Topologically controlled multiskyrmions in photonic gradient-index lenses

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(Received 24 September 2023; revised 21 November 2023; accepted 19 January 2024; published 13 February 2024)

Skyrmions are topologically protected quasiparticles, originally studied in condensed-matter systems and recently in photonics, with great potential in high-density information storage. Despite the recent attention, most optical solutions require complex systems yet produce limited topologies. Here we demonstrate an extended family of quasiparticles beyond normal skyrmions that are controlled in compact photonic gradient-index media, extending to higher-order members such as multiskyrmions and multimerons, with increasingly complex topologies. We introduce multiple topological numbers (centrality, radiality, vorticity, and polarity) in addition to the skyrmion number to describe these photonic quasiparticles. Our compact creation system lends itself to integrated and programmable solutions of complex particle textures, with potential impacts on both photonic and condensed-matter systems for revolutionizing topological informatics and logic devices.

DOI: 10.1103/PhysRevApplied.21.024025

I. INTRODUCTION

In the current age of information explosion, the pursuit of next-generation information carriers and data storage is endless, eminently benefiting our daily lives. Since Skyrme [1] proposed a model to unify a large class of fundamental particles by methods of topology in the 1960s, known today as skyrmions, these have emerged as one of the highest-potential information carriers due to their topologically robust spin textures localized in ultrasmall regions, in particular, which has revolutionized the high-density-data-storage technologies in magnetic materials in recent years [2–5]. To further expand the capacity in informatic applications, it is highly desired to manipulate complex quasiparticles with higher-order topological textures in addition to the fundamental skyrmions [6]. For instance, the meron textures with fractional skyrmion numbers can be controlled in chiral magnets [7], and recently in antiferromagnets at room temperature [8]. Novel quasiparticles such as skyrmion bags [9], skyrmion bundles [10], and skyrmion braids [11] with large range control of skyrmion numbers were created in magnetic materials. Moreover, exotic spatially knotted quasiparticles of hopfions [12–14], torons [15,16], heliknotons [17], and polyskyrmionomers [18] characterized by complex topologies were designed in magnets, colloids, and chiral liquid crystals. The realization of novel quasiparticles in diversified condensed-matter systems has triggered the development of static or quasistatic ultrahigh-density-data-storage techniques.

Optical or photonic skyrmions, which were recently realized, provide new degrees of freedom in terms of the construction of topological quasiparticles [19]. Following the initial demonstrations of the formation of photonic skyrmions in surface plasma by evanescent electromagnetic fields [20,21], as well as optical spins [22,23], photonic skyrmions have also been generated in, for example, optical polarization Stokes vector fields in free space [24–27], electromagnetic fields in space-time [28–30], and pseudospins in nonlinear crystals [31]. However, previous studies have demonstrated a very limited number of topological states, often requiring the use of complex, bulky, and expensive systems.

Here we demonstrate both experimentally and theoretically an extended class of photonic quasiparticles constructed by vectorial structured light in gradient-index

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(GRIN) lenses. The quasiparticles that are formed with use of vectorial structured light exhibit novel forms of higher-order configurations beyond elementary skyrmions with increasingly complex topology and geometry. These complex quasiparticles can be characterized by multiple topological numbers and their diversified topological states can be controlled by cascaded GRIN lenses, as a generation method with a highly symmetric and high-quality beam profile [32]. The compact generator of high-quality photonic quasiparticles offers the potential to be integrated with on-chip solid-state storage devices and the application of ultrahigh-capacity information coding and transfer [33].

II. RESULTS

A. Vectorial optics of GRIN lenses

GRIN lenses are a kind of optical element that possesses a spatially varying refractive-index profile as well as birefringence, which enable focusing, imaging, and shaping of light through a rodlike structure [32,34–36], as shown in Fig. 1. GRIN lenses can be fabricated with diameters on the submillimeter scale, which means that GRIN-lens arrays can be formed with dimensions on the order of millimeters [Fig. 1(a2)]. Each GRIN lens can be composed of different cascaded segments with different refractive-index-gradient designs, together with segments of quarter-wave plates (QWPs) and half-wave plates (HWPs). If we input a plane wave into one side of a cascaded GRIN lens, the output will be light with spatially varying polarization patterns [32,36]. Here we show that through judicious assembly of the cascades of GRIN lenses combined with tuning of the polarization of the input light, we can precisely generate a wide variety of vectorial light fields as high-order photonic skyrmions with complex topologies, which can be arranged in a very compact volume.

Note that the formation of a vector beam, such as a skyrmionic beam, in a GRIN lens is not similar to the abrupt topological phase transition in condensed matter. It is a continue transition (without abrupt change) process to form a polarization pattern fulfilling topological skyrmion mapping, i.e., unwrapping the Poincaré sphere ($\mathbb{S}^2$) to a localized real space ($\mathbb{R}^2$) [19]. For instance, when uniform circularly polarized light goes into the GRIN-lens end surface, the polarization states except that at the center will gradually change into various elliptical polarization states covering all the states on a Poincaré sphere. There is a distinction between such continuous deformation of the optical field and the smooth deformation preserving its topology. For the photonic skyrmion topology, it has just been shown that this is the smooth case if the deformation can be written as a coordinate transformation [37]. When the mapping function is not smoothly deformed by some system, the skyrmion number will change, even if the field changes continuously, as has been shown even in free space with mixed-order modes [38]. Our GRIN system is evidently such a system too.

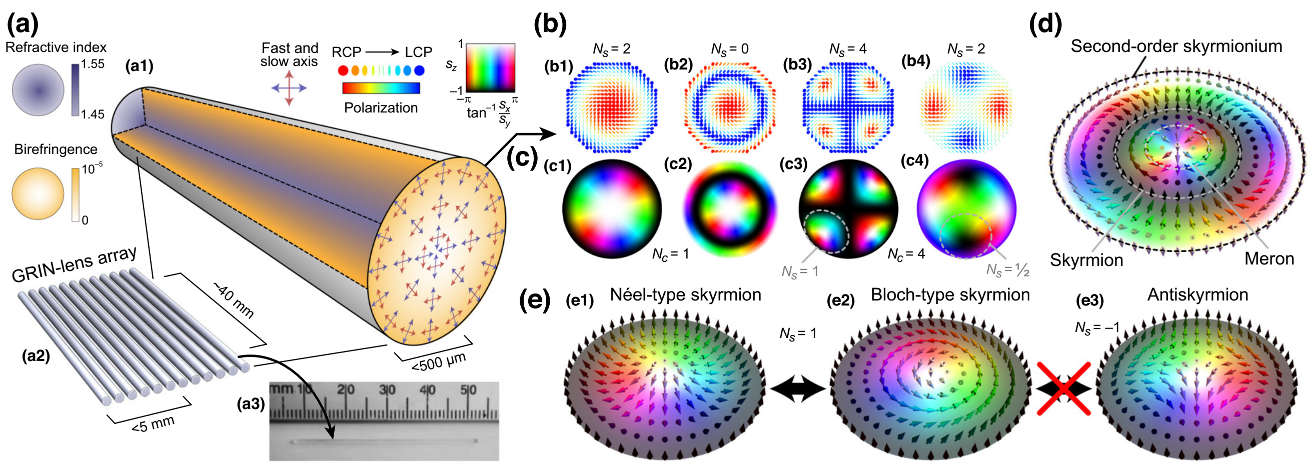

FIG. 1. GRIN lenses for photonic skyrmion generation. (a) Schematics of (a1) the GRIN lens and (a2) the GRIN-lens array, with radial and azimuthal distributions of the fast and slow axes at their cross section, the radius of which can be less than 500 μm, and (a3) a picture of a real GRIN lens. (b),(c) Diverse polarization vectorial light fields corresponding to the different complex photonic quasiparticles generated can be controlled by cascades of GRIN lenses, represented by (b) polarization-ellipse distributions and (c) Stokes-vector distributions, which include (b1),(c1) a skyrmion of $N_s = 2$, (b1),(c1) a skyrmion of $N_s = 0$, (b2),(c2) a skyrmionium of $N_s = 0$, (b3),(c3) a quadruskyrmion of $N_s = 4$ comprising four elementary skyrmions of $N_s = 1$, and (b2),(c2) a quadrumeron of $N_s = 2$ comprising four elementary merons of $N_s = 1/2$. (d) Conceptual schematic of the vector texture of a second-order skyrmionium ($N_r = 2, N_s = 0$), which includes a skyrmion ($N_r = 1, N_s = 2$) and meron ($N_r = 1/2, N_s = 1$) structure in its subspace. (e) Topological protection of the photonic skyrmions, where (e1) a Néel-type skyrmion can be transformed into (e2) a Bloch-type skyrmion on propagation and is resilient to perturbation under the same skyrmion number of $N_s = 1$. It is, however, impossible for it to be transformed into (e3) an antiskyrmion of different skyrmion number N_s of -1.

B. Topological photonic quasiparticles

A "skyrmion" refers to a three-component real vector or spin texture mapped from a two-sphere to a localized 2D real space, denoted as $\mathbf{s}(x,y) = [s_x(x,y), s_y(x,y), s_z(x,y)]$, with basic topology characterized by the skyrmion number N_s (which counts how many times the spins can wrap around a two-sphere with full azimuth):

$$N_s = \frac{1}{4\pi} \iint_\sigma \mathbf{s} \cdot \left(\frac{\partial \mathbf{s}}{\partial x} \times \frac{\partial \mathbf{s}}{\partial y}\right) \mathrm{d}x\mathrm{d}y, \qquad (1)$$

where σ represents the region in which the particle is confined. Photonic skyrmions or quasiparticles can be constructed by the polarization Stokes vectors, i.e., the vector defined by the three Stokes parameters, of the obtained structured light, analogously to the magnetic spin used to construct magnetic skyrmions [19,38,39]. Figure 1(b1) shows a theoretical result for the polarization-ellipse distribution of a skyrmionic light field generated by a customized GRIN lens, the Stokes vector field of which shows a fundamental skyrmion of $N_s = 2$ [Fig. 1(c1)]. Note that for the fundamental skyrmion, the signal of the skyrmion number was determined by its central spin with a spin-up state (+) or a spin-down state (−), which is defined as the polarity, $N_p = \mathrm{sgn}(N_s)$, and the absolute value of the skyrmion number determines the vortex charge of the transverse- (x,y) component distribution, defined as the vorticity, $N_v = |N_s|$. The result in Fig. 1(c1) shows a central spin with spin-up state and vortex charge of the transverse distribution of 2, thus resulting in $N_s = 2$.

In addition to the fundamental skyrmion, cascaded GRIN lens can also generate generalized quasiparticles, for instance, the skyrmionium [41], a skyrmion radially nested with another skyrmion with opposite topological number, resulting in a skyrmion number N_s of $(+2) + (-2) = 0$ [Figs. 1(b2) and 1(c2)]. If the radially nested number, N_r, is larger than 2, the extended configurations were previously called "$N_r\pi$ skyrmions" or "target skyrmions" [42,43], which, we argue, can also be easily realized in our cascaded GRIN-lens system. The versatility of GRIN lenses that are cascaded to form arrays allows us to access novel forms of photonic quasiparticles that are challenging to observe in other systems, for instance, the exotic multiskyrmions and multimerons, comprising multiple elementary skyrmions and merons in subspace with high symmetry. Figures 1(b3) and 1(c3) demonstrate a quadruskyrmion with four elementary skyrmions of unit skyrmion number, resulting in total skyrmion number $N_s = 4$. Figures 1(b4) and 1(c4), on the other hand, demonstrate a quadrumeron with four elementary merons of half skyrmion number, resulting in total skyrmion number $N_s = 2$. To describe the symmetry in multiskyrmions and multimerons, we define a number of centrality, N_c, which counts how many spin-up or spin-down center points there are. The centrality can also be interpreted as the numbers of singularities of the transverse-component distribution of the multiskyrmionic field, which thus can be detected by an algorithm for counting multiple singularities of a complex light field [44]. The quadruskyrmion and quadrumeron are both of centrality $N_c = 4$ [Figs. 1(c3) and 1(c4)]. On the other hand, the skyrmions, skyrmioniums, and target skyrmions are always of centrality $N_c = 1$ [Figs. 1(c1) and 1(c2)]. Figure 1(d) conceptually demonstrates the elementary relationship among a skyrmionium, a skyrmion, and a meron.

Demonstrated by prior work, such photonic Stokes skyrmions possess the property of topological stability, meaning that the skyrmion number remains invariant on propagation evolution [39,45], and is robust with regard to environmental perturbations [46]. Once a skyrmion is generated by a GRIN lens, its topology will be protected on further propagation in isotropic media or coupled in free space [39,45]. For instance, a Néel-type skyrmion with hedgehoglike texture [Fig. 1(e1)] can evolve into a Bloch-type skyrmion with vortexlike texture [Fig. 1(e2)], provided they are of the same skyrmion number $N_s = 1$; however, it is impossible for such a skyrmion to be transformed into an antiskyrmion of opposite skyrmion number $N_s = -1$ [Fig. 1(e3)]. The multiskyrmions composed of multiple elementary skyrmions can still possess the potential of topological stability, which can also be coupled with free-space paraxial propagation; see Supplemental Material [40].

C. Experimental generation and topological control

We have designed a series of GRIN-lens cascades to experimentally generate a diverse set of complex photonic quasiparticles with controlled multiple topological numbers in addition to skyrmion number N_s, which include polarity N_p, vorticity N_v, radiality N_r, and centrality N_c. Firstly, we show the topological control of the fundamental skyrmions. Using a GRIN-lens segment with input light of left-handed circular polarization (LCP), we can create a second-order skyrmion of $N_p = -1$ and $N_v = 2$; see the left side of Fig. 2(a). To control the polarity N_p, we tune the input light polarization from LCP to linear polarization and then to right-handed circular polarization (RCP); correspondingly, the skyrmion evolves into biskyrmion and quadrumeron configurations as intermediate states and finally into the skyrmion with opposite polarity [$N_p = 1$ and $N_v = 2$; see the right side of Fig. 2(a)]. Note that for the case of linear-polarization input, the output quasiparticle will exhibit a singular symmetry inducing a skyrmion number of zero, $N_s = 0$, where the polarity, $N_p = \mathrm{sgn}(N_s)$, cannot be defined.

To control radiality, we apply a cascade of two identical GRIN-lens segments, which results in a skyrmionium; see the left side of Fig. 2(b). Polarity control by tuning input

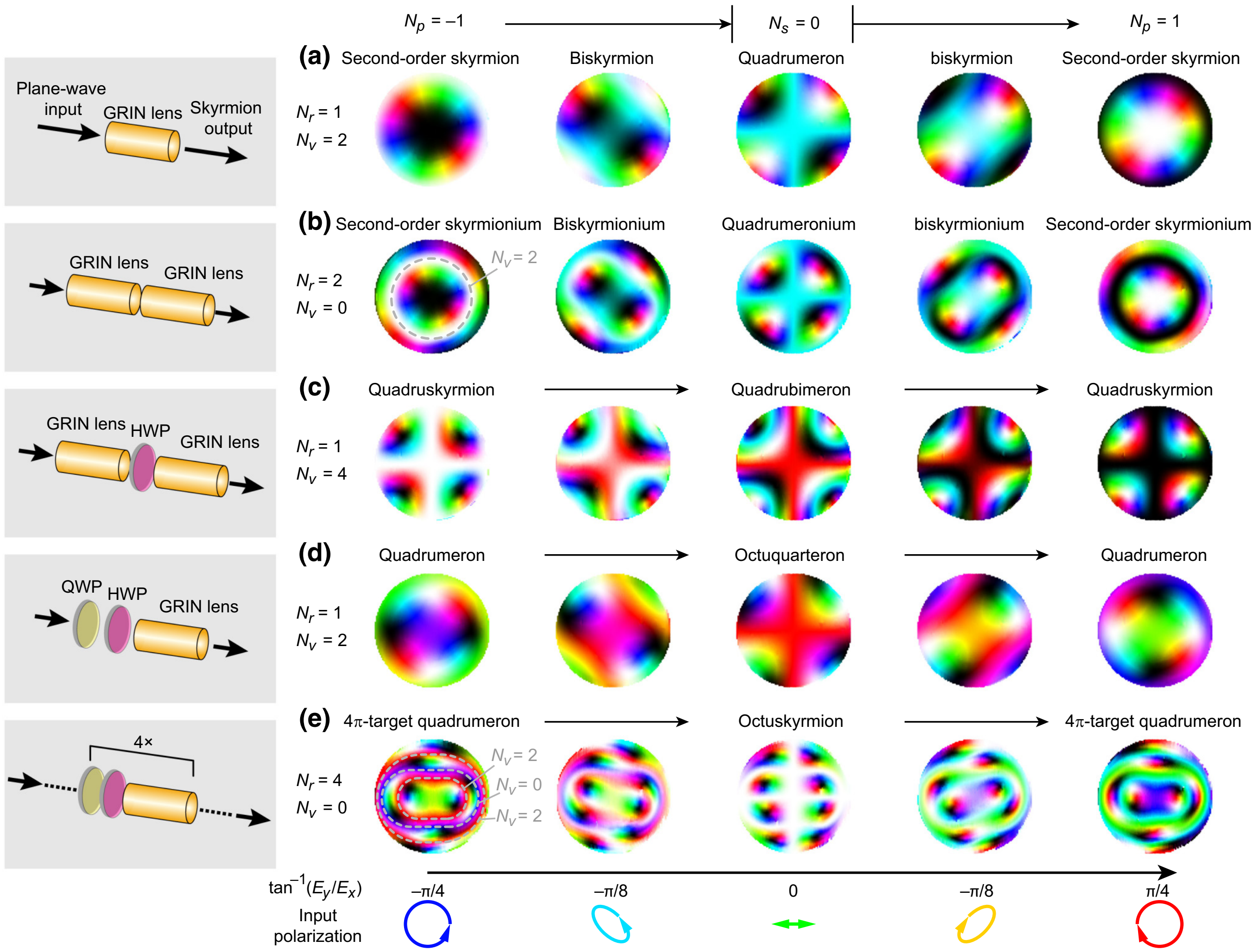

FIG. 2. Experimental results for a wide variety of topologically transformed quasiparticles obtained with use of cascaded GRIN lenses of different configurations (see corresponding simulated results in Supplemental Material [40]): (a) a skyrmion of $N_r = 1$ and $N_v = 2$ is transformed into a skyrmion of the same radiality and vorticity but opposite polarity, (b) a skyrmionium of $N_r = 2$ and $N_v = 0$ is transformed into a skyrmionium of the same radiality and vorticity but opposite polarity, (c) a quadruskyrmion of $N_r = 2$ and $N_v = 4$ is transformed into a quadruskyrmion of the same radiality and vorticity but opposite polarity, (d) a quadrumeron of $N_r = 1$ and $N_v = 2$ is transformed into a quadrumeron of the same radiality and vorticity but opposite polarity, and (e) a target quadrumeron of $N_r = 4$ and $N_v = 0$ is transformed into a target quadrumeron of the same radiality and vorticity but opposite polarity by our tuning the input light polarization from LCP to RCP. The insets show the layouts of optical element cascades for the generation of the diverse topological quasiparticle states in (a)–(e).

polarization works for this case as well, and the corresponding results are similar to those presented in Fig. 2(a) but with a layer of radially nested structure, resulting in a skyrmion number of zero.

To obtain more-complex quasiparticles with increased skyrmion number, we show experimental results for a quadruskyrmion obtained by our applying a cascaded configuration of a GRIN lens followed by an HWP followed by another GRIN lens; see the left side of Fig. 2(c), which comprises four skyrmions with the same unit vorticity, with centrality $N_c = 4$. Polarity control can also be applied to the quadruskyrmion. In addition, by our switching to a cascade involving a QWP followed by an HWP and then a GRIN lens, the quadrumeron can be generated, comprising four elementary merons of the same half charge, but with the same centrality N_c of 4; see Fig. 2(d) [distinct from the quadrumeron in the middle of Fig. 2(a)].

By varying the design of the cascaded GRIN-lens array, we can access increasingly complex topologies. With cascading multiple times in the style of a QWP followed by an HWP and then a GRIN lens, we can generate quasiparticles of a multi-radially-nested quadrumeronium and a target quadrumeron. Figure 2(e) shows an experimental quadrumeronium and target quadrumeron with radiality $N_r = 4$ obtained by a multiple of four cascades of the configuration described previously, i.e., a four-times nested matryoshka-like structure based on a basic central structure of a quadrumeron as in Fig. 2(d). All the distributions of the experimental quasiparticle fields and all the topological numbers can be detected by the methods of Stokes polarimetry of structured light, and details are provided in Supplemental Material [40] and Refs. [44,47,48].

On the basis of the principle outlined above, we can summarize a systematic rule for guiding the practical generation of these topological quasiparticles. We note that N_{cascade} is the number of combinations to be respectively cascaded together, and N_{GRIN}, N_{QWP}, and N_{HWP} are the numbers of GRIN lenses, QWPs, and HWPs in each combination, respectively. For the case of pure GRIN-lens cascading, i.e., each combination includes only one GRIN lens ($N_{\text{GRIN}} = 1$), the relation connecting the topological number and the cascading number can be expressed as $(N_r, N_v, N_c) = (N_{\text{cascade}}, 2 \times (N_{\text{cascade}} \mod 2), 1)$. For the cases of QWP-HWP–GRIN-lens cascading, the relation connecting the topological numbers and the elements in the GRIN system can be expressed as $(N_r, N_v, N_c) = (N_{\text{cascade}}, 2 \times (N_{\text{GRIN}} + N_{\text{QWP}}) \times (N_{\text{cascade}} \mod 2), 1)$. Also, the polarity $N_p = \pm 1$ and the sign are determined by RCP or LCP input. On the basis of this systematic rule, we can customize quasiparticles with on-demand topological numbers with related element cascading design.

Moreover, the quasiparticles that can be accessed are not limited to those demonstrated here. For example, we could easily generate and control the following photonic skyrmions: quadruskyrmioniums, octuskyrmions, octumerons, octuskyrmioniums, octumeroniums, etc. Furthermore, there is still a huge space to explore in terms of the creation of more-exotic and more-complex quasiparticles that have not existed before.

III. DISCUSSION

In summary, we have created a novel family of topological quasiparticles controlled by a very compact system, i.e., photonic-GRIN-lens cascades, extending the fundamental skyrmions to exotic radially nested skyrmioniums, multiskyrmions, and multimerons, which possess sophisticated spin textures characterized by multiple topological numbers. The scheme presented here could be further extended in terms of diversity of the photonic quasiparticles by taking advantage of the versatility of potential designs of the cascades of the GRIN lenses, which, importantly, can always be integrated and assembled within a very compact and confined transverse region. Therefore, this work opens a direction to control skyrmionic topologies of light in more-flexible and more-compact optical systems in the future; for instance, optical skyrmions in fibers.

The successful generation of skyrmionic beams from a GRIN system can also motivate more-extended studies on topological beam formation and topological light-matter interaction. One can create an optical field that is skyrmionic by changes that are continuous with respect to the field when it is propagating in the GRIN lenses, where the skyrmion number also changes, but that do not smoothly deform the mapping function. This might beg the question of why one should bother with such photonic topology. Here one must appreciate that care must be taken to ensure that once created, any change is a smooth deformation of the map, and here the rules of the game must still be developed—the topic is still very new. But the opportunity is huge: just as we know that matter responds to light with an enhanced effect when the light is structured (chiral light for chiral matter, orbital angular momentum (OAM) light for rotational probes, etc.), so too can we anticipate that topological matter will respond to topological light. The interaction and exchange of topologies from light to matter, not limited by GRIN lenses, and back holds exciting prospects for many fields.

Finally, the photonic quasiparticles proposed and discovered here in photonic fields are also highly desired for other communities and in other condensed-matter systems. On the one hand, quasiparticles were transferred to photonics from magnetic and condensed-matter systems to create interdisciplinary studies; on the other hand, photonic methods can inspire new forms of quasiparticles that have not existed before in any system, leading to new fundamental physical effects that could be explored and new applications such as topological optoelectronic devices.

IV. METHODS

A. Stokes polarimetry of structured light

The experimental reconstruction of the optical quasiparticles is achieved through Stokes polarimetry, more specifically, through the reconstruction of the distributions of three normalized Stokes parameters s_1, s_2, and s_3, which were computed from a set of polarization projection and intensity measurements. Figure 3 demonstrates the setup for Stokes-vector-field measurements. When the generated skyrmionic beams from a GRIN lens or GRIN-lens cascades pass through a QWP and a polarizer, spatial-variance intensity patterns are recorded by a CCD camera. We used the QWP, polarizer, and CCD camera to form a Stokes polarimeter to measure the polarization distribution by rotating the QWP to four different angles. This is a well-known process that was reported in previous technical reviews [47,48]. The principal equations for calculation of

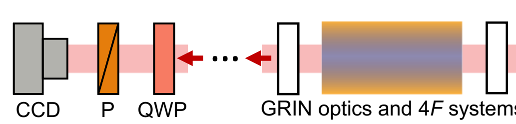

FIG. 3. The configuration setup used for Stokes polarimetry. The GRIN-lens cascades are combined with different types of GRIN lens and interstitial optical elements (such as a $4f$ system). Although the GRIN lens itself can be an imaging device, we use a $4f$ system for extended imaging purposes. P, polarizer.

the polarization field are as follows:

$$S^n_{\rm out} = M_{\rm P} \cdot M^n_{\rm QWP} \cdot S_{\rm in} = M_{\rm P} \cdot M^n_{\rm QWP} \cdot (A^{-1} \cdot I), \quad (2)$$

where $S_{\rm in}$ is the Stokes vector of the incident light field, $M_{\rm P}$ and $M^n_{\rm QWP}$ are Mueller matrices of the corresponding polarizer and QWP, $S^n_{\rm out}$ is the output Stokes vector for the nth fast-axis orientation state of the QWP, A is an instrument matrix, which is derived from $M_{\rm P}M^n_{\rm QWP}$, and $I = A \cdot S_{\rm in}$ is the intensity information recorded by the CCD camera. Note that a 2D distributed Stokes vector field $S_{\rm in}(x, y)$ consists of $[1, s_1(x, y), s_2(x, y), s_3(x, y)]$, where s_1, s_2, and s_3 are the vector components of the Stokes vector [47].

B. Detection of quasiparticle topological numbers

After reconstruction of the Stokes field of a quasiparticle, $\mathbf{s}(x, y) = [s_1(x, y), s_2(x, y), s_3(x, y)]$, the number N_r can be easily determined by counting the radially nested structures in the vector field. For the case of $N_r > 1$, we always care about the basic structure in the center, which we refer to here as the "core structure," and the outer layers are just the same as the core structure only with staggered polarity. Then we care about the number N_c of the core structure, i.e., how many spin-up or spin-down center points in the core region exist, which is equivalent to how many singularities of the transverse component $[s_1(x, y), s_2(x, y)]$ are contained in the core region. This singularity-counting problem can be solved by an algorithm for singularity searching of a complex multisingularity light field [44]. To further characterize structural details, the skyrmion density distribution of the quasiparticle is calculated by $\rho_s(x, y) = \mathbf{s}(x, y) \cdot [\partial_x \mathbf{s}(x, y) \times \partial_y \mathbf{s}(x, y)]$. If $N_r > 1$, the skyrmion density distribution will also correspondingly show a radially nested structure. Next we integrate the skyrmion density to the core region, and the skyrmion number N_s of the core structure is obtained. Finally, the polarity and vorticity can be obtained by $N_p = \text{sgn}(N_s)$ and $N_v = |N_s|$, and all four numbers (N_p, N_r, N_v, and N_c) required to characterize the photonic quasiparticles (multiskyrmions and multimerons) are measured; see the detailed analysis and calculation results for experiments in Supplemental Materail [40].

ACKNOWLEDGMENTS

C.H. acknowledges the support of the John Fell Fund from the University of Oxford and the Junior Research Fellowship from St John's College, University of Oxford. Y.S. acknowledges support from a Nanyang Technological University Start Up Grant and Singapore Ministry of Education via the AcRF Tier 1 grant (MOE-T1EP5P#023669-00001).

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