\eta_1$	b_1	γ_1	κ_1	$\xi_1^{G_1}$	$\xi_1^{\bar{G}_2}$	ξ_1^S	$\xi_1^{\bar{Y}}$
6.8	8	0.1	0.7	0	1	1	1
Insurer 2							
$a_2\eta_2$	b_2	γ_2	κ_2	$\xi_2^{G_2}$	$\xi_2^{\bar{G}_1}$	ξ_2^S	$\xi_2^{\bar{Y}}$
3	5	0.3	0.5	0	1	1	1

We set $Q_k = Q_k^{[0,1]}$ in our numerical studies, i.e., only proportional reinsurance is considered. However, the conclusion is the same for $Q_k = Q_k^{[0,+\infty)}$, as their solutions are usually the same for $q_k^* < 1$. We view the ambiguity aversion coefficients with values 0 and 100 as extreme cases corresponding to no robustness and full robustness (all alternative measures are equal), respectively. We report insurer 2's strategy and value function only because the two insurers are symmetric in terms of properties.

Figure 1 shows the effects of κ_2 and $\xi_1^{G_1}$ on equilibrium reinsurance strategy $q_2^*(0)$ at time $t = 0$ for $\rho = 0.2$ and $\rho = -0.2$. The left plot shows that if the correlation between the two insurers' surplus processes is positive, then the reinsurance proportion $(1-q)$ decreases if the level of relative concern (κ_2) increases and/or the competitor's ambiguity aversion ($\xi_1^{G_1}$) decreases. In other words, for a positive correlation, an insurer retains a greater proportion of her claim when her competitor is aggressive. The right plot shows the situation of a negative correlation, for which an opposite result

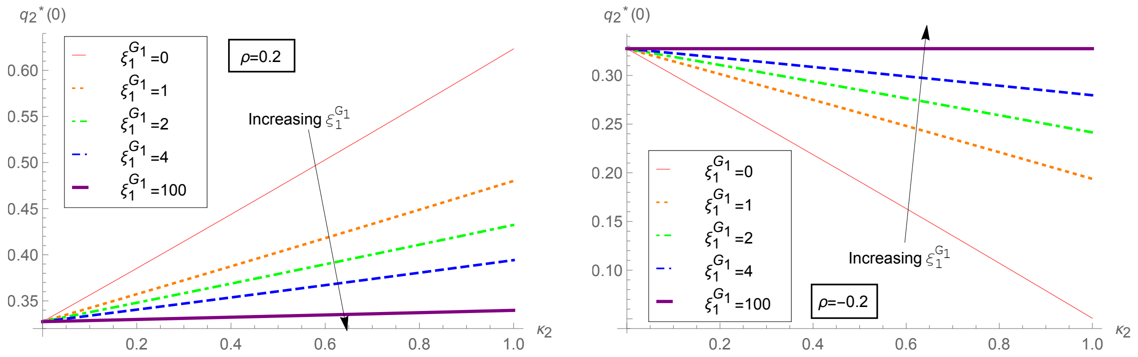

Figure 1: Effect of κ_2 on $q_2^*(0)$ with different $\xi_1^{G_1}$.

is obtained. Now, the reinsurance proportion increases with an increase in the level of relative concern and/or a decrease in the competitor's degree of ambiguity aversion. The underlying insurer becomes conservative (aggressive) when her competitor is aggressive (conservative). However, in both situations, if the underlying insurer is more concerned about her competitor's performance, then the ambiguity aversion of that competitor plays a more important role in her decision-making process.

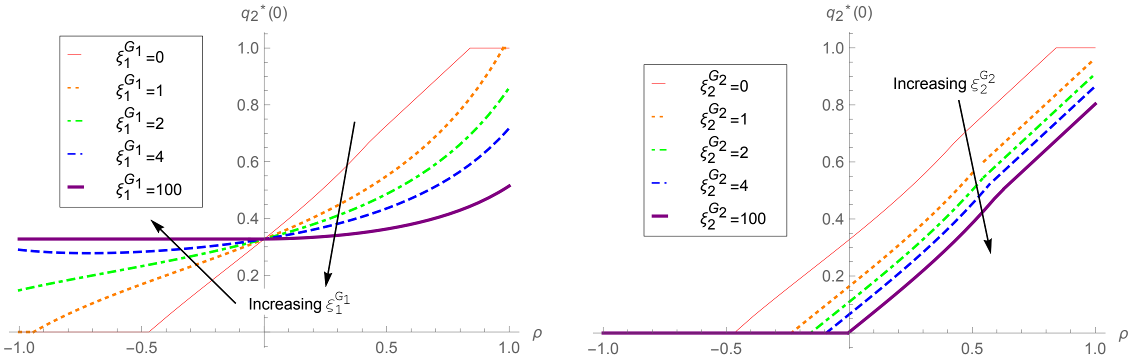

Figure 2: Effect of ρ on $q_2^*(0)$ with different $\xi_1^{G_1}$ and $\xi_2^{G_2}$.

The sign of the correlation is crucial to each insurer's decision. Figure 2 shows the sensitivity of ρ to equilibrium reinsurance strategy $q_2^*(0)$ with different $\xi_1^{G_1}$ and $\xi_2^{G_2}$. We can observe that $q_2^*(0)$ is an increasing function of ρ when relative performance is a concern. It can be seen from the right plot that the insurer, who has less confidence in her own surplus process, will increase her reinsurance proportion. The left plot further confirms our observation from Figure 1: the insurer's reinsurance strategy tends to be the optimal (single-agent) reinsurance strategy when her competitor is more conservative and there is a negative correlation between the two insurers' surplus processes.

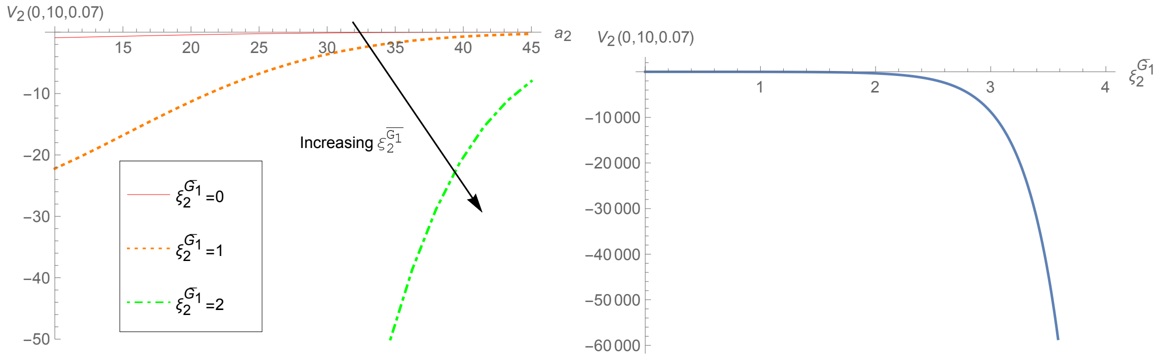

Figure 3: Effects of a_2 and $\xi_2^{\bar{G}_1}$ on $V_2(0,10,0.07)$.

Figure 3 shows the effects of a_2 and $\xi_2^{\bar{G}_1}$ on $V_2(0,10,0.07)$ at time $t=0$, the initial value function of insurer 2. Although the insurer 2's ambiguity aversion in terms of her competitor's surplus process does not affect her reinsurance strategy, it does diminish her value function, primarily because of the penalty for ambiguity. It is natural that the value function would be an increasing function of the claim rate.

5. Conclusion

This paper presents the general formulation of the robust non-zero-sum stochastic differential reinsurance game. The Nash equilibrium is characterized by a coupled nonlinear system and PDEs for the value functions. We remark here that the reinsurance game problem presented in this paper nests the single-agent optimization problem with or without model uncertainty as special cases. When the risk-free interest rate is zero ($r=0$), the equilibrium strategies also minimize the two insurers' ruin probabilities individually, as Browne (1995) points out.

The effects of ambiguity aversion and relative performance concerns are illustrated with numerical examples. We find that the more concerned an insurer is about her competitor, the more likely she is to retain claim and a risky portfolio. When there is a positive correlation between two insurers' claim processes, when one increases (decreases) her own reinsurance proportion, the other will follow suit. However, the opposite is the case for a negative such correlation. In general, the introduction of ambiguity aversion to the reinsurance game problem blurs the effects of the model parameters, the drift term in particular.

The correlation between the surplus processes of two insurers plays an important role in their reinsurance strategies, although the introduction of ambiguity aversion slightly reduces the effect of such correlation. The correlation robustness considered by Fouque et al. (2014) seems to be a promising and interesting direction for future research. Although the focus of this paper is on the reinsurance decision, the optimization techniques can be extended to the investment-reinsurance decision. Moreover, we can consider a generic model for risky assets, such as the one considered in Pun et al. (2015). A more involved correlation structure for all diffusion processes may be considered in the future research in order to study the impacts of economic situation on the reinsurance strategies.

Acknowledgement

The authors are grateful to the editor, the associated editor, and the anonymous referee for their valuable comments.

References

Anderson, E.L., L. Hansen, T. Sargent. 1999. Robustness, Detection and the Price of Risk. *Working paper of the University of Chicago*.

Bai, L., J. Guo. 2008. Optimal Proportional Reinsurance and Investment with Multiple Risky Assets and No-Shorting Constraint. *Insurance: Mathematics and Economics*. **42**, 968-975.

Bensoussan, A., C.C. Siu, S.C.P. Yam, H. Yang. 2014. A Class of Non-Zero-Sum Stochastic Differential Investment and Reinsurance Games. *Automatica*. **50**, 2025-2037.

Boyle, C. 2015. Reinsurance Industry Faces Uncertain Future as Time Moves On. *Insurance Journal*. July 2, 2015. www.insurancejournal.com/news/international/2015/07/02/373959.htm.

Browne, S. 1995. Optimal Investment Policies for a Firm With a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin. *Mathematics of Operations Research*. **20**, 937-958.

Cao, Y., N. Wan. 2009. Optimal Proportional Reinsurance and Investment Based on Hamilton-Jacobi-Bellman Equation. *Insurance: Mathematics and Economics*. **45**, 157-162.

Cole, C.R., K.A. McCullough. 2006. A Reexamination of the Corporate Demand for Reinsurance. *Journal of Risk and Insurance*. **73**, 169-192.

DeMarzo, P.M., R. Kaniel, I. Kremer. 2008. Relative Wealth Concerns and Financial Bubbles. *The Review of Financial Studies*. **21**, 19-50.

Ellsberg, D. 1961. Risk, Ambiguity, and the Savage Axioms. *Quarterly Journal of Economics*. **75**, 643-669.

Espinosa, G.-E., N. Touzi. 2015. Optimal Investment under Relative Performance Concerns. *Mathematical Finance*. **25**, 221-257.

Fouque, J.-P., C.S. Pun, H.Y. Wong. 2014. Portfolio Optimization with Ambiguous Correlation and Stochastic Volatilities. *Working paper of UC Santa Barbara and The Chinese University of Hong Kong*.

Garven, J.R., J. Lamm-Tennant. 2003. The Demand for Reinsurance: Theory and Empirical Tests. *Insurance and Risk Management*. **7**, 217-238.

Klugman, S.A., H.H. Panjer, G.E. Willmot. 2008. *Loss Models*. Hoboken, NJ: John Wiley.

Knight, F.H. 1921. *Risk, Uncertainty and Profit*. New York: Houghton Mifflin.

Liu, Y., J. Ma. 2009. Optimal Reinsurance/Investment Problems for General Insurance Models. *The Annals of Applied Probability*. **19**, 1495-1528.

Luo, S., M. Taksar, A. Tsoi. 2008. On Reinsurance and Investment for Large Insurance Portfolios. *Insurance: Mathematics and Economics*. **42**, 434-444.

Maenhout, P.J. 2004. Robust Portfolio Rules and Asset Pricing. *Review of Financial Studies*. **17**, 951-983.

Promislow, S.D., V.R. Young. 2005. Minimizing the Probability of Ruin when Claims Follow Brownian Motion with Drift. *North American Actuarial Journal*. **9**, 110-128.

Pun, C.S., S.F. Chung, H.Y. Wong. 2015. Variance Swap with Mean Reversion, Multifactor Stochastic Volatility and Jumps. *European Journal of Operational Research*. **245**, 571-580.

Pun, C.S., H.Y. Wong. 2015. Robust Investment-Reinsurance Optimization with Multiscale Stochastic Volatility. *Insurance: Mathematics and Economics*. **62**, 245-256.

Schmidli, H.. 2002. On Minimizing the Ruin Probability by Investment and Reinsurance. *The Annals of Applied Probability*. **12**, 890-907.

Yi, B., Z. Li, F.G. Viens, Y. Zeng. 2013. Robust Optimal Control for an Insurer with Reinsurance and Investment under Heston's Stochastic Volatility Model. *Insurance: Mathematics and Economics*. **53**, 601-614.

Zhang, X., T.K. Siu. 2009. Optimal Investment and Reinsurance of an Insurer with Model Uncertainty. *Insurance: Mathematics and Economics*. **45**, 81-88.